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# *CONTRIBUTION TO THE THEORY OF ELASTICITY OF CRYSTALS AT LOW TEMPERATURES*

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A microscopic derivation is given for a set of equations describing the propagation of heat and the motion of an elastic medium for pure crystals at low temperatures. Nonlinear effects in thermal conductivity are discussed.

## 1. INTRODUCTION

In very pure crystalline dielectrics at temperatures well below the Debye temperature, the relaxation time of the total quasimomentum of the phonon system is shown to be much greater than the characteristic interaction time of the phonons with one another. Thus the friction between the gas of excitations (phonons) and the lattice is seen to be weak and one can speak of a state of incomplete thermodynamic equilibrium, which is characterized not only by a temperature and volume (deformation tensor), but also by the total quasimomentum (or drift velocity) of the excitations.

It is clear that in this case the equations of elasticity theory should be altered. Such equations were simultaneously described in the work of Guyer and Krumhansl[1] and in the work of Gurevich and the author.[2] However, the equations given in[1,2] differ from one another.

The basic difficulty arising in the phenomenological derivation lies in the description of the equation for the phonons in the accelerated (inhomogeneously deformed) region of the crystal. The equations obtained in[1] and[2] differ at precisely this point. In this connection, a microscopic derivation of these equations is carried out in the present paper. In contrast with[1,2], we have not limited ourselves here to the linear approximation and to the lowest order in the anharmonicity. However, we have not taken dissipative terms into account. As expected, the obtained set of equations is completely analogous to the hydrodynamics of a superfluid. The principal difference is that there is a shear modulus in the crystal. In the linear approximation and with the lowest anharmonic terms, the equations obtained there are identical with the equations of[2] and differ from the equations of[1].

## 2. MICROSCOPIC EQUATIONS

We shall assume that the Hamiltonian density of the crystal has the following form:

$$H(\mathbf{r}) = \frac{\rho_0 \dot{s}_i^2}{2} + \psi(\bar{s}_{ik}), \tag{2.1}$$

where

$$\bar{s}_{ik} = \frac{1}{2}(s_{ik} + s_{ki} + s_{li}s_{lk}), \quad s_{ik} = \frac{\partial s_i}{\partial x_k}, \quad \dot{s}_i = \left(\frac{\partial s_i}{\partial t}\right)_{x_k}. \tag{2.2}$$

Here $s_i$ is the displacement from equilibrium, $x_i$ the Lagrangian coordinate of the body, $\rho_0$ the density of the crystal at the equilibrium position, and $\psi(\bar{s}_{ik})$ an arbitrary function of the deformation tensor. We shall assume here that the perturbations are long-wave phonons interacting with one another and we shall neglect transport processes completely.

The Lagrangian density has the form

$$\hat{\Lambda} = \frac{\rho_0 \dot{s}_i^2}{2} - \psi(\bar{s}_{ik}). \tag{2.3}$$

Following the usual procedure of quantum field theory, we easily obtain the equations of motion and the conservation laws corresponding to (2.3):

$$\rho_0 \ddot{s}_i = \frac{\partial}{\partial x_k}\frac{\partial \psi}{\partial s_{ik}}; \tag{2.4}$$

$$\dot{\hat{T}}_i + \frac{\partial \hat{T}_{ik}}{\partial x_k} = 0, \quad \hat{T}_i = -\frac{\rho_0}{2}[\dot{s}_k, s_{ki}]_+,$$

$$\hat{T}_{ik} = \delta_{ik}\left(\frac{\rho_0 \dot{s}_i^2}{2} - \psi\right) + s_{li}\frac{\partial \psi}{\partial s_{lk}}; \tag{2.5}$$

$$\dot{\hat{H}} + \operatorname{div}\mathbf{q} = 0, \quad \hat{q}_i = -\frac{1}{2}\left[\dot{s}_k, \frac{\partial \psi}{\partial s_{ki}}\right]_+. \tag{2.6}$$

Here $(A, B)_+ = AB + BA$.

In addition, we average the Eqs. (2.4)–(2.6) over a time that is small in comparison with the times of change in the macroscopic quantities, but large in comparison with the microscopic relaxation times of the phonons. We introduce the displacement $u_i$ and velocity $\dot{u}_i$ averaged in such fashion:

$$\begin{aligned} \langle s_i \rangle &= u_i, \quad s_i = u_i + Q_i, \quad \langle Q_i \rangle = 0; \\ \langle \dot{s}_i \rangle &= \dot{u}_i, \quad \dot{s}_i = \dot{u}_i + \dot{Q}_i, \quad \langle \dot{Q}_i \rangle = 0. \end{aligned} \tag{2.7}$$

Here the operators $Q_i$ describe the thermal oscillations superimposed on the microscopic displacements $u_i$. Like the $s_i$, they satisfy the commutation rule

$$\rho_0[\dot{Q}_i(\mathbf{r}), \quad Q_k(\mathbf{r}')] = -i\hbar\delta(\mathbf{r} - \mathbf{r}')\delta_{ik}. \tag{2.8}$$

We shall average by means of the Gibbs density matrix. By the mean at the point **r**, we shall understand the mean over all states of an (imaginary) homogeneously deformed crystal, where the deformation $u_{ik}$ is numerically equal to $u_{ik}(\mathbf{r})$, while the temperature is homogeneous over all points and is equal to $T(\mathbf{r})$. In such a method of averaging, we lose the higher order derivatives, i.e., the dissipation terms.

Thus the problem is reduced to the expression of the mean values of the operators $\partial\psi/\partial s_{ik}$, $\hat{T}_i$, $\hat{T}_{ik}$, $\hat{H}$ and $\hat{q}$ in terms of the thermodynamic functions of a homogeneously deformed crystal. In order to obtain the averaged Hamiltonian, it suffices to substitute (2.7) in (2.1) and integrate over the volume, assuming $u_{ik}$ to be homogeneous. Here we can expand $Q_i$ in the normal coordinates, omitting in this expansion the terms with a wave vector equal to zero (taken into account by the presence of $u_i$). Then the conditions $\langle Q_i \rangle = 0$, $\langle \dot{Q}_i \rangle = 0$ are automatically satisfied. Since there are no terms in this Hamiltonian with the product $\dot{u}_i \cdot \dot{Q}_i$, which vanishes on integration, it is easy to obtain the dependence of all quantities on $\dot{u}_i$:

$$\langle H\rangle = \frac{\rho_0 \dot{u}_i^2}{2} + E,$$

$$T_i = \langle \hat{T}_i \rangle = -\rho_0 \dot{u}_k u_{ki} + T_i^{(0)},$$

$$T_{ik} = \langle \hat{T}_{ik} \rangle = \delta_{ik} \frac{\rho_0 \dot{u}_l^2}{2} + T_{ik}^{(0)}$$

$$q_i = \langle \hat{q}_i \rangle = -\pi_{ki} \dot{u}_k + q_i^{(0)}, \tag{2.9}$$

where $\pi_{ki} = \langle \partial\psi/\partial s_{ki} \rangle$ does not depend on $\dot{u}_i$. Equations (2.9) are actually the consequence of the principle of Galilean relativity.

## 3. THE THERMODYNAMICS OF A HOMOGENEOUS DEFORMED CRYSTAL

In this section, we introduce the thermodynamic functions of a homogeneously deformed crystal with a drift of excitations, and obtain some identities which allow us to carry out averaging of the equation.

It is easy to establish the fact that the operator

$$\int \hat{T}_i^{(0)}\, d\mathbf{r} = -\frac{1}{2}\rho_0 \int \left[\dot{Q}_k, \frac{\partial Q_k}{\partial x_i}\right]_+ d\mathbf{r} \tag{3.1}$$

commutes with the averaged Hamiltonian. For this purpose, it suffices to expand $Q_k$ in the normal coordinates of the harmonic approximation and establish the fact that (3.1) is

$$\frac{1}{2} \sum_{\mathbf{k}j} k_i \left[b_{\mathbf{k}j}^{+} b_{\mathbf{k}j} + b_{\mathbf{k}j} b_{\mathbf{k}j}^{+}\right],$$

where k is the wave vector, j the index of the phonon branch, and $b_{\mathbf{k}j}^{+}$ and $b_{\mathbf{k}j}$ are the creation and annihilation operators. Since we have neglected Umklapp processes and have considered the macroscopic deformations to be homogeneous, the total quasi-momentum is conserved (with account of the anharmonicity).

It will be more convenient for us, however, to transform in (3.1) from the Lagrangian coordinate $x_i$ to the Eulerian $a_i$ by the formula

$$a_i = x_i + u_i(x_k, t). \tag{3.2}$$

Then

$$\int \hat{T}_i^{(0)} d\mathbf{r} = \int \hat{P}_k d\mathbf{r} \frac{\partial a_k}{\partial x_i} = \hat{\mathscr{P}}_k \frac{\partial a_k}{\partial x_i},$$

$$\hat{P}_k = -\frac{1}{2}\rho_0 \left[\dot{Q}_l, \frac{\partial Q_l}{\partial a_k}\right]_+. \tag{3.3}$$

The operator $\hat{\mathscr{P}}_k$ is an integral of the motion and we shall call it the quasimomentum operator, while the operator $\hat{P}_k$ is the quasimomentum density.

We shall carry out the averaging with the matrix density of the form $\exp[(F - \mathscr{H}')/T]$, where F is the free energy and

$$\mathscr{H}' = \int (H - \hat{\mathbf{P}}\mathbf{W})\, d\mathbf{r}, \tag{3.4}$$

where **W** is the drift velocity of the excitations. We assume that our crystal has unit volume before the deformation (e.g., the integration in (3.4) is carried out over the unit volume). Then

$$\langle \mathscr{H}' \rangle = E' = E - \mathbf{PW}, \tag{3.5}$$

where $\mathbf{P} = \langle \mathbf{P} \rangle$ (in this section, we assume $\dot{u}_i = 0$ for brevity). Making use of the identity

$$\langle (\partial \mathscr{H}'/\partial \mathbf{W})_{u_{ik}} \rangle = (\partial E'/\partial \mathbf{W})_{S, u_{ik}}, \tag{3.6}$$

where S is the entropy, we get

$$(\partial E'/\partial \mathbf{W})_{S, u_{ik}} = -\mathbf{P}. \tag{3.7}$$

Then

$$dE' = TdS + \sigma_{ik} du_{ik} - \mathbf{P}d\mathbf{W}, \tag{3.8}$$

$$dE = TdS + \sigma_{ik} du_{ik} + \mathbf{W}d\mathbf{P}, \tag{3.9}$$

where all the additive quantities refer to unit volume of the undeformed crystal.

We now use the identity

$$\sigma_{ik} = (\partial E'/\partial u_{ik})_{S, \mathbf{W}} = \langle (\partial \mathscr{H}'/\partial u_{ik})_{\mathbf{W}} \rangle. \tag{3.10}$$

We write $\mathscr{H}'$ in the form

$$\mathscr{H}' = \int \left( \frac{\rho_0 \dot{Q}_i^2}{2} + \psi + \frac{1}{2}\rho_0 \left[\dot{Q}_k, \frac{\partial Q_k}{\partial x_i}\right]_+ W_l \frac{\partial x_i}{\partial a_l} \right) d\mathbf{r}, \tag{3.11}$$

where the integration is carried out over unit volume, as before. In the differentiation of $\mathscr{H}'$ with respect to $u_{ik}$, we take into account the equality

$$\frac{\partial}{\partial u_{ik}} \left( \frac{\partial x_l}{\partial a_m} \right) = -\frac{\partial x_l}{\partial a_i} \frac{\partial x_k}{\partial a_m}, \tag{3.12}$$

which can be obtained by differentiating the identity

$$\frac{\partial x_l}{\partial a_m} \frac{\partial a_m}{\partial x_i} = \delta_{il}.$$

As a result, we get

$$\sigma_{ik} = \pi_{ik} + P_i W_l \frac{\partial x_k}{\partial a_l}, \tag{3.13}$$

where

$$\pi_{ik} = \langle \partial\psi / \partial u_{ik} \rangle.$$

We now introduce thermodynamic functions referred to unit volume of the deformed crystal. As is well known, the density of the deformed crystal $\rho$ is related to $\rho_0$ by the equation

$$\rho \mathscr{J} = \rho_0,$$

where $\mathscr{J} = |\partial a_i/\partial x_k|$ is the Jacobian of the transformation. Denoting the new thermodynamic functions by capital letters with tildes, we get

$$\tilde{E} = E/\mathscr{J},\ \tilde{E}' = E'/\mathscr{J},\ \tilde{S} = S/\mathscr{J},\ \tilde{\mathbf{P}} = \mathbf{P}/\mathscr{J}. \tag{3.14}$$

We substitute (3.14) in (3.8). Then

$$d\tilde{E}' = Td\tilde{S} - \tilde{\mathbf{P}}d\mathbf{W} + \left( \frac{\sigma_{ik}}{\mathscr{J}} - \frac{\tilde{F}}{\mathscr{J}} \frac{\partial \mathscr{J}}{\partial(\partial u_i/\partial x_k)} \right) du_{ik}, \tag{3.15}$$

where $\tilde{\mathbf{F}} = \tilde{\mathbf{E}} - \mathbf{T}\tilde{\mathbf{S}} - \tilde{\mathbf{P}} \cdot \tilde{\mathbf{W}}$.

We shall write the quantity $\mathscr{H}'$ for a homogeneous deformed crystal occupying unit volume. We note it by $\tilde{\mathscr{H}}'$. This quantity is given by (3.4) where, however, the integral is carried out over a volume which depends on the deformation tensor $\bar{u}_{ik}$. Therefore, it is convenient to transform to integration over da. Then

$$\tilde{\mathcal{H}}' = \int \left( \frac{\rho_0 \dot{Q}_i^2}{2} + \psi + \frac{1}{2} \rho_0 \left[ \dot{Q}_l, \frac{\partial Q_l}{\partial a_k} \right]_+ W_k \right) \frac{d\mathbf{a}}{\mathcal{J}}. \qquad (3.16)$$

The integration is now carried out over the unit volume. We further use the identity

$$\langle (\partial \tilde{\mathcal{H}}' / \partial u_{ik}) \rangle_{\mathbf{W}} = (\partial \tilde{E}' / \partial u_{ik})_{\mathbf{W}, \tilde{S}}. \qquad (3.17)$$

In the differentiation of (3.16), it is necessary to express the derivatives $\partial Q_i / \partial x_k$ in the argument of $\psi$ in terms of $\partial Q_i / \partial a_m$ and to consider the latter to be constant. It is further necessary to take it into account that the commutation relations (2.8) contain $\delta(\mathbf{r} - \mathbf{r}') = \mathcal{J} \delta(\mathbf{a} - \mathbf{a}')$, where $\mathcal{J}$ depends on $u_{ik}$. Therefore, it is useful to transform to a new momentum $Z_i = \rho Q_i = \rho_0 Q_i / \mathcal{J}$, which satisfies the commutation relations

$$[Z_i(\mathbf{a}), Q_k(\mathbf{a}')] = -i\hbar \delta_{ik} \delta(\mathbf{a} - \mathbf{a}').$$

The value of $Z_i$ can be assumed to be constant in the differentiation. As a result, we get the following equation:

$$\sigma_{ik} - F \frac{\partial \mathcal{J}}{\partial(\partial a_i / \partial x_k)} = \left\langle \left( \frac{\rho_0 \dot{Q}_l^2}{2} - \psi \right) \frac{\partial \mathcal{J}}{\partial(\partial a_i / \partial x_k)} \frac{1}{\mathcal{J}} + \frac{\partial \psi}{\partial \bar{s}_{lm}} \frac{\partial b_p}{\partial a_i} a_{pklm} \right\rangle, \qquad (3.18)$$

where

$$a_{pklm} = \frac{1}{2} \left( \frac{\partial b_p}{\partial x_m} \delta_{lk} + \frac{\partial b_p}{\partial x_l} \delta_{mk} \right),$$
$$b_p = a_p + Q_p = s_p + x_p. \qquad (3.19)$$

## 4. THE MACROSCOPIC EQUATIONS

Averaging Eq. (2.4) by means of Eq. (3.13), we get

$$\rho_0 \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial}{\partial x_k} (\tilde{P}_i W_l) \frac{\partial \mathcal{J}}{\partial(\partial a_l / \partial x_k)}. \qquad (4.1)$$

We use here the well-known identities[3]

$$\frac{\partial x_k}{\partial a_l} = \frac{1}{\mathcal{J}} \frac{\partial \mathcal{J}}{\partial(\partial a_l / \partial x_k)} \qquad \frac{\partial}{\partial x_k} \left( \frac{\partial \mathcal{J}}{\partial(\partial a_l / \partial x_k)} \right) = 0. \qquad (4.2)$$

We now proceed to Eq. (2.5). We write $T_{ik}^{(0)}$ in the form

$$T_{ik}^{(0)} = -\pi_{ik} + t_{ik},$$
$$t_{ik} = \left\langle \left( \frac{\rho_0 \dot{Q}_l^2}{2} - \psi \right) \delta_{ik} + \frac{\partial b_l}{\partial x_i} \frac{\partial \psi}{\partial s_{lk}} \right\rangle \qquad (4.3)$$

or

$$t_{ik} = \left\langle \left( \frac{\rho_0 \dot{Q}_l^2}{2} - \psi \right) \delta_{ik} + \frac{\partial b_p}{\partial x_i} \frac{\partial \psi}{\partial \bar{s}_{lm}} a_{pklm} \right\rangle. \qquad (4.4)$$

Multiplying (3.18) by $\partial a_i / \partial x_i$ and comparing the result with (4.4), we obtain

$$t_{ik} = \sigma_{lk} \frac{\partial a_l}{\partial x_i} - F \delta_{ik}. \qquad (4.5)$$

where $F = \tilde{F} \mathcal{J}$.

Taking (2.9), (3.3), (4.3), and (4.5) into account, we finally obtain the equation

$$\dot{P}_i \frac{\partial a_i}{\partial x_k} + P_i \frac{\partial \dot{u}_i}{\partial x_k} + S \frac{\partial T}{\partial x_k} + P_i \frac{\partial W_i}{\partial x_k} + \frac{\partial}{\partial x_i} (\tilde{P}_s W_l) \frac{\partial \mathcal{J}}{\partial(\partial a_l / \partial x_i)} \frac{\partial a_s}{\partial x_k} = 0. \qquad (4.6)$$

For the derivation of the last remaining equation, we must find the mean value of the operator $q_i$. However, obtain it by a somewhat different method. Finding the derivative of E from the thermodynamic identity (3.9) and substituting it in the averaged Eq. (2.6), we require that the derivative of the entropy density S with respect to the time be equal to the divergence of the vector. We then get

$$q_i^{(0)} = TSW_k \frac{\partial x_i}{\partial a_k} + P_l W_l W_k \frac{\partial x_i}{\partial a_k}. \qquad (4.7)$$

The equation for the entropy has the following form here:

$$\dot{S} + \frac{\partial}{\partial x_i} \left( SW_k \frac{\partial x_i}{\partial a_k} \right) = 0. \qquad (4.8)$$

Equations (4.1), (4.6), (4.8) are also the desired equations of the quantum theory of elasticity. If the drift velocity W is small in comparison with the sound velocity c, the relation between **P** and **W** is linear: $P_i = \eta_{ik} W_k$. In the linear approximation, the equations take the form

$$\rho_0 \ddot{u}_i = \partial \sigma_{ik} / \partial x_k, \quad \dot{\mathbf{P}} + S \nabla T = 0, \quad \dot{S} + S \operatorname{div} \mathbf{W} = 0,$$

which are entirely identical with the result of[2].

We shall now show how the variables introduced by us are connected with the quantities $v_n$ and $v_s$ which are required in the hydrodynamics of a superfluid. It is evident that $\rho \dot{u}_i$ is the density of the total momentum. The velocity of the excitations in the laboratory (Eulerian) set of coordinates is

$$\mathbf{v}_n = \dot{\mathbf{u}} + \mathbf{W}. \qquad (4.9)$$

Finally, $v_s$ should be defined so that the total momentum density in the set of coordinates in which $v_s = 0$ is identical with the excitation momentum density

$$\rho \dot{\mathbf{u}} = \rho \mathbf{v}_s + \mathbf{P}. \qquad (4.10)$$

Equations (4.9), (4.10), together with the condition $\tilde{P}_i = \eta_{ik} W_k$, allow us to express u and $\hat{\mathbf{P}}$ in terms of $\mathbf{v}_n$ and $\mathbf{v}_s$. As a result, we get

$$\rho \dot{\mathbf{u}} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \qquad (4.11)$$

where the tensors $\rho_n$ and $\rho_s$ are connected with $\eta$ by the relations

$$\rho_n = \left( 1 + \frac{\eta}{\rho} \right)^{-1} \eta, \qquad \rho_s = \rho \left( 1 + \frac{\eta}{\rho} \right)^{-1},$$
$$\rho_{nik} + \rho_{sik} = \rho \delta_{ik}, \qquad (4.12)$$

while the tensor $\eta$ is expressed in terms of $\rho_n$ and $\rho_s$ in the following way:

$$\eta = \rho \rho_s^{-1} \rho_n. \qquad (4.13)$$

The dispersion equation for second sound in a solid has the form (see[2])

$$\omega^2 = \frac{TS^2}{C_V} (\eta^{-1})_{ik} q_i q_k,$$

where q is the wave vector. With account of (4.13), it is entirely analogous to the corresponding equation of two-fluid thermodynamics. In the lowest order of the anisotropy, $\eta$ coincides with $\rho_n$ and the velocity of second sound is temperature independent. However, a dependence on T appears in the higher orders.

As a conclusion to this section, we write out the obtained formulas in Eulerian coordinates. The system has the following form: 1)

$$\frac{\partial}{\partial t}(\rho\dot{u}_i)+\frac{\partial\Pi_{il}}{\partial a_l}=0,\quad \Pi_{il}=\rho\dot{u}_i\dot{u}_l-\sigma_{ik}\frac{1}{\mathcal{J}}\frac{\partial a_l}{\partial x_k}+\tilde{P}_iW_l,$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\dot{\mathbf{u}}=0,\quad \frac{\partial\tilde{S}}{\partial t}+\operatorname{div}\tilde{S}\mathbf{v}_n=0,$$

$$\frac{\partial\tilde{P}_i}{\partial t}+\frac{\partial}{\partial a_k}(\tilde{P}_i v_{nk})+\tilde{P}_k\frac{\partial v_{nk}}{\partial a_i}+\tilde{S}\frac{\partial T}{\partial a_i}=0. \tag{4.14}$$

There should be a continuous heat flow on the surface of the crystal, while the tangential component of $v_n$ should be equal to the tangential component of $\dot{u}$.

A similar derivation can be made for a liquid. Here the result is obtained that $\sigma_{ik} = -p\partial\mathcal{J}/\partial u_{ik}$, where p is the pressure. If we substitute this expression in (4.14) and impose the condition curl $v_s = 0$, this system goes over into the ordinary equations of two-fluid hydrodynamics.

## 5. NONLINEAR THERMAL CONDUCTIVITY

As an application of the obtained nonlinear system of equations, we consider the thermal conductivity of thin rods in the absence of transfer processes. We shall assume that the drift velocity W is small in comparison with the sound velocity c, and that the resulting lattice deformations are also small ($\partial u_i/\partial x_k \ll 1$). Then the phonon fluid can be regarded as "incompressible," and Eqs. (4.6), (4.8), which describe the stationary propagation of heat, will have the following form:

$$\frac{\partial F}{\partial x_i}+\eta_{ik}W_l\frac{\partial W_k}{\partial x_l}-\gamma_{iklm}\frac{\partial^2 W_l}{\partial x_k\partial x_m}=0,$$
$$\operatorname{div}\mathbf{W}=0. \tag{5.1}$$

Here $\gamma$ is the viscosity tensor.[4,2] Let us consider the consequences of the presence of a nonlinear term. Equations (5.1) are similar to the ordinary equations of a viscous incompressible fluid. Here the Reynolds number R is defined in the following way:

$$R = W\eta d/\gamma, \tag{5.2}$$

where d is a characteristic dimension of the system. In the phonon approximation,

$$R=\frac{W}{c}\frac{d}{l}, \tag{5.3}$$

where $l$ is the free path length of the phonons. It is then seen that R can be large. For laminar flow,[4]

$$R\approx\frac{d}{T}\left(\frac{d}{l}\right)^2\overline{\nabla}T. \tag{5.4}$$

If the direction of the axis of the long rod (the x axis) is perpendicular to the plane of symmetry of the crystal, then the solution of the set (5.1) contains only a single non-zero component of the velocity, $W_x(y, z)$, which is determined by a linear equation of second order, and the nonlinear term vanishes. The investigation of this solution for stability is very complex. It is natural, however, to assume that the anisotropy changes the result of such an investigation slightly. Then, using the experimental data of hydrodynamics, we can confirm that the disruption of laminar flow for a circular rod takes place at $R \approx 10^3$. The equation for cubic crystals, averaged over the pulsations, has the same structure as in hydrodynamics. It is therefore natural to expect that in developed turbulence,[5] the heat flow will be proportional to $(\partial T/\partial x)^{1/2}\ln(\partial T/\partial x)$. However, such values of R are probably unattainable at the present time because of the quality of the crystals. Therefore, there is more interest in the case in which the axis of the rod is directed arbitrarily relative to the axis of the crystal. In this case, the components $W_y$ and $W_z$ are different from zero. Here the nonlinear terms begin to play a role, as only the condition $R \ll 1$ is violated. However, detailed consideration of this question lies beyond the framework of the present paper.

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1) It can be shown that all the thermodynamic functions depend on $\bar{u}_{ik}$ and $W_k\partial x_i/\partial a_k$. Starting out from this, we can see that the tensor $\Pi_{il}$ is symmetric. The tensor $\sigma_{ik}$ in nonlinear elasticity theory should be symmetric.