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CORRELATION FUNCTION OF A FINITE ISING MODEL

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We present a method of obtaining explicit expressions for the physical quantities in finite two-dimensional lattices of the Ising-model type, starting from the appropriate equations of infinite lattices. As an example we evaluate the correlations in a finite Ising lattice with periodic boundary conditions. The method described here enables us to expand the region of applicability of methods involving the direct summation of polygons.

KAUFMAN and Onsager[1] found the correlation function of the infinite lattice and later on this expression was obtained by other methods by Montroll, Potts, and Ward[2] and by Vdovichenko.[3] For a finite lattice we only know the expression for the partition function which was first obtained by Kaufman[4] and afterwards by Potts and Ward[5] and by Thompson.[6]

The aim of the present paper is to construct an expression for the correlation function of a finite lattice. We shall follow Vdovichenko's method.[3,7,8] Incidentally we shall show how we can obtain the partition function for the partition function of any finite lattice if the corresponding infinite lattice can be computed by the methods of[7-11]. For the simplest square lattice we shall write down an explicit analytical expression. Knowing the correlation function solves in principle the problem of the phase transition in a finite system.

However, the main fact which prompted us to solve the problem of a finite lattice is the necessity to make the methods described in[7-11] more precise. The partition function was in those papers written as a sum over loops. In the case of an infinite lattice this sum converges only above the transition point. The corresponding results are thus, strictly speaking, valid only above the transition point. Using the Kramers-Wannier[12] transformation (and similar transformations[11]) one can describe for many lattices also the region below the transition point. However, for more complicated lattices when the Kramers-Wannier transformation is not known, there remains one method: to evaluate the partition function of a finite lattice and to let the number of spins tend to infinity.

We perform the calculation using the example of the simplest square lattice containing L rows and M columns. Let the lattice be wound on a torus (periodic boundary conditions). If we try to use the methods of[7-10], in contrast the case of an infinite lattice, a finite contribution is made by loops wound round the torus and not containing a single winding in the plane tangential to the torus. These loops as well as the loops which are obtained from them by a continuous deformation occur in the partition function with an incorrect sign.

In this connection we remind ourselves of the main result of[7,8]. Let J be the energy of interaction of two spins, T the temperature, k Boltzmann's constant, $x = \tanh(J/kt)$, Z the partition function, A the matrix of the random walks on the lattice (with the Kac-Ward factor[9] $e^{i\varphi/2}$ for each winding), φ the angle of rotation of the tangential vector. Then

$$Z = [2^{LM} / (1-x^2)^{LM}]S, \quad S = \det{}^{1/2}(1-A). \tag{1}$$

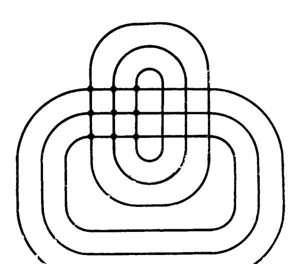
The figure given below illustrates the cause of the inapplicability of (1) to a finite lattice. First of all, the Kac-Ward factors[9] from windings on long (long in the figure) bonds (−1 from each bond) are not taken into account. Secondly, each loop occurs with a sign depending on the number of self-intersections[7,8] so that the intersections of long bonds give a superfluous factor $(-1)^s$, where s is the number of such fictitious intersections. We introduce the notation: S_{00} is the sum over the loops with an even number of long bonds along the x and the y axes, S_{10} the sum over loops with an odd number of long bonds along the x axis and with an even number of long bonds along the y axis, S_{01} the sum over loops with an even number of long bonds along the x axis and an odd number of long bonds along the y axis, and S_{11} the sum over loops with an odd number of long bonds along the x and the y axes.

Each configuration of loops is one of four possibilities and the required sum over all loops is thus

$$S = S_{00} + S_{10} + S_{01} + S_{11}. \tag{2}$$

We now consider $\det^{1/2}(1 - A)$ which according to[7,8] is equal to the sum over all loops with factors $e^{i\varphi/2}$ from each turn around an angle φ (when we go round the loop) and with a factor $(-1)^p$ where p is the number of loops. If the loops do not contain any long bonds then the factors $e^{i\varphi/2}$ give, due to the absence of self-intersections, exactly $(-1)^p$ and the corresponding terms in $\det^{1/2}(1 - A)$ will be the same as the analogous terms in S.

To retain this connection in the general case for configurations which contain n long bonds along the x axis and m long bonds along the y axis we must



multiply each configuration by a factor $(-1)^{n+m}$ (the Kac-Ward factor from winding along long bonds) and by a factor $(-1)^s$ where s is the number of intersections of long bonds (each intersection changes the sign, see [7,8]). Such intersections occur in a plane, but not on a torus so that we need not consider them. The additional factor $(-1)^s$ compensates their contribution.

The correct sign factor for each loop has thus the form $\exp\{i\Sigma\varphi/2\}(-1)^p(-1)^{n+m}(-1)^s$ while $\det^{1/2}(1-A)$ automatically takes into account only the first two factors. The above mentioned splitting into four sums allows us to take into account also the remaining two factors as the parity of s is the same as the parity of the product nm. Taking into account the parity of the quantity n + m + nm we get instead of (1)

$$S_{00}-S_{10}-S_{01}-S_{11}=\det{}^{1/2}(1-A). \tag{3}$$

We must note here that the derivation of (1) given in [7,8] assumes that the sum over the loops converges and is correct only for small x. However, in the case of a finite lattice the original sum over loops and the final expression for the partition function are polynomials in x of finite degree N. If these polynomials are the same for small x they will also be the same for all x. Therefore (3) is correct for all x although the calculation following [7,8] refers to small x only.

To evaluate the sum (2) which is of interest to us we must find all S_{ik}. We can obtain the necessary relations using the following method.

We assign to each step along the x axis in the positive direction a factor $e^{i\pi/L}$ and to each step in the opposite direction a factor $e^{-i\pi/L}$. In the general case this factor is equal to $e^{i\pi\Delta x/X}$ where Δx is the displacement along the x axis and X the dimension of the lattice along the x axis. After introducing this factor each configuration of loops will contain an additional factor $(-1)^l$ where l is the number of complete turns around the torus along the x-axis. The parity of the number l is clearly the same as the parity of the number of long bonds along the x-axis so that the introduction of this factor changes the sign of S_{10} and S_{11}, but not that of S_{00} and S_{01}.

The random walk matrix including the just introduced additional factors $e^{i\pi/L}$ from each step along the x-axis will be denoted by A_{10}. According to what we said earlier we have for A_{10} instead of (3)

$$S_{00}+S_{10}-S_{01}+S_{11}=\det{}^{1/2}(1-A_{10}). \tag{4}$$

If we introduce a similar factor along the y axis (the corresponding matrix will be denoted by A_{01}) we get for A_{01}

$$S_{00}-S_{10}+S_{01}+S_{11}=\det{}^{1/2}(1-A_{01}). \tag{5}$$

If, finally, we introduce such factors for both axes we get for A_{11}

$$S_{00}+S_{10}+S_{01}-S_{11}=\det{}^{1/2}(1-A_{11}). \tag{6}$$

From Eqs. (3) to (6) we find the S_{ik} and substitute into (2). The quantity S we are looking for turns out to be equal to ($A_{00}\equiv A$)

$$S={}^1/_2[-\det{}^{1/2}(1-A_{00})+\det{}^{1/2}(1-A_{10})+\det{}^{1/2}(1-A_{01})+\det{}^{1/2}(1-A_{11})]. \tag{7}$$

One can evaluate the determinants occurring on the right-hand side of (7) as in [7,8] by making a Fourier transform (changing from the x,y- to the p,q-representation). The additional factors $e^{i\pi/L}$ and $e^{-i\pi/L}$, in A_{10} for instance, occur as factors, respectively, in front of $e^{2\pi ip/L}$ and $e^{-2\pi ip/L}$ so that the introduction of these factors simply means replacing p by p + ½. Similarly we can evaluate $\det^{1/2}(1-A_{01})$ and $\det^{1/2}(1-A_{11})$. Bearing this in mind and taking the square roots in (7) we get for odd L = 2a + 1 and M = 2b + 1:

$$\begin{aligned}
\det{}^{1/2}(1-A_{00}) &= \prod_{p,q}\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}+\cos\frac{2\pi q}{M}\right)\right]^{1/2} \\
&= (1-2x-x^2)\prod_{p=1}^{a}\prod_{q=1}^{b}\left[(1+x^2)^2-2x(1-x^2)\left(1+\cos\frac{2\pi q}{M}\right)\right] \\
&\quad\times\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}+1\right)\right]\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}+\cos\frac{2\pi q}{M}\right)\right]^2, \\
\det{}^{1/2}(1-A_{10}) &= \prod_{p,q}\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}+\cos\frac{2\pi q}{M}\right)\right]^{1/2} \\
&= (1+x^2)\prod_{p=1}^{a}\prod_{q=1}^{b}\left[(1+x^2)^2-2x(1-x^2)\left(-1+\cos\frac{2\pi q}{M}\right)\right] \\
&\quad\times\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}+1\right)\right]\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}+\cos\frac{2\pi q}{M}\right)\right]^2, \\
\det{}^{1/2}(1-A_{01}) &= \prod_{p,q}\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}+\cos\frac{2\pi q+\pi}{M}\right)\right]^{1/2} \\
&= (1+x^2)\prod_{p=1}^{a}\prod_{q=1}^{b}\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}-1\right)\right]\left[(1+x^2)^2-2x(1-x^2)\right. \\
&\quad\left.\times\left(1+\cos\frac{2\pi q+\pi}{M}\right)\right]\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p}{L}+\cos\frac{2\pi q+\pi}{M}\right)\right]^2 \\
\det{}^{1/2}(1-A_{11}) &= \prod_{p,q}\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}+\cos\frac{2\pi q+\pi}{M}\right)\right]^{1/2} \\
&= (1+2x-x^2)\prod_{p=1}^{a}\prod_{q=1}^{b}\left[(1+x^2)^2-2x(1-x^2)\left(-1+\cos\frac{2\pi q+\pi}{M}\right)\right] \\
&\quad\times\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}-1\right)\right]\left[(1+x^2)^2-2x(1-x^2)\left(\cos\frac{2\pi p+\pi}{L}+\cos\frac{2\pi q+\pi}{M}\right)\right]^2.
\end{aligned} \tag{8}$$

For even L and odd M, odd L and even M, and even L and M we get similar expressions after taking the square root. Up to the point where the square root is taken Eqs. (8) are the same for all L and M.

The answer (7) must be completed by a rule for choosing the sign in front of the square roots in (3) to (6). For x = 0 all left-hand sides of (3) to (6) are equal to unity so that for small x all roots on the right-hand sides and also in (8) must be taken with the plus sign. From writing (8) in the form of polynomials (without square roots) and from the analycity of S in x it is clear that this choice of signs in front of the square roots remains unchanged for all x.

One can evaluate the partition function of different finite lattices in a similar way. In any case Fourier-transformation enables us to calculate $\det^{1/2}(1-A_{00})$ and the remaining three terms in (7) are obtained by the substitution $p \to p+\frac{1}{2}$ and $q \to q+\frac{1}{2}$.

We now turn to a calculation of the correlation. The correlation, like the partition function, is equal to the determinant of a matrix. For a finite lattice, as in (7), we get four determinants instead of one. For each of these determinants we must perform the same transformations as in[3]. In the simplest case of a square lattice we get for the correlation $G(t)$ of two spins in the same column at a distance t

$$G(t)=S^{-1}[-\det{}^{1/2}(1-A_{00})G_{00}(t)+\det{}^{1/2}(1-A_{10})G_{10}(t)+\det{}^{1/2}(1-A_{01})G_{01}(t)+\det{}^{1/2}(1-A_{11})G_{11}(t)], \quad (9)$$

where S is determined from Eqs. (7) and (8), $G_{rs}(t) = |C_{kl}^{rs}|$ ($r=0, 1$, $s=0, 1$) is the determinant of order t of the matrix elements C_{kl}^{rs}

$$C_{kl}^{rs}=\frac{1}{M}\sum_{q}f_{rs}(q)\exp\left\{\frac{2\pi iq(k-l)}{M}\right\},$$

$$f_{00}=\frac{1+y^L}{1-y^L}f(\omega),\quad f_{10}=\frac{1-y^L}{1+y^L}f(\omega),$$

$$f_{01}=\frac{1+z^L}{1-z^L}f'(\omega),\quad f_{11}=\frac{1-z^L}{1+z^L}f'(\omega),$$

$$y=\frac{1-g(\omega)}{1+g(\omega)},\quad z=\frac{1-g'(\omega)}{1+g'(\omega)},$$

$$f(\omega)=\left[\frac{(xx^*-e^{-i\omega})(x^*-xe^{i\omega})}{(xx^*-e^{i\omega})(x^*-xe^{-i\omega})}\right]^{1/2},$$

$$g(\omega)=\left[\frac{(x^*-xe^{i\omega})(x^*-xe^{-i\omega})}{(xx^*-e^{i\omega})(xx^*-e^{-i\omega})}\right]^{1/2},$$

$$f'(\omega)=f\left(\omega+\frac{\pi}{M}\right),\ g'(\omega)=g\left(\omega+\frac{\pi}{M}\right),\ \omega=\frac{2\pi q}{M},\quad x^*=\frac{1-x}{1+x}. \quad (10)$$

Or, in other words,

$$f=\exp\left\{i\arg\frac{x^*-xe^{i\omega}}{xx^*-e^{i\omega}}\right\},\quad g=\left|\frac{x^*-xe^{i\omega}}{xx^*-e^{i\omega}}\right|.$$

The square roots in (10) are taken with the plus sign. Writing out (10) shows how the usual expression[3] for the correlations is modified.

We can write Eq. (10) without square roots. The quantities f_{00} and f_{10} for odd $L=2a+1$ can be put in the form

$$f_{00}=\frac{u}{r}\frac{\sum_{s=0}^{a}C_L^{2s}r^s}{\sum_{s=0}^{a}C_L^{2s+1}r^s};\quad f_{10}=u\frac{\sum_{s=0}^{a}C_L^{2s+1}r^s}{\sum_{s=0}^{a}C_L^{2s}r^s};\quad u=\frac{x^*-xe^{i\omega}}{xx^*-e^{i\omega}};\quad r=|u|^2.$$

In the case of a finite lattice the square brackets in (9) as in (7) are polynomials of finite degree. The denominators, for instance, in C_{kl}^{00},

$$C_{kl}^{00}=\frac{1}{LM}\times\sum_{p,q}\frac{2x(1+x^2)-x^2(1-x^2)e^{2\pi iq/M}-(1-x^2)e^{-2\pi iq/M}}{(1+x^2)^2-2x(1-x^2)[\cos(2\pi p/L)+\cos(2\pi q/M)]}e^{2\pi iq(k-l)/M} \quad (11)$$

cancel the same factors in $\det^{1/2}(1-A_{00})$ in (9). The other C_{kl}^{rs} will be

$$C_{kl}^{rs}=\frac{1}{LM}\sum_{p,q}F_{rs}(p,q)e^{2\pi iq(k-l)/M}.$$

The quantities $F_{00}(p, q)$ are written down in (11) and

$$F_{10}(p,q)=F_{00}(p+\tfrac{1}{2},q);\quad F_{01}(p,q)=F_{00}(p,q+\tfrac{1}{2});$$
$$F_{11}(p,q)=F_{00}(p+\tfrac{1}{2},q+\tfrac{1}{2}).$$

If we let L and M in (7) to (10) tend to infinity we obtain the corresponding expressions for the infinite lattice. This transition is simple, but not trivial: as Kaufman[4] has already noted, the limiting behavior of the separate terms of the sum (7) below and above T_c turns out to be different. For large L the substitution $p \to p+\frac{1}{2}$ changes (8) only when $p\approx 0$ and $p\approx L$, but terms with these p occur in (8) as factors.

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