MODULATED WAVES IN NONLINEAR DISPERSIVE MEDIA

V. I. KARPMAN and E. M. KRUSHKAL'

Novosibirsk State University

Submitted January 10, 1968

Zh. Eksp. Teor. Fiz. **55**, 530–538 (August, 1968)

We consider nonstationary waves in nonlinear dispersive media (under the assumption that the amplitude, wavelength, etc., are sufficiently slow functions of time and space). Decay of a plane wave into separate wave packets is investigated for the case when the wave is unstable with respect to modulation.

1. FUNDAMENTAL EQUATIONS

WHITHAM[1,2] developed a general method for the averaged description of nonstationary waves in nonlinear dispersive media. We apply this method to those cases in which the nonstationary wave can be represented with sufficient accuracy as a quasistationary one, with slowly changing (with space and time) amplitude, wavelength, etc.

As is well known, the existence of nonlinear stationary waves (by the latter we mean waves in which all the oscillating quantities are functions only of $x - Vt$) is possible only because of the interaction of nonlinear and dispersive effects (for example, in ordinary gasdynamics, the exact stationary solutions of the Euler equations do not exist). In the case of a periodic stationary wave with wavelength λ, it is convenient to introduce the phase of the oscillations $\theta = kx - \omega t$, where $k = 2\pi/\lambda$, $\omega = kV$; then all the oscillating quantities can be represented as periodic functions of the phase θ (with period 2π). Here the frequency of the nonlinear wave ω depends not only on the wave number k, but also on other quantities which are assumed small in the linear approximation (for example, on the amplitude of the wave a and so forth). It is usually not difficult to find this dependence by considering periodic solutions of the entire set of equations, which depend on x and t only through the phase $\theta = kx - \omega t$. In what follows, we shall limit ourselves to the case in which the frequency ω depends only on a single nonlinear parameter—the amplitude a. We shall call such a dependence

$$\omega = \omega(k^2, a^2) \tag{1.1}$$

(for simplicity, the medium is assumed to be isotropic) the nonlinear dispersion equation. Here we shall consider waves of sufficiently small (but finite) amplitude, such that one can restrict oneself to the first two terms of the expansion of the dispersion equation (1.1) in powers of a^2, i.e.,

$$\omega \approx \omega^0(k^2) + (\partial\omega / \partial a^2)_0 a^2, \tag{1.2}$$

where $\omega^0(k^2)$ determines the dispersion law in the linear approximation.

In the present paper, we investigated modulation processes in nonlinear nonstationary waves where the amplitude, wavelength, etc., change sufficiently slowly at distances of the order of a wavelength and over a time of the order of the period of oscillation. Here the problems of fundamental interest to us occur in the region where the "adiabatic" approximation of Whitham-Lighthill[1-3] becomes applicable. First of all, we note that in the case in which one can neglect the nonlinear terms of order higher than a^2, the fundamental equations for the slowly changing quantities in the adiabatic approximation can be obtained by starting out not from the general formalism of Whitham, but from the more elementary eikonal method (see, for example,[4]).

For example, we consider a wave which differs little from the stationary wave (the latter has the parameters ω_0, k_0, a_0, which satisfy the dispersion equation (1.1)). When the phase of the wave can be represented in the form

$$\theta(\mathbf{r}, t) = k_0 x - \omega_0 t + \varphi(\mathbf{r}, t), \tag{1.3}$$

where $\varphi(r, t)$ is the contribution to the phase made by the nonstationarity. Substituting in Eq. (1.2)

$$\begin{gathered}\omega(\mathbf{r}, t) = -\theta_t = \omega_0 - \varphi_t,\\ k(\mathbf{r}, t) = \nabla\theta(\mathbf{r}, t) = \mathbf{k}_0 + \nabla\varphi(\mathbf{r}, t)\end{gathered} \tag{1.4}$$

and limiting ourselves to terms of second order in a, $\nabla\varphi/k_0$, we get

$$\varphi_t + u_0\varphi_x + {}^1/_2 u_0'\varphi_x^2 + \frac{u_0}{2k_0}(\nabla_\perp\varphi)^2 + \left(\frac{\partial\omega}{\partial a^2}\right)_0 (a^2 - a_0^2) = 0, \tag{1.5}$$

where

$$u_0 = \left(\frac{\partial\omega}{\partial k}\right)_{k=k_0,\, a=0}, \quad u_0' = \left(\frac{\partial^2\omega}{\partial k^2}\right)_{k=k_0,\, a=0} \tag{1.6}$$

As a second equation, we take the equation of energy transport, which can be described in the form $\partial a^2/\partial t + \operatorname{div}(u_E a^2) = 0$. In our approximation we can assume for the rate of energy transport u_E

$$u_E(k, a^2) \approx u_E(k, 0) = \partial\omega(k, 0) / \partial k,$$

i.e., we assume it to equal the group velocity in the linear approximation (in general these two velocities are different[3]). It must be recognized here that k is determined by (1.4). As a result we get, at the assumed degree of accuracy,

$$\frac{\partial a^2}{\partial t} + u_0\frac{\partial a^2}{\partial x} + u_0'\frac{\partial(\varphi_x a^2)}{\partial x} + \frac{u_0}{k_0}\nabla_\perp(\nabla_\perp\varphi a^2) = 0, \tag{1.7}$$

where u_0, u_0' are defined in (1.6).

The set of equations (1.5) and (1.7) completely describes the evolution of the quantities $a(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$. It is not difficult to show that it is equivalent to the corresponding equations for the waves of small amplitude, obtained by Lighthill,[3] who started out from the general formalism of Whitham.

In the one-dimensional case, where all the quantities depend only on x and t, it is convenient to transform to the new variables

$$\xi = x - u_0 t, \quad \tau = u_0' t \tag{1.8}$$

(ξ is the coordinate in the set of coordinates moving with the "unperturbed" group velocity u_0). Then Eqs. (1.5) and (1.7) take the form

$$\varphi_\tau + \frac{1}{2}\varphi_\xi^2 + \frac{1}{u_0'}\left(\frac{\partial\omega}{\partial a^2}\right)_0 (a^2 - a_0^2) = 0, \qquad \frac{\partial a^2}{\partial\tau} + \frac{\partial(a^2\varphi_\xi)}{\partial\xi} = 0, \tag{1.9}$$

which are identical with the hydrodynamic equations, where $\varphi(\xi, \tau)$ is the velocity potential, the role of the density is played by the quantity a^2, the "adiabatic compressibility" is determined by the equation

$$\alpha = -\frac{a_0^2}{u_0'}\left(\frac{\partial\omega}{\partial a^2}\right)_0, \tag{1.10}$$

the square of the velocity of "sound" is equal to $c_S^2 = -\alpha$. It then follows that, for the plane wave to be stable against changes in phase and amplitude, it is necessary that the quantity c_S^2 be positive, i.e., $\alpha < 0$. In the opposite case, the considered plane wave will be unstable. This result was first obtained by Lighthill.[3] It is significant that for application of this criterion it is necessary to know only the nonlinear dispersion equation in the form (1.2).

It is evident that in the case of instability ($\alpha > 0$), Eqs. (1.9) become inapplicable, generally speaking, even at small values of τ and, for a correct description of the nonstationary processes, terms should be added which contain derivatives of higher order of the slowly changing variables, which were omitted in the derivation of Eqs. (1.9) (and which are correspondingly omitted in the adiabatic approximation). It should also be noted that Eqs. (1.9) become inapplicable for rather large τ even in the case in which $c_S^2 = -\alpha$ is positive, inasmuch as the nonlinear increase in the steepness of the profile leads to the formation of "shock waves."[1,5] The appearance of large gradients in this case also requires allowance for terms with higher derivatives. These quantities are most easily obtained in the following way.

We consider the quantity $\Phi = \Psi(\mathbf{r}, t)\exp[i(k_0 x - \omega_0 t)]$, where

$$\Psi(\mathbf{r}, t) = a(\mathbf{r}, t)\exp[i\varphi(\mathbf{r}, t)]. \tag{1.11}$$

The real part of Φ describes the waves under consideration in the case of sufficiently small amplitudes. In the linear approximation, the quantity Φ satisfies the equation

$$[\hat{k}^2 - F_0(\hat{\omega})]\Phi = 0, \tag{1.12}$$

where $\hat{\mathbf{k}} = -i\nabla$, $\hat{\omega} = i\partial/\partial t$, and $k^2 = F_0(\omega)$ is the dispersion law in the linear approximation. Assuming that $\Psi(\mathbf{r}, t)$ is a slowly changing function, we expand $F_0(\omega)$ in powers of $\omega - \omega_0$ and limit ourselves to terms of second order. Substituting (1.11) in the resultant expression, and separating the real and imaginary parts, we first obtain

$$\varphi_t + u_0\varphi_x + \frac{1}{2}\left[u_0'\varphi_x^2 + \frac{u_0}{k_0}(\nabla_\perp\varphi)^2\right] - \frac{u_0'}{2a}a_{xx} - \frac{u_0}{2k_0 a}\Delta_\perp a = 0 \tag{1.13}$$

and second, Eq. (1.7).

Equation (1.13) differs from (1.5) in two respects. On the one hand, it does not contain terms proportional to a^2. This is associated with the fact that this equation was obtained from the linear Eq. (1.12). On the other hand, it contains second derivatives with respect to the coordinates, which are absent in (1.5). If we were to start out, as was done above, from the method of "geometric optices," substituting Eq. (1.4) in the linear dispersion equation, then we would have obtained Eq. (1.13) without second derivatives. Thus, it follows from a comparison of Eq. (1.13) with (1.5) that the equation for φ with account of the nonlinear terms of order a^2 and terms with derivatives up to second order, we should finally obtain the form

$$\varphi_t + u_0\varphi_x + {}^1/_2\left[u_0'\varphi_x^2 + \frac{u_0(\nabla_\perp\varphi)^2}{k_0}\right] + (\partial\omega/\partial a^2)_0(a^2 - a_0^2) - (u_0'/2a)a_{xx} - (u_0/2k_0 a)\Delta_\perp a = 0. \tag{1.14}$$

Thus, Eqs. (1.14) and (1.7) contain the complete set of equations which contain the correction terms which are not considered in the adiabatic approximation. Here the terms with second derivatives can generally have the same order as terms of order a^2 (in the case of sufficiently large gradients.[1])

We also note that the system (1.4) and (1.7) can be written in the form of a single equation for the complex amplitude of $\Psi(\mathbf{r}, t)$, determined in (1.11), that is,

$$i\left(\frac{\partial\Psi}{\partial t} + u_0\frac{\partial\Psi}{\partial x}\right) + \frac{u_0'}{2}\frac{\partial^2\Psi}{\partial x^2} + \frac{u_0}{2k_0}\Delta_\perp\Psi - \left(\frac{\partial\omega}{\partial a^2}\right)_0 (|\Psi|^2 - |\Psi_0|^2)\Psi = 0. \tag{1.15}$$

It is essential that the coefficients of the set (1.14) and (1.7) or of Eq. (1.15) are completely determined by the nonlinear dispersion equation (1.2). Therefore, waves of different nature can be considered from a single viewpoint. For example, in nonlinear optics, where the expression for the dielectric susceptibility has the form $\epsilon = \epsilon_0(1 + \epsilon'|E|^2)$, where E is the field intensity, the nonlinear dispersion equation is

$$\omega = kc/\sqrt{\varepsilon} \approx (kc/\sqrt{\varepsilon_0})[1 - (\varepsilon'/2)|E_0|^2],$$

so that (1.15) goes over into an averaged equation for the electric field, which describes the nonlinear self-action of electromagnetic waves (about the latter, see, for example, the reviews[6,7]).

The application of what has been developed above to other cases (waves in plasma, on the surface of a liquid, and so forth) also presents no difficulty. It is important here that, in the approximation considered, the corresponding equations have the form (1.15), which makes it possible to transfer directly in all these cases the results obtained in nonlinear optics.

---

[1]Of course, Eq. (1.14) could have been obtained by the method of averaging, considering the approximation which comes after the adiabatic one, but we have preferred to bring in a more descriptive derivation, which would allow us to make clear the physical meaning of the individual terms.

2. AUTOMODULATION OF AN UNSTABLE PLANE WAVE

Let us consider a wave modulated in its direction of propagation (the x axis). In the variables (1.8) (that is, in a set of coordinates moving with the group velocity u_0), Eqs. (1.14) and (1.7) take the form

$$\begin{gathered}\varphi_\tau + {}^1/_2\varphi_\xi{}^2 - \alpha(a^2 - a_0{}^2)\,/\,a_0{}^2 - a_{\xi\xi}\,/\,2a = 0,\\ (a^2)_\tau + (\varphi_\xi a^2)_\xi = 0,\end{gathered} \tag{2.1}$$

where α is the parameter defined in (1.10). If $\alpha < 0$, the set (2.1) is similar to the equations of "hydrodynamics with dispersion." The role of the density is played here by a^2. We first consider small oscillations of amplitude and phase ($a - a_0 \ll a_0$, $\varphi_0 \ll 1$). The corresponding dispersion equation has the form

$$\Omega = (\varkappa^2 - 4\alpha)^{1/2}\varkappa\,/\,2, \tag{2.2}$$

where Ω is the frequency and κ the wave number of the oscillations. For $\alpha < 0$, and for not too large $a - a_0$, the system (2.1) can be reduced to the Korteweg–de Vries equation[8]

$$\begin{gathered}v_\tau + \left(\sqrt{-\alpha} + \frac{3}{2}v\right)v_\xi - \frac{1}{8\sqrt{-\alpha}}v_{\xi\xi\xi} = 0,\\ v = \varphi_\xi,\end{gathered} \tag{2.3}$$

which has been studied in a number of papers (see, for example,[9,10]).

If we neglect the term with $a_{\xi\xi}$ in the first equation of (2.1) (which corresponds to the "adiabatic" approximation of Whitham), then (2.3) follows from (2.1) without the last term, that is, the equation for a simple wave. The profile of the latter, as is well known, for sufficiently large τ becomes non-single valued, just as the profile of the solution of the Korteweg-de Vries equation decays for large τ into solitary waves (solitons)[2)] and small-scale oscillations which have been described in detail in[9-10]. This gives a representation of what is obtained instead of shock waves when $\alpha < 0$ (although Eq. (2.3) follows from the set (2.1) under the condition of not too large values of $a - a_0$, the qualitative picture is preserved, even in the general case).

For $\alpha > 0$, the frequency Ω becomes purely imaginary for

$$\varkappa^2 < \varkappa_0{}^2 = 4\alpha, \tag{2.4}$$

i.e., modulation waves with wave numbers less than the critical value κ_0 are unstable. This result was obtained in nonlinear optics by Bespalov and Talanov.[11] For arbitrary dispersive media, a similar result, as was recalled above, was first obtained by Lighthill,[3] starting out from equations that were equivalent to the system (2.1), but without the term with $a_{\xi\xi}$. If we omit this term, then we get $\Omega = \kappa\sqrt{-\alpha}$ in place of (2.2), i.e., $\kappa_0 = \infty$. Thus account of the term with $a_{\xi\xi}$ leads only to the appearance of the upper boundary (2.4) for the unstable region. Nevertheless, without the term with $a_{\xi\xi}$ it is impossible to carry through a correct analysis of the development of the modulation instability. Qualitative results of this analysis, on the basis of Eqs. (2.1), were briefly set forth in[12]. This question will be considered in more detail below. We shall also give the results of numerical integration, which permits us to represent the general picture for sufficiently large τ.

We shall start out from the system (2.1), assuming that $\alpha > 0$. The solution of these equations, which correspond to the plane wave $\Phi = a_0 \exp[i(k_0x - \omega_0t)]$ has the form $a = a_0$, $\varphi = 0$. We now consider the solution of the set (2.1) for initial conditions of the form

$$\varphi(\xi, 0) = 0, \quad a(\xi, 0) = a_0[1 + f(\xi)], \tag{2.5}$$

where $f(\xi)$ is a function which vanishes as $|\xi| \to \infty$. We shall assume that $f(\xi)$ is sufficiently small that the initial evolution of the perturbation can be followed by starting out from the linearized equations (2.1). The solution of the latter for the initial conditions (2.5) has the form

$$a(\xi,\tau) = a_0 + \frac{1}{\pi}\int_0^\infty d\varkappa\, A(\varkappa)\cos[\varkappa\xi - \Omega(\varkappa)\tau], \tag{2.6}$$

where $\Omega(\kappa)$ is defined in (2.2) and

$$A(\varkappa) = 2a_0\int_0^\infty f(\xi)\cos(\varkappa\xi)\,d\xi \tag{2.7}$$

(here the function $f(\xi)$ in (2.5) is assumed to be even for simplicity). We consider the asymptotic behavior of the integral (2.6) for large τ and ξ. We shall not write out the cumbersome general expression here, but limit ourselves to two limiting cases of small and large ξ ($\xi \ll \kappa_0\tau$ and $\xi \gg \kappa_0\tau$, where κ_0 is defined in (2.4)). In the first case, applying the saddle point method, we get

$$\begin{gathered}a(\xi,\tau) \approx a_0 + \mathrm{const}\cdot e^{\alpha\tau}\cos(\sqrt{2\alpha}\xi + \delta),\\ \xi \ll \varkappa_0\tau,\end{gathered} \tag{2.8}$$

where the quantities const and δ are defined by the initial condition.[3)]

For large $\xi/\kappa_0\tau$, it is not difficult to obtain the asymptote of Eq. (2.6) by using the stationary phase method. The principal contribution to the integral in this case is made by the regions lying in the vicinity of the values of κ which satisfy the condition

$$\frac{d\Omega}{d\varkappa} = \frac{2\varkappa^2 - \varkappa_0{}^2}{2\sqrt{\varkappa^2 - \varkappa_0{}^2}} = \left|\frac{\xi}{\tau}\right|. \tag{2.9}$$

The solutions of Eq. (2.9) have the form

$$\varkappa \approx |\xi\,/\,\tau|, \quad \varkappa \approx \varkappa_0(1 + \varkappa_0{}^2\tau^2\,/\,8\xi^2). \tag{2.10}$$

As a result, we obtain the following asymptotic expression:

$$\begin{gathered}a(\xi,\tau) \approx a_0 + \mathrm{Re}\{(2\,/\,\pi\tau)^{1/2}A(\xi\,/\,\tau)\exp[i(\xi^2\,/\,2\tau - \pi\,/\,4)]\\ + (\varkappa_0{}^3\tau^2\,/\,2\pi|\xi|^3)^{1/2}A(\varkappa_0)\exp[i(\varkappa_0\xi - \varkappa_0{}^3\tau^2\,/\,4|\xi| + \pi\,/\,4]\},\\ |\xi| \gg \varkappa_0\tau.\end{gathered} \tag{2.11}$$

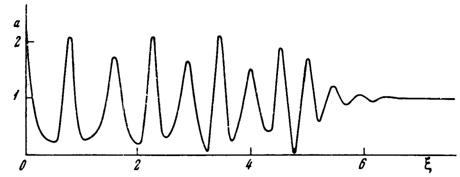
The contribution to a_0 describes modulation waves which propagate from the center with the group velocity equal to ξ/τ, in accord with (2.9).

Thus the general picture of the development of modulation (2.5) in the initial stages of the process appears as follows. In the central region ($|\xi| \ll \kappa_0\tau$), longwave

---

2) In the case considered, these will be "rarefaction" solitons, since the coefficient for $v_{\xi\xi\xi}$ in Eq. (2.3) is negative.[10].

3) The region of applicability of Eq. (2.8) is limited by the value of τ; however, one can always so choose the small perturbation $f(\xi)$, which defines the value of const in (2.8), that the second term remains sufficiently small even at $\alpha\tau \sim 1$ so that the region of ξ where Eq. (2.8) is applicable will span several oscillations with wavelengths of the order of $2\pi/\kappa_0$.



oscillations of the amplitude are formed with wave numbers $\kappa \sim \kappa_0$. With increase in τ, the linear approximation loses force and the exponential growth of the amplitude ceases. However, the strength of the modulation in this region increases with time and the wave splits into wave packets. Moreover, modulation waves are propagated from the boundaries of the strongly modulated region. For sufficiently large ξ, where the asymptote (2.11) is valid, their wave numbers (2.10) become larger than the critical value. Therefore they lie in the stable region (see (2.4)). Modulation waves in the intermediate region have components with $\kappa < \kappa_0$ and are therefore unstable; at first they give rise to new wave groups. Thus the width of the region splitting into packets increases with increase in τ.

The general form of the profile for not very large τ, when the width of the strongly modulated region is still not large, appears to be the same as is shown in the drawing of[12].

For large τ, the process of automodulation was studied with the aid of numerical integration of the initial equations (for a brief description, see the Appendix). For the initial perturbation $f(\xi)$ in (2.5) the expression

$$f(\xi) = \gamma \exp(-\xi^2 / l^2). \tag{2.12}$$

was taken. The profile of the solution, shown in the drawing given here, corresponds to the following values of the parameters: $a_0 = 1$, $\gamma = 1$, $\kappa_0 l = 2\alpha^{1/2} l = 2$, $\alpha = 36$, $\tau = 0.4$.[4)] Upon increase in τ, the width of the modulated region increases.

The amplitudes of the packets tend to stationary values (except for the central one, which has a peak at $\xi = 0$, and which pulsates markedly). The minima decrease with passage of time. The distances between the packets increase slowly (in comparison with the rate of expansion of the region of modulation). The most interesting fact is that the profiles of packets with sufficiently established amplitudes A are very well described by the expression

$$a = A \operatorname{sech}\left[\frac{A\sqrt{\alpha}}{a_0}(\xi - \xi_0)\right]. \tag{2.13}$$

As an illustration, we have Table I, where we compare the data obtained from numerical integration with those computed from formula (2.13) for the third and fourth packets (the packet which has a vertex at $\xi = 0$ is assumed to be the first).

As is well known,[13,5-7] Eq. (2.13), together with the expression for the phase

$$\varphi = \alpha(2A / a_0^2 - 1)\tau + \text{const} \tag{2.14}$$

is an exact solution of Eq. (2.1), which describes the stationary wave packet which is fixed in the set of coordinates used (i.e., which moves with the group velocity u_0 relative to the medium). By analogy with the solitary

Table I. Comparison of numerical values $\alpha_{\text{num}}(\xi, \tau)$ with those calculated (α_{calc}) from formula (2.13)

$A = 1{,}73;\ \xi_0 = 1{,}59$			$A = 2{,}07;\ \xi_0 = 2{,}26$		
$\xi-\xi_0$	a num	$a$calc	$\xi-\xi_0$	$a$num	$a$calc
0,06	1.46	1.44	0.05	1.72	1.72
0.1	1.15	1.1	0.1	1.07	1.10
0.12	1.0	0.92	0.12	0.84	0.89
0.15	0.8	0.7	0.15	0.58	0.62

waves that can propagate in nonlinear dispersive media,[9,10] these packets can be called solitons (of the envelopes). Thus, the numerical solution considered shows that in the case in which the initial plane wave is unstable ($\alpha > 0$), small perturbations lead to its decay into solitons (2.13), (2.14). Although the distance between the individual packets, shown in the drawing, increases very slowly, the "soliton-like" shape of the individual packets is established very quickly (apparently, in the vicinity of the minima, the departures from Eq. (2.13) become considerable).

Furthermore, it is interesting to note the rather clear separation of the "solitons" into greater and lesser ones, which is clearly seen in the drawing. The amplitudes of the "greater" solitons have here the value $A \approx 2.1$ and the lesser, $A \approx 1.7$. The numerical integration was carried out up to $\tau = 1$. Here the amplitudes of the formed solitons remain constant in time with high accuracy, while their number grows rapidly from the expansion of the perturbed region (see the drawing, where the solution is shown for $\tau = 0.4$ and Table II, left half, which contains the values of the maxima and minima $a(\xi, \tau)$ in a sufficiently modulated region for $\tau = 0.9$).

The phase (2.14) in the limits of extension of the separate solitons is seen to be approximately constant (for a given τ). However, on the boundary between the two single tones, it changes rather sharply, so that one can represent it as a step function in ξ.

For an initial perturbation of the form

$$f(\xi) = \gamma \operatorname{sech}(\xi / l) \tag{2.15}$$

the amplitudes of the solitons are obtained as about the same as for the case (2.12) (see Table II, right half). Thus the structure of the quasistationary region, where the solitons are already formed, evidently does not depend on the detailed shape of the initial perturbation.

In conclusion, we take this opportunity to thank V. I. Talanov for useful discussions.

APPENDIX

For numerical analysis, it is convenient to start from the equation for the complex amplitude

$$i\Psi_\tau + {}^1/_2\Psi_{\xi\xi} + \alpha(|\Psi|^2 / |\Psi_0|^2 - 1)\Psi = 0, \tag{A.1}$$

which is equivalent to the system (2.1). This equation was solved by the explicit difference scheme (see, for example,[14]). As the boundary condition, we used $\Psi(\xi) = \Psi(\xi + 2L)$, $-L \le \xi \le L$, where 2L is the size of the region of integration. In order that the boundary conditions not influence the character of the solution, the value of L should be much greater than the dimensions of the perturbed region. The calculations were carried out for L = 20 and L = 40. Complete agreement

[4)] All the solutions with the same values of the parameters $\kappa_0 l$ and γ are similar (for a given form of $f(\xi)$).

Table II. Values of maxima and minima in the quasi-stationary region

$f(\xi)=\exp(-\xi^2/l^2)$; $\tau=0,9$			$f(\xi)=\text{sech}(\xi/l)$; $\tau=0,5$			$f(\xi)=\exp(-\xi^2/l^2)$; $\tau=0,9$			$f(\xi)=\text{sech}(\xi/l)$; $\tau=0,5$		
ξ	a_{max}	a_{min}	ξ	a_{max}	a_{min}	ξ	a_{max}	a_{min}	ξ	a_{max}	a_{min}
0	1.97	—	0	—	0.26	3.75	—	0,21	3.2	2.14	—
0.61	—	0.07	0.68	1.73	—	4.05	1.98	—	3,43	—	0,24
0,86	2,11	—	1,03	—	0.31	4.35	—	0,11	3.81	1,58	—
1,12	—	0,02	1.27	2.29	—	4.71	1,82	—	4,16	—	0,08
1.78	1.71	—	1,63	—	0.03	5,05	—	0.26	4.37	2,17	—
2.32	—	0,16	1.87	2.21	—	5,36	1.89	—	4.57	—	0.21
2.6	2,04	—	2,10	—	0.09	5.66	—	0,08	4.9	1,43	—
2.88	—	0,10	2,58	1.65	—	5.95	1,88	—	5.24	—	0.44
3,34	1.77	—	2.97	—	0.09	6,26	—	0,23	—	—	—

of the solutions for the two cases testifies to the fact that the effect of the boundaries is lacking.

The course of calculation was controlled by the conservation of the energy

$$E=\int_{-L}^{L}|\Psi|^2\,d\xi;$$

the smallness of the round-off errors was verified by preservation of the symmetry of the solution relative to the point $\xi = 0$ in the range $-L \le \xi \le L$. The results plotted in the drawing were obtained for the following values of the steps: $\Delta\tau = 0.5\times10^{-4}$, $\Delta\xi = 0.01$. Reduction of the time step $\Delta\tau$ by a factor of two did not change the character of the solution.

[1] G. B. Whitham, Proc. Roy. Soc. (London A283, 238 (1965)).

[2] G. B. Whitham, J. Fluid Mech. 22, 273 (1965).

[3] M. J. Lighthill, J. Inst. Math. Appl. 1, 269 (1965); Proc. Roy. Soc. (London) A299, 28 (1967).

[4] L. D. Landau and E. M. Lifshitz, Teoriya polya (Theory of Fields) Fizmatgiz, 1967 [Addison-Wesley, 1962].

[5] L. A. Ostrovskiĭ, Zh. Eksp. Teor. Fiz. 51, 1189 (1966) [Sov. Phys.-JETP (U.S.S.R.) 24, 797 (1967).

[6] V. I. Bespalov, A. G. Litvak and V. I. Talanov, Samovozdeĭstvie élektromagnitnykh voln v kubichnykh izotropnykh sredakh (Self-action of Electromagnetic Waves in Cubic Isotropic Meida). Works of the II All Union Symposium on Nonlinear Optics. Novosibirsk, 1967.

[7] S. A. Akhmanov, A. P. Sukhorukov and R. V. Khokhlov, Usp. Fiz. Nauk 93, 19 (1967) [Sov. Phys.-Uspekhi 10, 609 (1968)].

[8] D. J. Korteweg and G. de Vries, Phil. Mag. 39, 422 (1895).

[9] N. J. Zabusky, M. D. Kruskal, Phys. Rev. Lett. 15, 240 (1965); C. Gardner, J. Greene, M. Kruskal and R. Miura, Phys. Rev. Lett. 19, 1095 (1967).

[10] V. I. Karpman, Phys. Lett. 25A, 708 (1967), V. I. Karpman, V. P. Sokolov, Zh. Eksp. Teor. Fiz. 54, 1568 (1968) [Sov. Phys.-JETP 27, 839 (1968)]. Yu. A. Berezin and V. I. Karpman, Zh. Eksp. Teor. Fiz. 51, 1557 (1966); [Soviet Phys.-JETP 24, 1049 (1967)].

[11] V. I. Bespalov and V. I. Talanov, ZhETF Pis. Red. 3, 471 (1966) [JETP Lett. 3, 307 (1966)].

[12] V. I. Karpman, ZhETF Pis. Red. 6, 829 (1967); [JETP Lett. 6, 277 (1967)].

[13] R. Y. Chiao, E. Garmire and C. H. Townes, Phys. Rev. Lett. 13, 479 (1964).

[14] S. K. Godunov and V. C. Ryaben'kiĭ, Vvedenie v teoriyu raznostnykh skhem (Introduction to the Theory of Difference Schemes) Fizmatgiz, 1962.

Translated by R. T. Beyer
62