SOVIET PHYSICS JETP VOLUME 27, NUMBER 5 NOVEMBER, 1968

CONTRIBUTION TO THE THEORY OF THE THERMAL CONDUCTIVITY OF DIELECTRIC SAMPLES OF LIMITED DIMENSION

R. N. GURZHI and S. I. SHEVCHENKO

Kharkov Polytechnic Institute

Submitted November 5, 1967

Zh. Eksp. Teor. Fiz. 54, 1613–1616 (May, 1968)

A hydrodynamic mechanism of the thermal conductivity of dielectrics is considered under conditions when the scattering of the phonons by the boundaries of the sample is close to specular. For an arbitrary scattering law, the boundary condition is obtained for the velocity of ordered motion of the phonons. The thermal conductivity of a dielectric plate is calculated by way of an application.

AT sufficiently low temperatures, when phonon-phonon collisions accompanied by Umklapp processes are much rarer than normal collisions, the phonons in pure dielectrics behave like an ordinary gas. The thermal conductivity of samples of limited dimensions has under such conditions the character of a Poiseuille flow[1-4].

As already indicated[1,2], the problem reduces to a solution of a hydrodynamic equation of the Navier-Stokes type for the velocity of the ordered motion of the phonon gas $\mathbf{u}(\mathbf{r})$, with corresponding boundary conditions. It was assumed here that $\mathbf{u} = 0$ on the sample boundary, corresponding to diffuse scattering of the phonons. However, with decreasing temperature, the average phonon wavelength λ_T becomes inevitably larger than the characteristic dimension of the roughness of the sample surface ($\lambda_T \approx hs/T$, s—speed of sound). It is clear that in this case the scattering of the phonons by the boundary will be close to specular. We obtain below a corresponding boundary condition and consider by way of an application the thermal conductivity of a plate.

We denote by l^N and l^V respectively the mean free paths relative to normal collisions and collisions with loss of quasi-momentum, and by d the characteristic transverse dimension of the sample. The hydrodynamic situation arises under the condition that

$$l^N \ll d, l^V. \tag{1}$$

The phonons are characterized in this case by the drift distribution function

$$f_0(\varepsilon - \mathbf{up}) \approx f_0(\varepsilon) - \mathbf{up}\partial f_0 / \partial\varepsilon, \tag{2}$$

and the velocity of the ordered motion $\mathbf{u}(\mathbf{r})$ satisfies the equation

$$\gamma\nabla T = \nu\Delta\mathbf{u} - \mathbf{u} / \tau^V. \tag{3}$$

Here $\epsilon(\mathbf{p})$ is the phonon energy, $f_0(\epsilon) = (e^{\epsilon/T} - 1)^{-1}$, $\gamma \approx s^2/T$, the kinematic viscosity is $\nu \approx sl^N$, and $\tau^V = l^V/s$. For simplicity, the phonon dispersion law is assumed isotropic: $\epsilon = sp$.

The boundary condition for $\mathbf{u}(\mathbf{r})$ can be obtained by starting from the condition of mechanical equilibrium of a thin layer of gas adjacent to the surface of the sample. Let the layer thickness L satisfy the inequalities $l^N \ll L \ll d$. The force exerted by the boundary on the layer is equal to the momentum absorbed by the wall per unit time. Acting from the opposite side, where the hydrodynamic treatment is valid, is the friction force. We choose the z axis perpendicular to the surface of the sample and the x axis along the temperature gradient. The equilibrium condition yields

$$\left\{\nu\rho\frac{\partial u_x}{\partial z} + \frac{1}{h^3}\left[\int\limits_{s_z>0} d\mathbf{p}\, p_x s_z f(\mathbf{p}, \mathbf{r}) + \frac{1}{h^3}\int\limits_{s_z<0} d\mathbf{p}\, p_x s_z \int\limits_{s_z'>0} d\mathbf{p}'\, R_{\mathbf{p}'}^{\mathbf{p}} f(\mathbf{p}', \mathbf{r})\right]\right\}_{\mathbf{r}=\mathbf{r}_0} = 0. \tag{4}$$

Here $\rho = -h^{-3}\int d\mathbf{p}\, p_x^2 f_0'(\epsilon)$, $f_0'(\epsilon) = \partial f_0/\partial\epsilon$, $\mathbf{r}_0$—coordinates of the points of the surface, $R_{\mathbf{p}'}^{\mathbf{p}}$—probability of transition from the state with momentum $\mathbf{p}'$ to a state with momentum $\mathbf{p}$ as a result of scattering from the boundary.

The phonon distribution function $f(\mathbf{p}, \mathbf{r})$ contained in (4) cannot be obtained for arbitrary $R_{\mathbf{p}'}^{\mathbf{p}}$, inasmuch as the kinetic equation cannot be solved near the boundary. The problem simplifies in the case of scattering that is close to specular. In the case of pure specular scattering of the phonons by the sample boundary, in the general case of an arbitrary dispersion law, a change takes place only in the momentum component that is normal to the wall (the longitudinal component of the momentum and the energy are conserved, cf.[5]). Therefore, in purely specular scattering, the kinetic equation will have the drift solution (2) up to the very boundary. From physical considerations it is clear that in nearly specular scattering the distribution function $f(\mathbf{p}, \mathbf{r})$ will differ little from the drift solution. Substituting (2) in (4) we obtain the following boundary condition:

$$l^N|\partial u_x / \partial z| = (1 - P)u_x,$$

where

$$1 - P = \left[\int\limits_{s_z>0} d\mathbf{p}\, p_x^2 s_z f_0'(\varepsilon) + \int\limits_{s_z<0} d\mathbf{p}\, p_x s_z \frac{1}{h^3}\int\limits_{s_z'>0} d\mathbf{p}' R_{\mathbf{p}'}^{\mathbf{p}} p_x' f_0'(\varepsilon')\right]\left[s\int d\mathbf{p}\, p_x^2 f_0'(\varepsilon)\right]^{-1} \tag{5}$$

has the meaning of the probability of diffuse scattering upon one collision with the wall[1)].

We present by way of an example the result of the

---

1)As will be shown below, the boundary condition (5), which was obtained for the case of almost specular reflection (1 – P << 1), is actually valid for any character of the scattering.

solution of Eq. (3) with boundary condition (5) for a plate. If we represent the thermal conductivity coefficient κ in the usual form:

$$\kappa = {}^1/_3 C l^{\text{eff}} s, \tag{6}$$

where C is the specific heat, then

$$l^{\text{eff}} = l^V\left\{1 - \frac{\operatorname{sh} v}{v \operatorname{ch} v + 2v^2\alpha^{-1}\operatorname{sh} v}\right\} \quad v = \frac{d}{2\sqrt{l^V l^N}}, \quad \alpha = \frac{(1-P)d}{l^N}, \tag{7}$$

Here d—thickness of the plate. Let us consider the following possible limiting cases:

a) when $\alpha \gg 1 \gg v^2$

$$l^{\text{eff}} \approx {}^1/_{12} d^2 / l^N; \tag{8}$$

b) when $1 \gg \alpha \gg v^2$

$$l^{\text{eff}} \approx d/2(1-P); \tag{9}$$

c) when $v^2 \gg \alpha$ or $v \gg 1$

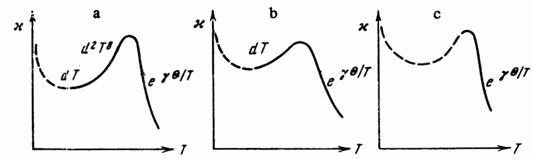
$$l^{\text{eff}} \approx l^V. \tag{10}$$

The results can be readily explained by intuitive considerations. Since $l^N \ll d$, the normal collision will cause the phonon to move from the wall to the wall like a Brownian particle. From the formulas for Brownian motion it follows that in this case the phonon covers a path with a total length on the order of d^2/l^N. It is also clear that it collides with the wall on the average d/l^N times before it moves away from the wall a distance comparable with d, and thus, α yields the number of diffuse collisions during that time. On the other hand, $\tau^{\text{eff}} = l^{\text{eff}}/s$ is the free path time relative to collisions with loss of quasi-momentum (in the x-axis direction), and l^{eff} is the length of the corresponding trajectory. It is clear from the foregoing that if we disregard exchange collisions with loss of quasi-momentum, then $l^{\text{eff}} \approx d^2/l^N$ when $\alpha \gg 1$ and $l^{\text{eff}} \approx d^2/l^N\alpha \approx d/(1-P)$ when $\alpha \ll 1$ (see formulas (8) and (9)). Finally, if d^2/l^N or $d/(1-P)$ is much larger than l^V, then collisions with the boundaries are insignificant, and, in accordance with (9), $l^{\text{eff}} \approx l^V$.

We note that expression (8) has exactly the same form as in the case of pure diffuse scattering of phonons by the boundaries (see[2]). Consequently, the condition (5) becomes equivalent to the condition $u(r_0) = 0$ even within the limits of applicability of the approach developed above, i.e., when $1 - P \ll 1$. This is caused by the fact that the phonon returns many times to the wall as the result of normal collisions with other phonons. It is clear therefore that the boundary condition (5) is valid for any character of the scattering.

We note that when the scattering is close to specular it can be easily shown (see, for example,[6] that $1 - P = a(\eta/\lambda_T)^2$, where η is the characteristic dimension of the roughness of the surface, $\lambda_T \approx hs/T$ is the wavelength of the temperature phonons, and a is a dimensionless factor on the order of unity, the value of which depends on the detailed structure of the surface of the sample.

Using formula (6)—(10) we can easily obtain the temperature variation of the thermal-conductivity coefficient. Depending on the ratio of the parameters of the problem, three cases can occur, as shown in the figure. Case a corresponds to relatively bulky samples with perfect boundaries, on which $\alpha \gg 1$ when $v \sim 1$. Then, the possibilities (10), (8), and (9) are



realized with decreasing temperature. On the other hand, if $\alpha \ll 1$ when $v \sim 1$, then the result depends on whether the inequality $d \gg l^N$ is violated when $v^2 \ll \alpha$ (case b) or when $v^2 \gg \alpha$ (case c).

The dashed lines in the figure show the regions where $l^N \gg d$, and therefore the hydrodynamic method of describing is not applicable. This case was considered by the authors earlier[7] (see Fig. 2b in[7], where the dashed lines, to the contrary, denote the region investigated in the present paper).

We note that within the limits of applicability of the hydrodynamic approach, the thermal conductivity is not very sensitive to the shape of the cross section of the sample (see[2]), and therefore formulas (8)—(10) are correct, in order of magnitude, for samples of arbitrary form.

The obtained results make it possible to explain qualitatively the experimental data of Mezhov-Deglin[8] on the thermal conductivity of crystalline He^4. The measurements were made on cylindrical samples in the temperature interval approximately from 0.4 to 1.5°K. In the thinnest of the investigated samples, at temperatures close to 0.5°K, there was observed a relatively small growth (by a factor of approximately 1.5) in the value of l^{eff} with decreasing T. On the other hand, a decrease of l^{eff} was observed in the bulkier samples under the same conditions. To explain these regularities, it is natural to assume that when $T \approx 0.5$°K the scattering of the phonons by the boundaries is nearly specular. In thin samples, according to (9), we have $l^{\text{eff}} \propto T^{-2}$, while in thick samples, owing to multiple collisions between the phonons and the boundaries, the case (8), corresponding to effective diffuse scattering, is realized, and $l^{\text{eff}} \propto T^5$.

[1] J. A. Sussman and A. Thellung, Proc. Phys. Soc. 81, 1122 (1963).

[2] R. N. Gurzhi, Zh. Eksp. Teor. Fiz. 46, 719 (1964) [Sov. Phys.-JETP 19, 490 (1964)].

[3] L. P. Mezhov-Deglin, ibid. 46, 1926 (1964) [19, 1297 (1964)].

[4] L. P. Mezhov-Deglin, ibid 49, 66 (1965) [22, 47 (1966)].

[5] P. J. Price, IBM J. 4, 152 (1960).

[6] J. M. Ziman, Electrons and Phonons, Oxford, 1960.

[7] R. N. Gurzhi and S. I. Shevchenko, Zh. Eksp. Teor. Fiz. 52, 814 (1967) [Sov. Phys.-JETP 25, 534 (1967)].

[8] L. P. Mezhov-Deglin, ibid. 52, 866 (1967) [25, 568 (1967)].

Translated by J. G. Adashko
185