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ON THE CONSTRUCTION OF A COMPLETE SET OF FUNCTIONS IN A SPACE-LIKE REGION

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Using the methods of integral geometry, a complete set of functions in a space-like region is constructed and proven to be complete. It is shown that a complete set of functions with respect to the norm is formed by a space of pairs of functions, viz., the functions which realize the representations of the Lorentz group of the fundamental series $(i\rho, i\rho)$ and of the discrete series $(n, -n)$.

Ginzburg and Tamm[1] have introduced the internal particle degrees of freedom x_i, where $x_i^2 = 1$. By introducing such variables it was possible to modify the wave equation and to obtain a "mass spectrum." The expansion of a square-integrable function f(x) in terms of solutions of the Laplace equation defined on a single-sheeted hyperboloid[1] was carried out on an incomplete set of functions. In the present paper we present a method for constructing a complete set of functions using the results of Gel'fand and Graev[2] (cf. also [3]).

According to [2,3] a square-integrable even function f(x) can be expanded in irreducible components of the Lorentz group:

$$f(x)=\frac{(-)}{4i(2\pi)^3}\int_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\int_L F(\xi,\sigma)\,|(x\xi)|^{-\sigma-2}d^2\xi d\sigma$$

$$+\frac{2}{\pi^2}\sum_{n=1}^{\infty}2n\int_L F(\xi,x;2n)\,\delta((x\xi))\,d^2\xi. \tag{1}$$

Here $x^2 = x_0^2 - \mathbf{x}^2 = -1$, $\xi^2 = 0$, and L is the contour of integration on the cone (sphere, if $\xi_0 = 1$). The numbers σ and n are the weights of the representations of the Lorentz group of the fundamental and the discrete series, respectively, where $\sigma = -1 + i\rho$ in the unitary case.

In order to write an expansion for an odd function $f(x) = -f(-x)$ ($x_0 \to -x_0$, $\mathbf{x} \to -\mathbf{x}$), one must replace, in (1), the expression $|(x\xi)|^{-\sigma-2}$ by $|(x\xi)|^{-\sigma-2}\operatorname{sign}(x\xi)$ and 2n in the second term by $2n - 1$. The proof for this assertion is given in the Appendix.

The functions $F(\xi, \sigma)$ and $F(\xi, x; 2n)$ transform according to the irreducible representations of the Lorentz group $(\sigma + 1, \sigma + 1)$ and $(2n, -2n)$, respectively. We introduce the new variables

$$\begin{aligned} x_0 &= \operatorname{sh}\alpha, & \xi_0 &= 1,\\ x_3 &= \operatorname{ch}\alpha\cos\theta, & \xi_3 &= \cos\vartheta,\\ x_2 &= \operatorname{ch}\alpha\sin\theta\cos\varphi, & \xi_2 &= \sin\vartheta\cos\Phi,\\ x_1 &= \operatorname{ch}\alpha\sin\theta\sin\varphi, & \xi_1 &= \sin\vartheta\sin\Phi. \end{aligned} \tag{2}$$

Then

$$\begin{aligned}(x\xi) &= \operatorname{sh}\alpha - \operatorname{ch}\alpha[\cos\theta\cos\vartheta + \sin\theta\sin\vartheta\cos(\varphi-\Phi)]\\ &= \operatorname{sh}\alpha - \operatorname{ch}\alpha\cos\Theta; \quad d^2\xi = \sin\vartheta d\vartheta d\Phi.\end{aligned} \tag{3}$$

Let us expand $F(\xi, \sigma)$ and $F(\xi, x; 2n)$ in a series,

$$F(\xi,\sigma)=\sum_{lm}a_{lm}(\sigma)Y_{lm}(\vartheta,\Phi), \tag{4}$$

$$F(\xi,x;2n)=\sum_{lm}c_{lm}D_{mk}^{l}(\xi/|\xi|), \quad k=2n. \tag{5}$$

We rotate the coordinate system such that the z axis coincides with the vector **x**. Under this rotation the functions Y_{lm} and D_{mk}^l transform like

$$Y_{lm}(\vartheta,\Phi)=\sum_i D_{mi}^l(\varphi,\theta,\chi)Y_{li}(\Theta,\psi), \tag{6}$$

$$D_{mk}^l(\Phi,\vartheta,0)=\sum_j D_{mj}^l(\varphi,\theta,\chi)D_{jk}^l(\psi,\Theta,0), \tag{7}$$

and because of the invariance of the measure on the sphere under rotations,

$$d\cos\vartheta d\Phi = d\cos\Theta d\psi = d^2\xi. \tag{8}$$

We substitute (4) and (5) in (1) and integrate over $d^2\xi$, using (6) to (8). We then obtain

$$f(x)=\frac{(-)}{4i(2\pi)^3}\int_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\Gamma(-\sigma-1)\cdot$$

$$\times\sum_{lm}a_{lm}(\sigma)Y_{lm}(\theta,\varphi)\frac{P_l^{\sigma+1}(\operatorname{th}\alpha)+(-)^lP_l^{\sigma+1}(-\operatorname{th}\alpha)}{\operatorname{ch}\alpha}d\sigma$$

$$+\frac{2}{\pi^2}\sum_{n=1}^{\infty}2n\sum_{lm}\frac{4\pi}{2l+1}c_{lm}\frac{P_l^{2n}(\operatorname{th}\alpha)}{\operatorname{ch}\alpha}Y_{lm}(\theta,\varphi), \tag{9}$$

where $P_l^{\sigma+1}(\tanh\alpha)$ and $P_l^{2n}(\tanh\alpha)$ are the associated Legendre functions. The first term in this expression is analogous to the expansion of a function on a two-sheeted hyperboloid, obtained by Bilenkin and Smorodinskiĭ.[4]

The appearance of the signature $(-)^l$ in this term arises from our restriction to even functions, $f(x) = f(-x)$ for $x_0 \to x_0$, $\mathbf{x} \to -\mathbf{x}$. For odd functions $(-)^l$ must be replaced by $-(-)^l$ in (9), and the summation over even numbers must be changed to one over odd numbers. The characteristic feature of the single-sheeted hyperboloid is that an arbitrary function $\varphi(x)$ defined on it can be expanded in the pairs of functions $P_l^{\sigma+1}(\tanh\alpha)$ and $P_l^n(\tanh\alpha)$, where $n = 1, 2, 3\ldots$.

The second term in (9) is exactly the expansion obtained by Ginzburg and Tamm with the same restrictions on the quantum numbers

$$l = n, n+1, n+2, \ldots, n = 1, 2, 3, \ldots \tag{10}$$

These restrictions are clearly seen from (9) and the properties of $P_l^n(\tanh\alpha)$. Using only discrete representations, Ginzburg and Tamm obtained a "mass spectrum" $m_0^2 = m_0^2(l, n)$. The complete set of functions with respect to the norm

$$\int |f(x)|^2\, dx = N^2 < \infty$$

is formed by the space of pairs of functions

$$\mathrm{ch}^{-1}\alpha \begin{pmatrix} P_l^{n+1}(\mathrm{th}\,\alpha) \\ P_l^{n}(\mathrm{th}\,\alpha) \end{pmatrix} Y_{lm}(\theta, \varphi).$$

Thus, together with the "mass spectrum" corresponding to the eigenvalues of the Laplace operator $\lambda = -(n^2 - 1)$,

$$m_0^2 = \frac{\varkappa^2 - \beta(-n^2 + 1)}{1 + \varepsilon(l^2 + l - n^2 + 1)}, \tag{11}$$

there also exists a "mass spectrum" corresponding to $\lambda = \rho^2 + 1$:

$$m_0^2 = \frac{\varkappa^2 - \beta(\rho^2 + 1)}{1 + \varepsilon(l^2 + l + \rho^2 + 1)}, \tag{12}$$

where β and ε are certain constants (cf. [1]), and κ is the mass corresponding to the usual wave equation. Since we have only chosen the vector x as an internal coordinate, the second invariant of the Lorentz group $\Delta_1 = \varepsilon_{\mu\nu\alpha\beta} M_{\mu\nu} M_{\alpha\beta}$ is equal to zero.

The second term in (1) arises from the integration over the region $|(ax)| < 1$ in the derivation of the inversion formula (cf. the Appendix), where the vector a corresponds to the origin of the system, for example, $a_3 = (0, 1, 0, 0)$. It follows from this that the discrete series occurs in a subregion of the hyperboloid, viz., for $|x_3| < 1$, or $|x_3| = |\cosh\alpha \cos\theta| < 1$ in the parametrization introduced above. Here $|x_0| = |\sinh\alpha|$ may be larger or smaller than $|x_3|$. If $|x_0| = |\sinh\alpha| > x_3 = |\cosh\alpha\cos\theta|$, then this means that the velocity of the interaction propagating along the x_3 direction is smaller than the velocity of light. In the opposite case, $|x_0| < |x_3|$, the velocity of the interaction is larger than the velocity of light. Let us cut out the subregion $|\tanh\alpha| < |\cos\theta|$ from the manifold $x_0^2 - \mathbf{x}^2 = -1$, $|x_3| < 1$. As a result we obtain a rotational body which is symmetric with respect to the x_3 axis and has the form shown in the figure in cross section; we also get rid of the velocities larger than the velocity of light in the x_3 direction.

In the resulting manifold it suffices to consider only the discrete series of representations.

Since the x are internal degrees of freedom of the particle, the figure shown describes, in principle, the shape of the particle. For $\theta \to 0$ we obtain a sphere with hollow half-axes, and for $\theta \to \pi/2$ we obtain a disk. If the "internal time" x_0 is reasonably restricted, then x_2 and x_1 will be finite. A particle in motion with the shape shown in the figure must have a momentum perpendicular to the (x_1, x_2) plane and a spin polarization (if the spin is nonzero) along the direction of motion or opposite to it.

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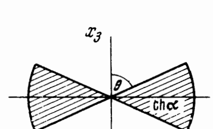

APPENDIX

The derivation of the integral representation of an odd function defined on a single-sheeted hyperboloid in terms of a function given on a cone is essentially a modification of the derivation of the integral representation of an even function given in [3]. We therefore use some of the results of [3] without dwelling on the details of the regularization of the divergent integrals by the method of analytic continuation in the coordinates.

Let f(x) be an odd function defined on $\{X\}$, $x \in X$, $x^2 = -1$. We associate with it the functions $h(\xi)$ and $\varphi(l)$. The function $h(\xi)$ is defined on the cone $\xi^2 = 0$ by

$$h(\xi) = \int f(x)\{\delta[(x\xi) - 1] - \delta[(x\xi) + 1]\}\, dx \tag{A.1}$$

The function $\varphi(l)$ is given on the set of isotropic straight lines $x = b + \xi t$ and is defined by

$$\varphi(l) = \int_l f(x)\, dl \equiv \int_{-\infty}^{\infty} f(b + \xi t)\, dt, \tag{A.2}$$

where

$$b^2 = -1, \quad (b\xi) = 0,\ \xi^2 = 0.$$

Our task is to derive an inversion formula for the integral representation (A.1), using (A.2).

We multiply both parts of (A.1) by $(|(a\xi)| - 1)_+^{\mu}$ sign $(a\xi)$. Integrating the resulting expression over $d\xi$ on the cone, we find

$$\int h(\xi)\,(|(a\xi)| - 1)_+^{\mu}\,\mathrm{sign}\,(a\xi)\, d\xi = \int \Phi(a, x, \mu) f(x)\, dx, \tag{A.3}$$

where

$$\Phi(x, a, \mu) = \int (|(a\xi)| - 1)_+^{\mu}\,\mathrm{sign}(a\xi)[\delta((x\xi) - 1) - \delta((x\xi) + 1)]\, d\xi. \tag{A.4}$$

Let us write (A.4) in more detail:

$$\Phi(a, x, \mu) = \int [(a - x, \xi)_+^{\mu} - (-a - x, \xi)_+^{\mu}]\,\delta((x\xi) - 1)\, d\xi$$

$$+ \int [(-a + x, \xi)_+^{\mu} - (a + x, \xi)_+^{\mu}]\,\delta((x\xi) + 1)\, d\xi \equiv \Phi_1 + \Phi_2. \tag{A.5}$$

Thus the task consists in the regularization and evaluation of integrals of the type

$$I_+^{\mp}(x, b; \mu) = \int (b\xi)_+^{\mu}\,\delta((x\xi) \mp 1)\, d\xi. \tag{A.6}$$

Formula (A.5) has been written down with account of the fact that the integrand has its support on the planes $(x\xi) = \pm 1$.

Using (A.6) we find for $\Phi_1(a, x; \mu)$ the expression

$$\Phi_1(x, a; \mu) = I_+^{-}(x, a - x; \mu) - I_+^{-}(x, -a - x; \mu). \tag{A.7}$$

The function $\Phi_2(x, a; \mu)$ is determined analogously. The integrals $I_+^{\mp}(x, b; \mu)$ are defined by

$$I_+^{\mp} = \frac{1}{2i\sin\mu\pi}\{e^{i\mu\pi} I^{\mp}(x, b - i0; \mu) - e^{-i\mu\pi} I^{\mp}(x, b + i0; \mu)\}, \tag{A.8}$$

and they have the values, according to [3],

$$I^{\mp}(x, b - i0; \mu) = -\frac{2\pi}{\mu + 1} e^{-i\pi\mu/2}(-P \pm i0)^{-1/2}\{(-P \pm i0)^{1/2} \pm ib_3\}^{\mu+1}, \tag{A.9}$$

$$I^{\mp}(x, b + i0; \mu) = -\frac{2\pi}{\mu + 1} e^{i\pi\mu/2}(-P \mp i0)^{-1/2}\{(-P \mp i0)^{1/2} \mp ib_3\}^{\mu+1}, \tag{A.10}$$

where the upper sign of the term i0 refers to the case $b_0 > 0$, and the lower sign to $b_0 < 0$, $P = b_0^2 - b_1^2 - b_2^2$.

Using (A.7) to (A.10), we find for $\Phi_1(a, x; \mu)$ the following expression [for $x = (0, 1, 0, 0)$]:

$$\Phi_1(x, a; \mu) = \frac{i\pi}{(\mu+1)\sin\mu\pi}[e^{i\pi\mu/2}(-P \pm i0)^{-1/2}$$
$$\times\{(-P \pm i0)^{1/2} + i(a_3 - 1)\}^{\mu+1} - e^{-i\pi\mu/2}(-P \mp i0)^{-1/2}$$
$$\times\{(-P \mp i0)^{1/2} - i(a_3 - 1)\}^{\mu+1} - e^{i\mu\pi/2}(-P \mp i0)^{-1/2}$$
$$\times\{(-P \mp i0)^{1/2} - i(a_3 + 1)\}^{\mu+1} + e^{-i\mu\pi/2}(-P \pm i0)^{-1/2}$$
$$\times\{(-P \pm i0)^{1/2} + i(a_3 + 1)\}^{\mu+1}], \qquad (A.11)$$

where $P = a_0^2 - a_1^2 - a_2^2$ and the upper sign refers to the case $a_0 > 0$ and the lower sign to $a_0 < 0$. The function $\Phi_2(x, a; \mu)$ has the form

$$\Phi_2(x, a; \mu) = \frac{i\pi}{(\mu+1)\sin\mu\pi}[e^{i\pi\mu/2}(-P \mp i0)^{-1/2}$$
$$\times\{(-P \mp i0)^{1/2} - i(1 - a_3)\}^{\mu+1} - e^{-i\pi\mu/2}(-P \pm i0)^{-1/2}$$
$$\times\{(-P \pm i0)^{1/2} + i(1 - a_3)\}^{\mu+1} - e^{-i\pi\mu/2}(-P \pm i0)^{-1/2}$$
$$\times\{(-P \pm i0)^{1/2} - i(a_3 + 1)\}^{\mu+1} + e^{-i\mu\pi/2}(-P \mp i0)^{-1/2}$$
$$\times\{(-P \mp i0)^{1/2} + i(a_3 + 1)\}^{\mu+1}]. \qquad (A.12)$$

Using

$$(-P \pm i0)^{1/2} = P_-^{1/2} \pm iP_+^{1/2}$$
$$(-P \pm i0)^{-1/2} = P_-^{-1/2} \mp iP_+^{-1/2}, \qquad (A.13)$$

we simplify (A.11) and (A.12) and obtain the form of the function $\Phi(x, a; \mu)$ for $P > 0$, $a_0 > 0$, $a_0 < 0$, $x = (0, 1, 0, 0)$:

$$\Phi = \frac{-2\pi P^{\mu/2}}{\mu+1}\left\{\left(1 + \frac{a_3 - 1}{P^{1/2}}\right)^{\mu+1} - \left(1 - \frac{a_3 + 1}{P^{1/2}}\right)^{\mu+1}\right\}, \qquad (A.14)$$

and in the region $P < 0$, i.e., $(ax)^2 = \cos^2 kr \equiv \cos^2\theta = a_3^2 < 1$:

$$\Phi(a, x; \mu) = \frac{-4\pi\sin^\mu kr}{(\mu+1)\sin\mu\pi}\cos kr. \qquad (A.14')$$

Here r is the distance between the points a and x in the geometry of the imaginary Lobachevskiĭ space. Since the value of $\Phi(a, x; \mu)$ remains unchanged if the points a and x are shifted simultaneously, the expressions obtained are valid for any two points of the single-sheeted hyperboloid.

The generalized function $(x_0^2 - x_1^2 - x_2^2)_+^{\mu/2}$ has a simple pole at $\mu = -3$ with the residue

$$\operatorname*{Res}_{\mu=-3}(x_0^2 - x_1^2 - x_2^2)_+^{\mu/2} = -4\pi\delta(x_0, x_1, x_2).$$

Then

$$\operatorname*{Res}_{\mu=-3}\int\Phi(x, a; \mu)f(x)dx = \operatorname*{Res}_{\mu=-3}\int_{|(ax)|>1}\Phi(x, a; \mu)f(x)dx$$
$$+ \operatorname*{Res}_{\mu=-3}\int_{|(ax)|<1}\Phi(x, a; \mu)f(x)dx = -8\pi^2 f(a)$$
$$-2\int_{|(ax)|<1}(ax)[1 - (ax)^2]^{-3/2}f(x)\,dx. \qquad (A.15)$$

The residue of the left-hand side of (A.3) at $\mu = -3$ is, because of

$$\operatorname*{Res}_{\mu=-3} t_+^\mu = {}^1/_2\delta''(t),$$

equal to

$$\frac{1}{2}\int h(\xi)\delta''(|(a\xi)| - 1)\operatorname{sign}(a\xi)\,d\xi. \qquad (A.16)$$

Equating the resulting expressions, we have

$$f(a) = -\frac{1}{16\pi^2}\int h(\xi)\delta''(|(a\xi)| - 1)\operatorname{sign}(a\xi)\,d\xi$$
$$-\frac{1}{4\pi^2}\int_{|ax|<1}(ax)[1 - (ax)^2]^{-3/2}f(x)\,dx. \qquad (A.17)$$

In the second term we introduce the following parametrization for x:

$$x_0 = t, \quad x_2 = -\sin\theta\cos\alpha + t\sin\alpha,$$
$$x_3 = \cos\theta, \quad x_1 = \sin\theta\sin\alpha + t\cos\alpha; \qquad (A.18)$$
$$0 \leqslant \alpha \leqslant 2\pi, \quad 0 \leqslant \theta \leqslant \pi, \quad -\infty < t < \infty.$$

Since in the region $|(ax)| < 1$, the vector x can always be written in the form $x = b + \xi t$, where, by (A.18), $\xi = (1, 0, \sin\alpha, \cos\alpha)$, $\xi^2 = 0$, $b^2 = -1$, $b = (0, \cos\theta, -\sin\theta\cos\alpha, \sin\theta\sin\alpha)$,

$$dx = \frac{dx_1dx_2dx_3}{x_0} = d\cos\theta\,dt\,d\alpha,$$

the second term has the form

$$I_2 = -\frac{1}{4\pi^2}\int_0^\pi\frac{\cos\theta d\theta}{\sin^2\theta}\int_L\varphi(\xi, \theta)\delta((a\xi))\,d\Omega, \qquad (A.19)$$

where $\varphi(\xi, 0) \equiv \varphi(\xi, b)$ is given by (A.2), L is a sphere, and $d\Omega$ is the surface element on the sphere. We expand $\varphi(\xi, b)$ in the series (cf. [3])

$$\varphi(\xi, b) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}F(\xi, b; n) \equiv \frac{1}{\pi}\sum_{n=-\infty}^{\infty}e^{in\theta}F(\xi, a; n). \qquad (A.20)$$

We substitute (A.20) in (A.19), interchange integration and summation, and take into account that we are expanding an odd function; we obtain

$$I_2 = -\frac{1}{2\pi^3}\sum_{n=-\infty}^{\infty}\alpha_n\int_L F(\xi, a; n)\delta((a\xi))\,d\Omega, \qquad (A.21)$$

where the summation goes over odd n and

$$\alpha_n = \int_0^\pi\cos\theta\sin^{-2}\theta\cos n\theta d\theta. \qquad (A.22)$$

The integral (A.22) is understood as the value of the integral

$$\int_0^\pi\cos\theta\sin^\lambda\theta\cos n\theta d\theta.$$

for $\lambda = -2$. Using (3.631) (8) of [5], we find that

$$\alpha_n = -2\pi|n|, \qquad n = 2m - 1,$$

i.e.,

$$I_2 = \frac{2}{\pi^2}\sum_{m=1}^{\infty}(2m-1)\int_L F[\xi, a; (2m-1)]\delta((a\xi))\,d\Omega. \qquad (A.23)$$

Expanding $h(\xi)$ in the homogeneous components $F(\xi, \sigma)$,

$$h(\xi) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}F(\xi, \sigma)\,d\sigma, \qquad (A.24)$$

we substitute (A.24) in (A.17) and carry out the integration. Then we obtain the final form of the rotation formula, using (A.23):

$$f(x) = \frac{(-)}{4i(2\pi)^3}\int_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\int_L F(\xi, \sigma)\,|(x\xi)|^{-\sigma-2}\operatorname{sign}(\xi x)\,d^2\xi d\sigma$$
$$+\frac{2}{\pi^2}\sum_{m=1}^{\infty}(2m-1)\int_L F[\xi, x; (2m-1)]\delta((\xi x))\,d^2\xi. \qquad (A.25)$$

Here $x^2 = -1$, $\xi^2 = 0$, L is the contour of integration on the cone (sphere), and $d^2\xi = d\Omega$ is the surface element on the sphere.

[1] V. L. Ginzburg and I. E. Tamm, Zh. Eksp. Teor. Fiz. **17**, 227 (1947).
[2] I. M. Gel'fand and M. I. Graev, Trudy Mosk. matem. obshchestva **11**, 243 (1962).
[3] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, Integral'naya geometriya i svyazannye s neĭ voprosy predstavleniĭ (Integral Geometry and Related Problems of the Theory of Representations), Fizmatgiz, 1962.
[4] N. Ya. Vilenkin and Ya. A. Smorodinskiĭ, Zh. Eksp. Teor. Fiz. **46**, 1793 (1964) [Sov. Phys.-JETP **19**, 1209 (1964)].
[5] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1963).

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