DEVIATIONS FROM A MAXWELLIAN ELECTRON ENERGY DISTRIBUTION IN A WEAKLY IONIZED PLASMA

È. B. SONIN

Semiconductor Institute, USSR Academy of Sciences

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This paper considers the distribution function for fast electrons and the concentration of atoms in the first excited level in the case in which the first excitation potential exceeds the mean electron energy. It is shown that in a number of cases the transitions of atoms between excited levels can lead to a Maxwellian electron energy distribution for arbitrarily weak electron-electron collisions; on the other hand, if these transitions are not taken into account, it is possible to obtain arbitrarily large deviations from a Maxwellian distribution. The electron temperature below which the spatial inhomogeneity of the fast electron distribution function can be neglected is also estimated.

THE problem of determining the electron energy distribution in a weakly ionized plasma has been considered by a number of authors.[1-4] The basic factor that causes deviations from a Maxwellian distribution in such a plasma is the inelastic collision mechanism for collisions between electrons and atoms. It was assumed there that the plasma is homogeneous (the necessity for this condition is treated in the Appendix).

It was also assumed in the work cited above that the slow electrons, with energies much below the first excitation potential E_1, exhibit a distribution function that is essentially Maxwellian at temperature T with $kT \ll E_1$; on the other hand, in the kinetic equation for electrons with energies $\sim E_1$ account is taken of inelastic electron-atom collisions, characterized by transitions between the ground level and the first excited level, as well as diffusion of electrons in energy space as a consequence of electron-electron collisions which lead to the production of fast electrons, i.e., electrons with energy greater than E_1.

Moĭzhes et al.[5] have used a similar method to solve the self-consistent problem for a cesium plasma, that is to say, they found the electron distribution function together with the atomic distribution over the excited levels under conditions in which the mechanism for violation of equilibrium in the system is the nonequilibrium ionization. In this case the density of free electrons is smaller than the equilibrium value given by the Saha formula. The number of fast electrons and the associated number of atoms in the first excited level was determined from the balance in the later production of fast electrons due to diffusion in energy space and the loss of fast electrons due to atomic transitions between the ground level and the first excited level. The diffusion is due to electron-electron collisions and is proportional to N_e^2 (N_e is the density of free electrons) while the loss of fast electrons is proportional to $N_0 N_e$ (N_0 is the concentration of atoms in the ground level). Hence, as the degree of ionization N_e/N_0 is reduced, the number of fast electrons approaches zero in this formulation of the problem. This result, which follows from the work of Moĭzhes et al.[5] is found to be incompatible with a more rigorous numerical calculation carried out in[6] for the same conditions on an electronic computer, in which all transitions for 40 excitation levels were taken into account in the kinetic equation. These calculations show that the number of fast electrons remains the same as for a Maxwellian distribution function even in the case $N_e/N_0 \to 0$, in which electron-electron collisions are not important (radiation effects are not considered in[5] or[6]).

The purpose of the present work is to show that the results of[6] can also be obtained by means of the model used in[1-5]; it is shown, however, that it is necessary to take account of the production of fast electrons in collisions of electrons with excited atoms, these collisions leading to the transition of the atom from one excited level to another (by analogy with optical transitions we will call these nonresonant transitions, in contrast with the resonant transition between the first excited level and the ground level). In general, the production of fast electrons in nonresonant transitions is much weaker than the production or loss of a fast electron due to resonance transitions which were taken into account earlier. However, the change in the number of fast electrons due to resonant transitions is not determined entirely by the total number of such transitions, but by the difference in the number of transitions in the upward and downward directions. This difference is equal to the flux of atoms over the spectrum from the first excited level to the continuum, that is to say, the ionization rate (this follows directly from[5]). If the ionization rate is smaller than the number of nonresonant transitions of atoms due to collisions with electrons with energy $\sim E_1$, as is the case in cesium, the nonresonant transitions in collisions are found to produce a Maxwellian electron distribution function. Although the simplified model used below is obviously not quantitatively as rigorous as the numerical method used in the earlier work[6] it is more lucid and provides the possibility of determining the physical pattern of the effect which is responsible for producing the Maxwellian electron distribution function solely by means of inelastic electron-atom collisions. Using this approach without repeating the numer-

ical calculations such as those in[6], we can also estimate the distribution function for a gas which is different from cesium.

Within the framework of this model, as will be seen below, it is also an easy matter to take account of radiation.

We introduce the quantity $\nu(\epsilon) = f(\epsilon)/f_M(\epsilon)$, which characterizes the deviation of the electron energy distribution function $f(\epsilon)$ from a Maxwellian distribution $f_M(\epsilon)$, and the quantity $y_i = N_i/N_i^B$ which characterizes the deviation of the atomic population N_i in the excited level from the equilibrium Boltzmann value

$$N_i^B = N_0 \frac{g_i}{g_0} \exp\left(-\frac{E_i}{kT}\right).$$

Here, g_0 and g_i are the statistical weights of the ground level and the i-th level. In the first excited level $i = 1$. As shown in[5], for weak electron-electron collisions $\nu(\epsilon)$ is given approximately by

$$\nu(\varepsilon) = \begin{cases} 1, & \varepsilon < E_1 \\ y_1, & \varepsilon > E_1 \end{cases}, \tag{1}$$

while the rate of production of fast electrons due to diffusion in energy space is

$$I_e(E_1) = \frac{D(E_1)}{kT} f_M(E_1)(1 - y_1), \tag{2}$$

where $D(\epsilon)$ is the diffusion coefficient, which determines the diffusion flux $I_e(\epsilon) = -D(\epsilon) f_M(\epsilon) d\nu/d\epsilon$ in the energy space ϵ.

As we have already indicated, the rate of production of fast electrons $I_e(E_1)$ is equated in[5] to the loss due to inelastic collisions associated with the resonant transition, which in turn, is equal to the flux of excited atoms from the first level to the continuum, that is to say, the generation rate. If recombination is neglected the generation rate J can be written in the form

$$J = QN_0N_e v y_1, \tag{3}$$

where $v = (8kT/\pi m)^{1/2}$ is the thermal velocity, k is the Boltzmann constant, m is the electron mass, and the cross-section Q depends only on the temperature.

It can be shown that if approximate account is taken of the nonresonant transitions, Eqs. (1)–(3) which follow from[5], still apply. However, the diffusion coefficient $D(\epsilon)$ in Eq. (2) must be written in the form of a sum of coefficients $D_e(\epsilon)$ which determine the contribution from electron-electron collisions already taken into account earlier [cf. Eq. (2) for $D(\epsilon)$ in[5]] and the coefficient $D_{inel}(\epsilon)$, which determines the total flux of electrons through the energy surface ϵ due to nonresonant atomic transitions:

$$D(\varepsilon) = D_e(\varepsilon) + D_{inel}(\varepsilon). \tag{4}$$

In particular, transitions between excited levels i and k produce an electron flux I_{ik} in energy space:

$$I_{ik} = \frac{N_e v}{(kT)^2} \int_{\varepsilon}^{\varepsilon + E_{ik}} \left[\nu(\varepsilon' - E_{ik}) \exp\left(-\frac{\varepsilon' - E_{ik}}{kT}\right) N_k \frac{g_i}{g_k} - N_i \nu(\varepsilon') \exp\left(-\frac{\varepsilon'}{kT}\right)\right] \sigma_{ik}(\varepsilon')\varepsilon' d\varepsilon', \tag{5}$$

where $\sigma_{ik}(\epsilon)$ is the cross-section for a transition from the level i to the level k; N_i and g_i are the concentration and statistical weight in the level i; E_{ik} is the energy for the transition from level i to level k. The calculations show[6,7] that for a Maxwellina distribution function in which the electron temperature is not too high a number of the lower excited levels characterized by $i \leq i_p$ have a population which is close to the equilibrium population given by the concentration in the ground level and the electron temperature, that is to say, $y_i \approx 1$. In this case the energy of the transition between these levels is comparable with kT and smaller than E_1. Thus, transitions between these levels occur by virtue of slow electrons with $\epsilon < E_1$ and for deviations from a Maxwellian distribution in the region of fast electrons $\epsilon > E_1$ [cf. Eq. (1)] the excited levels $i \leq i_p$ (including the first excited level $i = 1$) remain in equilibrium between themselves and the ground level. In other words, for these levels $y_i \approx y_1$ ($y_1 < 1$). The sum I_{ik} over the level indicated above for the group of low excited levels can be expressed in terms of $d\nu/d\epsilon$ if we use the following approximation in Eq. (5): $\nu(\epsilon' - E_{ik})$ is replaced by $\nu(\epsilon') - E_{ik} d\nu(\epsilon)/d\epsilon$:

$$\sum_{1 \leq i < k \leq i_p} I_{ik} = -D_{inel}(\varepsilon) f_M(\varepsilon) \frac{dv}{d\varepsilon}, \tag{6}$$

where

$$D_{inel}(\varepsilon) = y_1 \sum_{1 \leq i < k \leq i_p} \frac{N_e N_i^B v}{(kT)^2 f_M(\varepsilon)} \int_{\varepsilon}^{\varepsilon + E_{ik}} \exp\left(-\frac{\varepsilon'}{kT}\right) \sigma_{ik}(\varepsilon')\varepsilon' d\varepsilon'.$$

Substituting Eq. (4) in Eq. (2) and taking account of Eq. (6), we obtain the following expression for $I_e(E_1)$:

$$I_e(E_1) = (Q_e N_e^2 + Q_{inel} N_0 N_e y_1) v (1 - y_1), \tag{7}$$

where the cross-sections Q_e and Q_{inel} depend only on the temperature and are related to the appropriate diffusion coefficients:

$$Q_e = \frac{D_e(E_1) f_M(E_1)}{N_e^2 kTv}, \quad Q_{inel} = \frac{D_{inel}(E_1) f_M(E_1)}{y_1 N_e N_0 kTv}. \tag{8}$$

Using the relation $I_e(E_1) = J$ for y_1, we obtain a quadratic equation which has the single positive solution

$$y_1 = \frac{1}{2}\left[1 - \gamma - \frac{Q_e N_e}{Q_{inel} N_0} + \left\{\left(1 - \gamma - \frac{Q_e N_e}{Q_{inel} N_0}\right)^2 + 4\frac{Q_e N_e}{Q_{inel} N_0}\right\}^{1/2}\right], \tag{9}$$

where

$$\gamma = Q / Q_{inel} \tag{10}$$

The behavior of y_1 will be different as $N_e/N_0 \to 0$ for different values of γ. When $\gamma \gg 1$ Eq. (9) is equivalent to the expression obtained in[5] for weak electron-electron collisions:

$$y_1 = \frac{1}{1 + QN_0/Q_e N_e}. \tag{11}$$

In this case it is not necessary to take account of nonresonant transitions and $y_1 \sim N_e/N_0$ when $N_e/N_0 \to 0$. If $\gamma < 1$, then when $N_e/N_0 \to 0$ the quantity y_1 approaches the limiting value $1 - \gamma$. For this reason, if the parameter γ is small the deviation from the Maxwellian distribution does not appear in spite of the fact that electron-electron collisions become very infrequent as $N_e/N_0 \to 0$.

In order to estimate γ, in addition to knowing Q_{inel}, we must know the effective cross section Q which characterizes the ionization rate J [cf. Eq. (3)]. In cesium

the value of Q can be taken to be that obtained by the numerical calculations[6,8] for a Maxwellian distribution function, in which case y_1 is close to unity. In Fig. 1 we show (solid curve) the dependence of Q on temperature in accordance with curve 9 of Fig. 2 of[6], where the ionization probability and the probability for transitions between excited levels have been computed by the classical Thomson relation. In the same figure the dashed curve shows the values of Q as computed by means of detailed balance and the formula for the recombination coefficient α_p taken from Gurevich and Pitaevskiĭ[9].

In accordance with the principle of detailed balancing we have

$$QN_0N_e^S v = \alpha_p (N_e^S)^2,$$

where

$$N_e^S = \left\{ \left(\frac{2\pi mkT}{h^2} \right)^{3/2} N_0 \exp\left(-\frac{E_{ion}}{kT} \right) \right\}^{1/2}$$

is the equilibrium density of electrons or an atomic concentration N_0 in the ground level.

Using Eq. (5) for α_p from[9] and substituting in it the value N_e^S in place of N_e we obtain the following expression for Q:

$$Q = \frac{4\sqrt{2}\,\pi^{7/2} e^{10} Z^3 m^{3/2} \ln \sqrt{Z^2+1}}{9 h^3 (kT)^{7/2}} \exp\left(-\frac{E_{ion}}{kT} \right),$$

where Z is the ion charge (in cesium Z = 1).

In the work of Gurevich and Pitaevskiĭ[9] the discrete spectrum of energy levels for the excited states has been replaced by a continuum and this is valid for transition energies $E_{ik} \ll kT$. For the temperature shown in Fig. 1 in cesium the quantity Q is limited by the transition probability in the upper part of the spectrum where E_{ik} is comparable with kT (cf.[6]) while the energy spectrum differs from a hydrogen-like spectrum. The small differences between the curves in Fig. 1 show, nonetheless, that taking account of the actual discrete structure of the spectrum under these conditions yields small corrections and one can use the formulas from[9] for those gases for which the calculations similar to those carried out in[6] have not been carried out. In accordance with Fig. 1, the quantity $Q \sim \exp(-E^*/kT)$ where $E^* \sim 3$ eV and is close to the ionization potential for cesium $E_{ion} = 3.98$ eV. The quantity $Q_{inel} \sim \exp(-2E_1/kT)$ in accordance with Eq. (8) so that γ and its dependence on temperature are a function of the ratio E_{ion}/E_1. The trend toward a Maxwellian distribution due to nonresonant transitions must be stronger for larger ratios of E_{ion}/E_1, in which case γ is small. The ratio E_{ion}/E_1 for the alkali metals is larger, for example, than for the inert gases or hydrogen. Thus, for cesium γ is smaller than unity up to a temperature of 4000° K and this explains the results of the numerical calculations in[6].

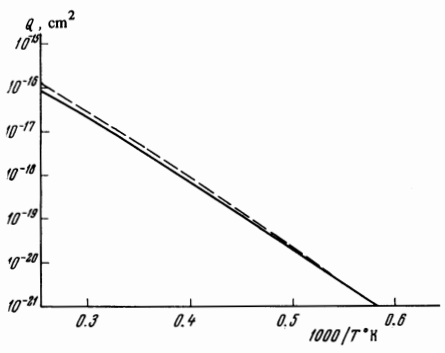

FIG. 1

We now consider the situation in which radiative transitions from the first level are important (resonance lines) but in which radiative transitions from higher levels remain rare compared with the collision transitions to these same levels. This situation will apply in a plasma which does not have high optical thickness, in which case the absorption of the resonance lines is not very large. In this case, as before, we can use Eq. (3) for the flux of atoms from the first level to the continuum but it is now necessary to take account of radiation in the balance relations; in place of the relation $I_e(E_1) = J$ we now have $I_e(E_1) = J + G$ where G determines the loss of excited atoms due to radiation:

$$G = N_1/\tau_1 = y_1 z N_0 v,$$
$$z = \frac{g_1}{g} \frac{\exp(-E_1/kT)}{\tau v}; \qquad (12)$$

where τ_1 is the effective lifetime in the first level (taking account of absorption). As before, y_1 will be determined by a quadratic equation whose solution is of the form given in (9); the expression for γ given by (10) now becomes

$$\gamma = \frac{QN_e + z}{Q_{inel} N_e}. \qquad (13)$$

It is evident that as N_e is reduced the value of γ increases and when $\gamma > 1$ the nonresonant transitions between excited levels can not bring about a Maxwellian distribution for the electrons and need not be taken into account. For large values of γ, taking account of the change from (10) to (13), we find that (11) is replaced by

$$y_1 = \left(1 + \frac{QN_eN_0 + zN_0}{Q_e N_e^2} \right)^{-1}. \qquad (14)$$

In all of the relations given above it is not necessary to know the cross section for transitions between the ground level and the first excited level. These are actually assumed to be infinitely large. In those cases in which the intensity of Coulomb collisions exceeds the intensity of collisional transitions from the ground level to the first excited level and back, the optical deexcitation does not have an effect on the shape of the distribution function.

APPENDIX

EVALUATION OF SPATIAL TRANSPORT OF FAST ELECTRONS

In the present work, as in[1-5], it is assumed that for the fast electrons the spatial transport is small compared with the transport in energy space. In order to evaluate the validity of this assumption we consider two equations[2] for the isotropic $f_0(\epsilon)$ and anisotropic $f_1(\epsilon)$ parts of the electron distribution function $f_0(\epsilon) + f_1(\epsilon)v_x/v$ where v and v_x are the modulus of the electron velocity

and the component in the x direction:

$$\frac{1}{3}\sqrt{\frac{2\varepsilon}{m}}\left(\frac{\partial f_1}{\partial x}+\frac{d\varphi}{dx}\frac{1}{\varepsilon}\frac{\partial(f_1\varepsilon)}{\partial\varepsilon}\right)=S(f_0(\varepsilon)), \tag{A.1}$$

$$\frac{\partial f_0}{\partial x}+\frac{d\varphi}{dx}\frac{\partial f_0}{\partial\varepsilon}=-\frac{f_1}{L(\varepsilon)}. \tag{A.2}$$

Here, $S[f_0(\epsilon)]$ is a collisional term, which takes account of electron collisions and inelastic electron atom collisions, $\varphi(x)$ is the potential associated with the electric field, and $L(\epsilon)$ is the relaxation length for the momentum. We will assume the highest general power dependence of $L(\epsilon)$ on energy, that is

$$L(\varepsilon)=L_p\varepsilon^p. \tag{A.3}$$

In (A.2) we substitute a Maxwellian distribution function in place of $f_0(\epsilon)$. Then, using (A.2), we can express $f_1(\epsilon)$ in terms of the gradients of N_e, T and φ, relating the current

$$I_e\sim\int_0^\infty f_1(\varepsilon)\varepsilon d\varepsilon$$

and the flux of electron kinetic energy

$$P_e\sim\int_0^\infty f_1(\varepsilon)\varepsilon^2 d\varepsilon$$

with $f_1(\epsilon)$ and expressing $f_1(\epsilon)$ in terms of I_e, P_e and $d\varphi/dx$. Substituting $f_1(\epsilon)$ in (A.1) we have

$$\begin{aligned}&\frac{f_M(\varepsilon)}{(p+1)!}\left\{\frac{dI_e}{dx}\left[(p+3)\left(\frac{\varepsilon}{kT}\right)^{p+1/2}-\left(\frac{\varepsilon}{kT}\right)^{p+3/2}\right]\right.\\&-\frac{1}{kT}\left(\frac{dP_e}{dx}-I_e\frac{d\varphi}{dx}\right)\left[\left(\frac{\varepsilon}{kT}\right)^{p+1/2}-\frac{1}{p+2}\left(\frac{\varepsilon}{kT}\right)^{p+3/2}\right]\\&+\frac{3[(p+2)I_e-P_e/kT]^2}{N_eL(kT)v(p+2)!}\left[-(p+3)(p+2)\left(\frac{\varepsilon}{kT}\right)^{p+1/2}\right.\\&+2(p+3)\left(\frac{\varepsilon}{kT}\right)^{p+3/2}-\left(\frac{\varepsilon}{kT}\right)^{p+5/2}\Big]+\frac{1}{kT}\frac{d\varphi}{dx}\left[I_e(p+3)-\frac{P_e}{kT}\right]\\&\left.\cdot\left[(p+1)\left(\frac{\varepsilon}{kT}\right)^{p-1/2}-2\left(\frac{\varepsilon}{kT}\right)^{p+1/2}+\frac{1}{p+2}\left(\frac{\varepsilon}{kT}\right)^{p+3/2}\right]\right\}=S(f_0(\varepsilon)).\end{aligned} \tag{A.4}$$

The first two terms in (A.4) are related to the ionization rate J, the loss of excited atoms by radiation G, and the balance of particles and electron energy:

$$\frac{dI_e}{dx}=J,$$

$$\frac{dP_e}{dx}-I_e\frac{d\varphi}{dx}=-JE_{ion}-GE_1. \tag{A.5}$$

Thus, two terms in (A.4) determine the spatial transport associated with the nonconservation of particle flux and energy flux associated with the ionization and radiation. These are always present and depend on the concentration and temperature at a given point. The remaining terms on the left in (A.4) depend on the proximity of the boundaries and external electric fields applied to the plasma, and will not be considered further.

The number Γ of fast electrons that appear at a given point by virtue of spatial transport is obtained by integrating the first two terms on the left side of (A.4) with respect to ϵ, using the limits E_1 and infinity. For integer p we have

$$\begin{aligned}\Gamma=-J\left[\sum_{i=0}^{p+1}\frac{1}{i!}\left(\frac{E_1}{kT}\right)^i-\frac{1}{(p+1)!}\left(\frac{E_1}{kT}\right)^{p+2}\right]\exp\left(-\frac{E_1}{kT}\right)\\+\left(J\frac{E_{ion}}{kT}+G\frac{E_1}{kT}\right)\frac{1}{(p+2)!}\left(\frac{E_1}{kT}\right)^{p+2}\exp\left(-\frac{E_1}{kT}\right).\end{aligned}$$

The quantity Γ can be neglected if it is small compared with J and G. In Fig. 2 we show the dependence of Γ/G on E_1/kT for p = 0 and p = 2 for small values of J. The case p = 0 corresponds to scattering from hard spheres and the case p = 2 approximates Coulomb scattering. It follows from Fig. 2, for example, that for Coulomb scattering with $E_1/kT < 10.4$ the ratio $\Gamma/G > 1$, that is to say, it is not possible to neglect the spatial inhomogeneities in considering the deviation from Maxwellian distributions due to radiation. A corresponding temperature for hydrogen, for which $E_1 = 10.2$ eV, is $T = 11{,}400^\circ$K while for cesium $T = 1540^\circ$K ($E_1 = 1.38$ eV). It should be noted that the spatial flow of fast electrons associated with ionization and radiation leads to enhanced loss of fast electrons, this being the mechanism that produces the deviation from a Maxwellian distribution. Hence, situations in which these flows must be taken into account do not eliminate the possibility that the distribution function can be a good approximation to a Maxwellian distribution; rather, they only indicate the possibility of a deviation.

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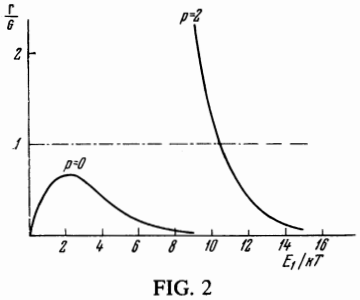

FIG. 2

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179