# QUASI-SHOCK WAVES IN ELECTRON BEAMS

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It is shown that quasi-shock waves with an oscillatory structure can propagate in an electron beam moving against an immobile ion background or in a definite waveguide system.

1. A single-velocity electron beam is described by a system of Maxwell's equations and the equation of motion. We shall write the equation of motion in the Euler form

$$\partial \mathbf{v} / \partial t + (\mathbf{v}\nabla)\mathbf{v} = -\eta \mathbf{E} - [\mathbf{v}\omega_H], \qquad (1)^*$$

where $\mathbf{v} = \mathbf{V}(\mathbf{r}, t)$—velocity of the electron beam; $\mathbf{E}$—electric field intensity; $\omega_H = \eta c^{-1}\mathbf{H}$. Here $\eta$—absolute magnitude of the specific charge of the electron, c—speed of light, and $\mathbf{H}$—intensity of the magnetic field. We call attention to the following circumstance. The only nonlinear term in (1), if $\omega_H$ is specified by the external magnetic field, is the hydrodynamic term $(\mathbf{v}\cdot\nabla)\mathbf{v}$. In this connection, it is natural to suggest the existence of hydrodynamic analogies in the nonlinear motions of the electron beam. In particular, it is of interest to consider the possible existence of motions of the shock-wave type in electron beams. This is the subject of the present article.

2. We consider first an unbounded beam moving against an immobile ion background. The initial system of equations is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \eta\frac{\partial \varphi}{\partial x},$$
$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0,$$
$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n - n_0), \qquad (2)$$

where v—velocity of the electron beam, $\varphi$—electrostatic potential, n—electron density, $n_0$—ion density, and e—absolute value of the electron charge.

We seek the solution of (2) in the form of a plane stationary wave:

$$v = v(\xi), \quad \varphi = \varphi(\xi), \quad n = n(\xi),$$

where $\xi = x - ut$.

From (2) we get

$$\tau^2 - 2\eta\varphi = \tau_0^2, \quad n\tau = n_0\tau_0,$$
$$\frac{d^2\varphi}{d\xi^2} = 4\pi e(n - n_0), \qquad (3)$$

where $\tau = v - u$—velocity of the electron beam in a coordinate system moving with velocity u; $\tau_0 = v_0 - u$—velocity of unperturbed electron beam in the moving system of coordinates.

We note that $\tau_0 > 0$ if the wave velocity is smaller than the velocity of the electron beam (slow waves) and

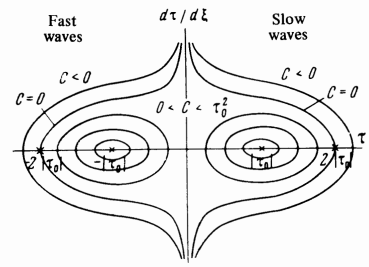


FIG. 1.

$\tau_0 < 0$ if the wave velocity exceeds the velocity of the electron beam (fast waves), sign $\tau$ = sign $\tau_0$.

The system (3) yields

$$\frac{d^2}{d\xi^2}(\tau^2) = 2\omega_p^2\left(\frac{|\tau_0|}{\tau}\operatorname{sign}\tau - 1\right), \qquad (4)$$

where $\omega_p^2 = 4\pi e\eta n_0$. Integrating (4), we get

$$\frac{1}{\omega_p^2}\left(\frac{d\tau}{d\xi}\right)^2 + (\tau^2 \mp 2|\tau_0|\tau + C)\tau^{-2} = 0, \qquad (5)$$

where the minus sign is taken for positive values of $\tau$ and the plus sign for negative values, and C is an arbitrary constant. Using the first equation of (3), we can rewrite (5) in the form

$$\tfrac{1}{4}(d\Phi / dl)^2 + \Phi - 2\sqrt{1 + \Phi} + 1 + C' = 0, \qquad (6)$$

where

$$\Phi = \frac{2\eta\varphi}{\tau_0^2}, \quad l = \frac{\omega_p}{|\tau_0|}\xi, \quad C' = \frac{C}{\tau_0^2}.$$

The phase trajectories of Eqs. (5) and (6) are shown schematically in Figs. 1 and 2 respectively. It is seen from these figures that in the case of a single-velocity electron beam against an immobile ion background there can exist no steady-state periodic waves with amplitude larger than $\tau_{\lim} = 2|\tau_0|$ or $\Phi_{\lim} = 3$. The limiting periodic solution is realized at C = 0.

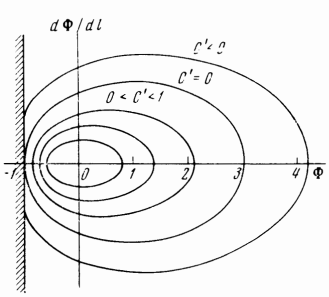


FIG. 2.

*$[\mathbf{V}\omega_H] \equiv \mathbf{V} \times \omega_H$.

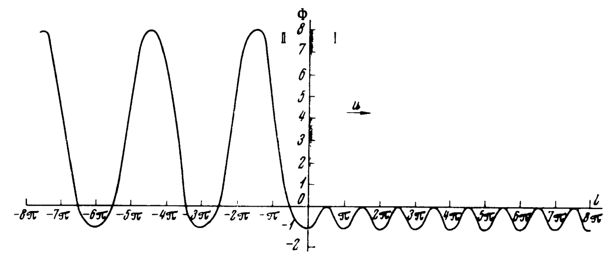

FIG. 3.

We can note, however, the following. If in some spatial region there exists a reflected electron beam, then we get in lieu of (6) the equation

$$\tfrac{1}{4}(d\Phi/dl)^2+\Phi-2\alpha\sqrt{1+\Phi}+1+C''=0, \tag{7}$$

where $\alpha > 1$. This can be verified by taking into account the reflected beam in the third equation of (2), and then integrating the obtained equation. The limiting amplitude of the periodic solution (7) equals $\Phi_{lim} = 4\alpha^2 - 1 > 3$. This circumstance allows us to find unique steady-state waves in the electron beam. Figure 3 shows schematically such a typical wave. On the boundary between the regions I and II, the electron beam is reflected. Region I is the region of the single-velocity electron beam, while region II is the region of a two-velocity beam.

The potential in region I is described by the equation

$$\tfrac{1}{4}(d\Phi/dl)^2+\Phi-\sqrt{1+\Phi}+1=0, \tag{8}$$

and the potential in region II is given by

$$\tfrac{1}{4}(d\Phi/dl)^2+\Phi-3\sqrt{1+\Phi}+1=0. \tag{9}$$

The coefficient $\alpha$ in region I equals $\frac{1}{2}$; this means that half of the electron beam passes into region I. For region II, the coefficient $\alpha$ equals $\frac{3}{2}$, meaning that half of the electron beam is reflected from the boundary between regions I and II. Owing to dissipative processes, the oscillations in I and II attenuate slowly.

The stationary wave of the type of Fig. 3 can be treated as a unique shock wave that couples two different states of the electron beam. On the boundary between regions I and II, a jump takes place in the average electron density. Such a shock wave is analogous in a certain sense to the collisionless shock wave produced in a non-isothermal plasma without an external magnetic field[1,2]. The difference lies in the fact that in the electron beam there cannot exist a solitary wave, so that the oscillatory structure of the quasi-shock wave occurs not only behind the front but also ahead of the front of the wave.

It is shown in[3] that in the presence of an external magnetic field, there are possibly such states of a purely electronic beam, in which nonlinear longitudinal waves oriented along the magnetic field behave just as in an electron beam on an immobile ion front in the absence of a magnetic field. This offers evidence that the quasi-shock waves with oscillatory profile can exist also in such electronic beams.

3. Let now the electron beam move in a simple waveguide system. By way of the latter we assume the following model. We assume that the waveguide system is a multiconductor line filling all of three-dimensional space. The conductors of the line are parallel to the xy plane and make an angle $\alpha$ with the yz plane. The electron beam moves along the z axis. In the absence of a beam, a plane wave with components $E_x$, $E_y$, and $H_z$ can propagate in the system.

The initial system of equations is in this case

$$\begin{gathered}
\partial \mathbf{v}/\partial t+(\mathbf{v}\nabla)\mathbf{v}=-\eta\mathbf{E}-\eta c^{-1}[\mathbf{v}\mathbf{H}_0],\\
\partial n/\partial t+\operatorname{div}(n\mathbf{v})=0,\\
\Delta\mathbf{E}-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}-4\pi e\frac{\partial n}{\partial \mathbf{r}},\\
\mathbf{j}=-en\mathbf{v}+\sigma\mathbf{E},
\end{gathered} \tag{10}$$

where $\mathbf{H}_0$—external magnetic field, assumed to be much stronger than the magnetic field produced in the system; $j_{con} = \sigma\mathbf{E}$—density of the conduction current flowing along the multiconductor line.

Assuming that a wave with components $E_x$, $E_y$, and $H_z$ propagates in the system, and that $\partial/\partial y = \partial/\partial z = 0$, we get from (10)

$$\begin{gathered}
\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\eta E_x,\quad \frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0,\\
\frac{\partial^2 E_x}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial j_x}{\partial t}-4\pi e\frac{\partial n}{\partial x},\\
\frac{\partial^2 E_y}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial j_y}{\partial t},\\
j_x=-env+j_{con}\sin\alpha,\quad j_y=j_{con}\cos\alpha,\\
E_x\sin\alpha+E_y\cos\alpha=0.
\end{gathered} \tag{11}$$

The last equation of (11) is the consequence of the assumption that the conductivity $\sigma$ along the multiconductor line is infinitely large. We note that from the Poisson equation it follows that

$$\frac{\partial^2 E_x}{\partial x^2}=-4\pi e\frac{\partial n}{\partial x}. \tag{12}$$

The system (11), with account of (11), yields

$$\begin{gathered}
\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\eta E_x,\quad \frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0,\\
\frac{\partial^2 E_x}{\partial x^2}-\frac{1}{v_{ph}^2}\frac{\partial^2 E_x}{\partial t^2}=-\frac{4\pi e\cos^2\alpha}{v_{ph}^2}\frac{\partial}{\partial t}(nv),
\end{gathered} \tag{13}$$

where $v_{ph} = c\sin\alpha$.

The system (13) is best treated as that describing the propagation of electromagnetic waves in a certain nonlinear medium with an effective polarization P. To this end, we represent all the unknown quantities in the form

$$\begin{gathered}
v=v_0+v_1(x,t),\quad n=n_0+n_1(x,t),\\
E_x=E_{x0}+E_{x1}(x,t),\quad x=x_0+x_1(x,t),
\end{gathered} \tag{14}$$

where the zero index denotes the quantities pertaining to the unperturbed beam, and unity denotes the perturbations. The function $x_1(x,t)$ describes the displacement of the electrons from their positions in the unperturbed beam. The velocity perturbation $v_1(x,t)$ is connected with the function $x_1(x,t)$ by the relation $v_1 = dx_1/dt$, where

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+(v_0+v_1)\frac{\partial}{\partial x}.$$

We define the polarization P as follows:

$$P=-en_0x_1(x,t), \tag{15}$$

It can be shown that

$$en_1 = \frac{\partial P}{\partial x}, \quad e(nv)_1 = -\frac{\partial P}{\partial t}. \tag{16}$$

Substituting (16) in (13) we get, with allowance for (14),

$$\frac{d^2P}{dt^2} = \frac{\omega_p^2}{4\pi} E_{x1},$$

$$\frac{\partial^2 E_{x1}}{\partial x^2} - \frac{1}{v_{\text{ph}}^2}\frac{\partial^2 E_{x1}}{\partial t^2} = \frac{4\pi \cos^2\alpha}{v_{\text{ph}}^2}\frac{\partial^2 P}{\partial t^2}, \tag{17}$$

where

$$\frac{d^2P}{dt^2} \equiv \ddot{P}\left(1 + \frac{1}{en_0}\frac{\partial P}{\partial x}\right)^{-1} - \frac{1}{en_0}\frac{\partial}{\partial x}(\dot{P})^2\left(1 + \frac{1}{en_0}\frac{\partial P}{\partial x}\right)^{-2}$$
$$+ \frac{1}{e^2 n_0^2}\frac{\partial^2 P}{\partial x^2}(\dot{P})^2\left(1 + \frac{1}{en_0}\frac{\partial P}{\partial x}\right)^{-3},$$

$$\dot{P} \equiv \frac{\partial P}{\partial t} + v_0 \frac{\partial P}{\partial x},$$

$$\ddot{P} \equiv \frac{\partial^2 P}{\partial t^2} + 2v_0\frac{\partial^2 P}{\partial x\, \partial t} + v_0^2 \frac{\partial^2 P}{\partial x^2}.$$

We seek the solution of (17) in the form of a plane stationary wave:

$$P = P(\xi), \quad E_{x1} = E_{x1}(\xi), \quad \xi = x - ut. \tag{18}$$

Substituting (18) in (17) we get

$$-\frac{1}{2}e^3 n_0^3 (v_0 - u)^2 \frac{d}{d\xi}\left(en_0 + \frac{dP}{d\xi}\right)^{-2} = \frac{\omega_p^2}{4\pi}E_{x1},$$

$$\frac{d^2 E_{x1}}{d\xi^2} = \frac{4\pi u^2 \cos^2\alpha}{v_{\text{ph}}^2 - u^2}\frac{d^2 P}{d\xi^2} \tag{19}$$

From (19) we get

$$\frac{d^2}{d\xi^2} e^2\left(en_0 + \frac{dP}{d\xi}\right)^{-2} = \frac{2\omega_p^2 u^2 \cos^2\alpha}{en_0^3 (u^2 - v_{\text{ph}}^2)(v_0 - u)^2}\frac{dP}{d\xi}. \tag{20}$$

We note that

$$en_0 + dP/d\xi = en. \tag{21}$$

On the other hand, from the continuity equation it follows that

$$n\tau = n_0\tau_0, \tag{22}$$

where, as above, $\tau = v - u$ and $\tau_0 = v_0 - u$. Taking (21) and (22) into account, (20) yields

$$\frac{d^2}{d\xi^2}(\tau^2) = 2\omega_p^2 a\left(\frac{|\tau_0|}{\tau}\operatorname{sign}\tau - 1\right), \tag{23}$$

where $a = u^2(u^2 - v_{\text{ph}}^2)^{-1}\cos^2\alpha$.

Equation (23) coincides with (4), accurate to the constant a. However, if $a < 0$, i.e., $u^2 < v_{\text{ph}}^2$, the solution (23) differ qualitatively from the solutions (4). Integrating (23), we get

$$\frac{1}{\omega_p^2}\left(\frac{d\tau}{d\xi}\right)^2 + a(\tau^2 \mp 2|\tau_0|\tau + C)\tau^{-2} = 0, \tag{24}$$

where the minus sign is taken for positive values of $\tau$ and the plus sign for negative values.

4. In the case when $a > 0$, the phase trajectories (24) are shown in Fig. 1. The period of the oscillations is now equal to

$$\lambda = \frac{2\pi}{\omega_p}|\tau_0|a^{1/2}. \tag{25}$$

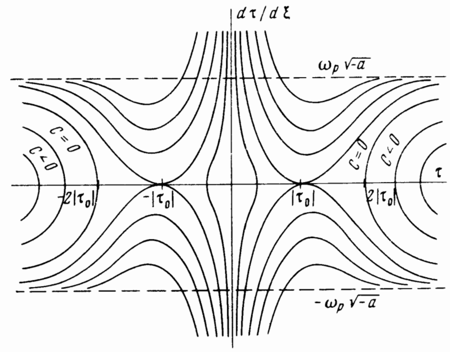


FIG. 4.

Relation (25) determines the dispersion properties of the nonlinear-medium model under consideration. From (25) we get

$$(v_0 - u)^2(u^2 - v_\Phi^2) = \left(\omega_p \frac{\lambda}{2\pi}\right)^2 u^2 \cos^2\alpha \tag{26}$$

or

$$u = v_0 \pm \Delta(u), \tag{27}$$

where

$$\Delta = \omega_p \frac{\lambda}{2\pi} a^{1/2}.$$

Equation (27) determines, for specified values of $v_0$, $v_{\text{ph}}$, and $\lambda$, the phase velocity of the wave u. Solving (27) graphically, we can readily determine all the dispersion properties of the considered model.

Comparing (24) with (25), we can see for the case when $a > 0$, in an electron beam moving in the considered waveguide system, there can exist quasi-shock waves with an oscillatory structure, essentially the same as in an electron beam against an immobile ion background. The difference lies in the fact that the period of the oscillations is different ahead of the front and behind the front of the wave (see (25)).

5. If $a < 0$, we can see from (26) or (27) that in this case u is a complex quantity. This offers evidence of the presence of instability in the system.

Figure 4 shows qualitatively the phase trajectories (24) in the case when $a < 0$. This case calls for a special study. Motions of the shock-wave type are apparently in this case solitary waves with captured electrons.

In conclusion, the author thanks V. G. Leĭman for useful discussions.


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[3] A. D. Gladun and V. G. Leĭman, ibid. 52, 154 (1967) [25, 98 (1967)].


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