ENHANCEMENT OF A MAGNETIC FIELD BY A CONDUCTING FLUID

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A simple model of turbulent motion of a conducting liquid fluid is considered, in which the flow velocity has a Gaussian distribution function and the time for establishment of diffusion of the liquid particles is zero. In this case an exact solution of the problem of amplification of a spontaneous magnetic field can be derived. The instability criterion and magnetic field increment are obtained.

INTRODUCTION

WE discuss in the present article the question of the enhancement of a spontaneous magnetic field by a turbulent conducting liquid. The general picture of the interaction between the magnetic field and the conducting liquid was proposed by Batchelor[1], who started from the analogy with a velocity vortex in an incompressible liquid. The turbulent motion of the liquid influences the magnetic field in two ways: On the one hand, the matter stretches the force lines and increases the magnetic energy, and on the other hand the matter increases the diffusion coefficient[2,3] and by the same token it increases the rate of damping of the magnetic field. The latter effect is determined by the diffusion velocity of the liquid particles, and the former by the diffusion velocity of the liquid particles relative to one another (the runaway velocity). We consider below a certain artificial model, which, however, enables us to find the exact solution and in which it is possible to trace in detail the roles of the noted effects.

The magnetic field is described by the diffusion equation

$$\partial \mathbf{H} / \partial t = \text{rot}[\mathbf{uH}] + \lambda \Delta \mathbf{H}, \quad \text{div } \mathbf{H} = 0, \qquad (1)*$$

where $\mathbf{u}$ is the liquid flow velocity and λ is the magnetic diffusion coefficient. The turbulent motion of the liquid is assumed stationary, isotropic, and possessing sufficiently simple correlation properties. For the mean values of the Fourier components of the velocity

$$\mathbf{u}(\mathbf{r}, t) = \int d\mathbf{k} e^{i\mathbf{kr}} \mathbf{u}(\mathbf{k}, t), \qquad (2)$$

$$\mathbf{u}(\mathbf{k}, t) = (2\pi)^{-3} \int d\mathbf{r} e^{-i\mathbf{kr}} \mathbf{u}(\mathbf{r}, t)$$

we have the relations

$$\langle u_i(\mathbf{k}, t) \rangle = 0,$$
$$\langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', t') \rangle = u(k, |t - t'|) \sigma_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}'), \qquad (3)$$
$$\sigma_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2.$$

We assume that the mean value of the products of an even number of velocities breaks up into a sum of products of all possible pairs of mean values (Gaussian distribution). This enables us to use a diagram technique. The magnetic field at the initial instant of time $H_i^{(0)}(\mathbf{k})$ will also be assumed to be a random quantity uncorrelated with the velocity of the liquid, with

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*$[\mu \mathbf{H}] \equiv \mu \times \mathbf{H}$.

$$\langle H_i^{(0)}(\mathbf{k}) H_j^{(0)}(\mathbf{k}') \rangle = H_0(k) \sigma_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}'). \qquad (3')$$

The problem consists of determining the quantity

$$\langle\!\langle H_i(\mathbf{k}, t) H_j(\mathbf{k}', t) \rangle\!\rangle = H(k, t) \sigma_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}'), \qquad (4)$$

where the double angle brackets denote averaging over the velocity distributions of the liquid and of the magnetic field $\mathbf{H}^{(0)}$. For H(k, t), obviously, there is the boundary condition H(k, 0) = H₀(k).

In such a formulation, the problem is still very complicated. Therefore we shall consider first only a limiting case in which the characteristic time u(k, t) tends to zero, and approximate the time-dependent correlation function by a δ function

$$u(k, t) = v(k)\delta(t). \qquad (5)$$

In this approximation the problem can already be solved exactly.

The statistical properties of the liquid particles, having a velocity correlator in the form (5), turn out to be very simple. We deal essentially with a continuous generalization of the discrete model of random walks, in which each displacement does not depend on the preceding one. In fact, if we consider the particle trajectory $\mathbf{r}(t)$, then the displacement $\delta \mathbf{r}(t) = \mathbf{r}(t + \delta t) - \mathbf{r}(t)$ is not correlated with $\mathbf{r}(t)$:

$$\langle \mathbf{r}(t) \delta \mathbf{r}(t) \rangle = 0, \qquad (6)$$

so that $\mathbf{r}(t)$ is a random quantity with Gaussian distribution, and

$$\frac{d}{dt} \langle \mathbf{r}(t)^2 \rangle = 2v_0, \qquad v_0 = \int d\mathbf{k} v(k). \qquad (7)$$

The distance between two liquid particles $\boldsymbol{\rho}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)$ is also a random Gaussian quantity. For the rate of change of $\langle \boldsymbol{\rho}^2 \rangle$, by virtue of the relation

$$\langle \exp\{i\mathbf{k}\boldsymbol{\rho}(t)\} \rangle = \exp\{-\tfrac{1}{2} k^2 \langle \boldsymbol{\rho}^2 \rangle\}$$

we have the equation

$$\frac{d\langle \boldsymbol{\rho}^2 \rangle}{dt} = 4 \int d\mathbf{k} v(k) \left[1 - \exp\left\{ -\frac{1}{2} k^2 \langle \boldsymbol{\rho}^2 \rangle \right\} \right]. \qquad (8)$$

The runaway velocity of two close particles is proportional to the distance between them

$$\frac{d\langle \boldsymbol{\rho}^2 \rangle}{dt} = 12 v_2 \langle \boldsymbol{\rho}^2 \rangle, \quad v_2 = \frac{1}{3!} \int d\mathbf{k} \, k^2 v(k). \qquad (9)$$

At large distances, the particles move apart with a velocity governed by double the diffusion coefficient (7). Henceforth, the most important will be only the first two

moments of the distribution function $v(k)$, namely v_0 and v_2; they have the meaning, as can be readily seen, of the turbulence diffusion coefficient and of the reciprocal runaway time of the close liquid particles, respectively.

We now proceed to derive an equation for $H(k, t)$.

EQUATION FOR THE DISTRIBUTION FUNCTION OF THE MAGNETIC FIELD

We expand the magnetic field in a series in the powers of the velocity

$$H(k, t) = g_0(k, t) H^{(0)}(k) + \sum_{n=1}^{\infty} H^{(n)}(k, t). \quad (10)$$

We then get from (1), in the momentum representation, the recurrence relation

$$H^{(n)}(k, t) = i \int_c^t dt_1 \int dk_1 g_0(k, t - t_1)[k[u(k - k_1, t_1) H^{(n-1)}(k_1, t_1)]], \quad (11)$$

where $g_0(k, t) R \exp(-\lambda k^2 t)$ is the zero-order Green's function. Substituting (10) in (4), we obtain an aggregate of diagrams; some of them (of second order) are shown in Fig. 1.
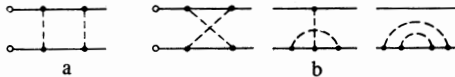

FIG. 1

The horizontal directions in the diagrams represent the time. The thin line with momentum k_1, joining the instants t_1 and $t_2 \le t_1$ (Fig. 1a) corresponds to $g_0(k_1, t_1 - t_2)$, and the integration with respect to t_2 is carried out from zero to t_1. The dashed line with momentum $p = k_1 - k_2$ corresponds to $v(p)\sigma_{j_2 j_2'}(p)$ and the point t_2 corresponds to $i(\delta_{i_1 j_2} k_{1 i_2} - \delta_{i_1 i_2} k_{1 j_2})$, the summation with respect to j_2 being carried out from 1 to 3. Finally, the circle at the beginning of the line denotes the initial magnetic field.

It is easy to see that diagrams with crossed dashed lines, such as shown in Fig. 1b, vanish. Indeed, instead of four integrals with respect to time, there remains in these diagrams, by virtue of the time ordering and of condition (5), only one integration with respect to t. But since there are only two δ functions, such diagrams vanish.
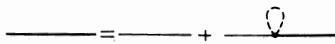

FIG. 2

Let us set up first the equation for the complete Green's function $g(k, t)\delta_{ij} + g_{||}(k, t)k_i k_j/k^2$ (Fig. 2). Separating the tensor indices, we get

$$\text{F} \quad g(k_1, t_1 - t_2) = g_0(k, t_1 - t_2) - g_{||}(k, t_1 - t_2), \quad (12)$$

$$g_{||}(k, t_1 - t_2) = \frac{1}{3} v_0 \int_{t_0}^{t_1} dt\, g_0(k, t_1 - t) g(k, t - t_2).$$

The solution of all these equations is elementary:

$$g(k, t) = \exp(-\bar{\lambda} k^2 t), \quad \bar{\lambda} = \lambda + \frac{1}{3} v_0. \quad (13)$$

Thus, the complete Green's function $g(k, t)$, which characterizes the rate of decay of the initial magnetic
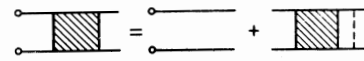

FIG. 3

field, is determined (apart from a number) by the summary diffusion coefficient—of the magnetic field and of the liquid particles.

We note that the purely longitudinal part of the Green's function $g_{||}$ does not enter in the subsequent equations. The equation for $H(k, t)$ is obtained by summing diagrams of the Ladder type (Fig. 3). Acting on both sides of the equation with the operator $(\partial/\partial t + 2\bar{\lambda} k^2)$, we get

$$(\dot{o}/\partial t + 2\bar{\lambda} k^2) H(k, t) = \int dq H(p, t) v(q) \left[k^2 - \frac{(kq)(kp)(qp)}{q^2 p^2} \right] \quad (p = k - q). \quad (14)$$

This is the sought equation for the distribution function.

Equation (14) can be considered also in r-space. If we introduce new correlations functions

$$\bar{H}(r, t) = r^{-3} \int_0^r d\rho\, \rho^2 H(\rho, t), \quad \bar{v}(r) = r^{-3} \int_0^r d\rho\, \rho^2 v(\rho),$$

$$H(r, t) = \int dk e^{ikr} H(k, t), \quad v(r) = \int dk e^{ikr} v(k), \quad (15)$$

then we can obtain from (14) the following equation for $\bar{H}(r, t)$:

$$\left(\frac{\partial}{\partial t} + 2\hat{\mathcal{L}} \right) \bar{H} = 0, \quad \hat{\mathcal{L}} = -\Lambda \frac{\partial^2}{\partial r^2} - \frac{\Lambda_1}{r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{dv}{dr},$$

$$\Lambda(r) = \bar{\lambda} - \bar{v}(r), \quad \Lambda_1(r) = \Lambda(r) + 3\bar{\lambda} - v(r). \quad (16)$$

The new diffusion coefficients $\Lambda(r)$ and $\Lambda_1(r)$ are positive for arbitrary $v(r)$, since the liquid-velocity correlator obviously satisfies the inequalities $v(r) \le v_0$ and $3\bar{v} \le v_0$.

The rest of the problem reduces essentially to a determination of the eigenfunctions and the eigenvalues of the operator $\hat{\mathcal{L}}$:

$$\hat{\mathcal{L}} \bar{H}_E = E \bar{H}_E, \quad (17)$$

since the general solution of (16) can be represented in the form

$$\bar{H}(r, t) = \sum_E \bar{H}_E(r) e^{-2Et}. \quad (18)$$

It is clear therefore that the magnetic field increases with time only if the operator $\mathcal{L}$ has negative eigenvalues (discrete spectrum).

After making the substitution

$$\bar{H}_E(r) = \exp\left(-\int_{r_0}^r \frac{d\rho \Lambda_1(\rho)}{2\rho \Lambda(\rho)} \right) h(r) \quad (19)$$

Eq. (16) takes on the form of a Schrödinger equation for a particle with effective mass $\Lambda^{-1}(r)$ in a potential $U(r)$:

$$\frac{d^2 h}{dr^2} + \Lambda^{-1}(r)[E - U(r)] h = 0, \quad (20)$$

$$U(r) = \frac{1}{r} \frac{dv}{dr} + \frac{1}{4r^2} \frac{\Lambda_1^2}{\Lambda} + \frac{1}{2} \Lambda \frac{d}{dr}\left(\frac{\Lambda_1}{r\Lambda} \right). \quad (21)$$

INSTABILITY CRITERION AND GROWTH INCREMENT OF THE MAGNETIC FIELD

Let us consider now the magnetic-field instability criterion. In this case this criterion coincides with the condition for the appearance of bound states in the field of the potential $U(r)$.

The asymptotic values of the potential U(r) and of the diffusion coefficients Λ and Λ_1 are:

$$\Lambda = \lambda, \quad \Lambda_1 = 4\lambda, \quad U = 2\lambda/r^2, \quad r \to 0; \quad (22)$$
$$\Lambda = \bar{\lambda}, \quad \Lambda_1 = 4\bar{\lambda}, \quad U = 2\bar{\lambda}/r^2, \quad r \to \infty. \quad (22')$$

At small distances corresponding to large **k**, the particle behaves like a "bare" particle, and at large distances the renormalized diffusion coefficient $\bar{\lambda}$ enters throughout. At both large and small distances the potential is repulsive. It is easy to see that at small values of v(r) the potential is positive also for intermediate values of r.

A sufficiently deep potential well in a certain region of r arises only under the condition

$$v_0 \gtrsim \lambda. \quad (23)$$

This is the criterion for the enhancement of the magnetic field: the turbulent diffusion coefficient must be sufficiently large compared with the magnetic-diffusion coefficient.

The exact threshold value of v_0, which is the instability limit, cannot be easily indicated. We shall therefore consider v_0 far from the instability boundary, i.e., the case $v_0 \gg \lambda$. In this case we can assume that the characteristic dimension r_0 of the function h(r) will be small compared with a—the characteristic length of v(r). It is then sufficient to substitute in (20) and (21) the expansion of v(r) near zero:

$$v(r) = v_0 - r^\alpha v_\alpha, \quad v_\alpha > 0, \quad 0 < \alpha \leqslant 2. \quad (24)$$

We do not assume beforehand that v(r) is analytic at zero. If the distribution function v(k) has a second moment, then v(r) is analytic at zero and $\alpha = 2$.

We introduce the following notation:

$$r = x[\lambda(3 + \alpha)/v_\alpha]^{1/\alpha}, \quad E = \varepsilon\lambda[v_\alpha/\lambda(3 + \alpha)]^{2/\alpha}. \quad (25)$$

Equations (20) and (21) then can be reduced to the form

$$\frac{d^2 h}{dx^2} + \Lambda^{-1}(x)(\varepsilon - \varphi(x))h = 0, \quad (26)$$

$$\Lambda(x) = 1 + x^\alpha, \quad \varphi(x) = x^{\alpha-2}\left(2 - \frac{3}{2}\alpha - \frac{1}{2}\alpha^2\right) + \frac{2}{x^2} - \frac{\alpha^2}{4}\frac{x^{2\alpha-2}}{1 + x^\alpha}. \quad (27)$$

If the characteristic dimension h(x) is of the order of unity, then $r_0/a \sim (\lambda/v_0)^{1/\alpha} \ll 1$, and the expansion (24) is justified.

When $x \to 0$ we have $\varphi(x) = 2/x^2$, and when $x \gg 1$ we have

$$\Lambda^{-1}(x)\varphi(x) = \frac{1}{x^2}\left(2 - \frac{3}{2}\alpha - \frac{3}{4}\alpha^2\right). \quad (28)$$

We see therefore that the effective potential is attracting when α is not too small. When $\alpha = 1$ the coefficient of $1/x^2$ in (28) is $-1/4$, corresponding, as is well known[4], to a critical value, starting with which the particle begins to "fall" towards the center. Therefore bound states should arise in the potential (28) when $\alpha > 1$. Here, however, ε may turn out to be very small, since the potential $\varepsilon\Lambda^{-1}(x) \sim \varepsilon x^{-\alpha}$ is repelling when $\varepsilon < 0$. In fact, numerical calculation shows that within the limits of the calculation accuracy the minimum value of ε turns out to be equal to zero up to $\alpha = 1.25$.

Of greatest interest in this problem is the determination of the maximum magnetic-field growth increment or
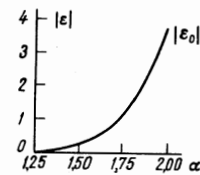

FIG. 4

of the ground state energy of the particle obeying Eq. (26), since it is precisely this state which will play the principal role in the expansion (18) in the case of large values of the time.

To find the energy of the ground state we can use a variational principle, which can be written for Eq. (23) in the form

$$\varepsilon = \int_0^\infty dx\, f\left[-\frac{d^2 f}{dx^2} + \Lambda^{-1}(x)\varphi(x)f\right]\Big/\int_0^\infty dx\, f^2\Lambda^{-1}(x), \quad \delta\varepsilon = 0. \quad (29)$$

We used as the trial function in (29) $f(x) = x^2 e^{-\beta x}$ (the function should decrease at zero like in the field of a centrifugal potential with unity moment), and ε was numerically minimized as a function of the parameter β. The results of the calculations for several values of α are shown in Fig. 4. When $\alpha < 2$, the radius of the bound state β^{-1} is of the order of unity. As α approaches 2, the value of β^{-1} increases. At $\alpha = 2$, the picture is complicated somewhat, and this case calls for a separate analysis.

As seen from (26), when $x \gg 1$ the particle is in the field of an effective potential $-(4 + \varepsilon)/x^2$, and its total energy is equal to zero. The wave function of such a state, obviously, does not decrease at large x and the assumption made above concerning the smallness of the radius of the bound state r_0 is no longer valid. In this case the exact value of ε depends on the details of the correlator v(r) at $r \sim a$.

We can indicate, however, a rather narrow region of values of ε, in which the energy of the ground state is concentrated. It is clear that we should have $\varepsilon > -4$. A bound state must certainly appear when $\varepsilon > \varepsilon_0 = -15/4$. In this case ε cannot greatly exceed the value of ε_0, since the wave function of the ground state should not have any nodes. Thus, the minimum ε lies in the range

$$-4 < \varepsilon < \varepsilon_0 + \delta, \quad \delta \sim 1/\ln(v_0/\lambda) \ll 1. \quad (30)$$

The fact that the characteristic dimension of the correlation potential H(r, t) becomes of the order of a when $\alpha = 2$ makes it difficult to apply the diffusion approximation to Eq. (14). In fact, if we assume formally that the wave packet v(k) is narrower than H(k, t), then, expanding the right side of (14) in powers of q, we obtain the diffusion equation in k-space

$$\frac{\partial H}{\partial t} + 2\lambda k^2 H = \frac{2}{5}v_2\left\{4H + k^2\frac{\partial^2 H}{\partial k^2} + 2k\frac{\partial H}{\partial k}\right\}. \quad (31)$$

This equation describes approximately the following picture: In the region of wave numbers $k < 2k_0 = \sqrt{4v_2/5\lambda}$, the magnetic field, in general, increases exponentially with time until the center of gravity of the wave packet H(k, t) shifts as a result of diffusion into the region $k > 2k_0$, after which the magnetic field attenuates.

For a "standing" packet $H(k, t) = e^{-2Et}h(k)/k$ we have the equation

$$\frac{d^2h}{dk^2} + \left(\frac{4+\varepsilon}{k^2} - k_0^{-2}\right)h = 0, \tag{32}$$

where ϵ is connected with E by (22). This again raises the question of the bound state (in momentum space) for a particle with total energy $-k_0^{-2}$ in the field of a potential $-(4 + \epsilon)/k^2$. The boundary conditions are not known for small k, just as was the case in the preceding problem for large r. We can therefore indicate only the interval of values in which ϵ lies and which coincides with the condition (30). The upper limit of this interval is also determined from the condition that h(k) have no nodes, since the distribution function h(k) must be positive.

For large values of ϵ, the function h(k) becomes of indefinite sign, which physically of course is inadmissible. This is apparently connected with the fact that the particle "falls" in this case into the region of small k, and Eq. (31) no longer holds.

Thus, the diffusion equation (31) is suitable only for minimal E and for not too small k. With the aid of this equation it is possible to obtain the distribution function of the magnetic field as $t \to \infty$:

$$H(k,t) = \frac{\mathrm{const}}{\sqrt{k}} K_\nu\left(\frac{k}{k_0}\right)\exp\left\{\frac{1}{5}|\varepsilon| v_2 t\right\}, \tag{33}$$

where $K_\nu(z)$ is the Macdonald function of order ν, $\nu = \sqrt{\epsilon_0 - \epsilon}$, and ϵ lies in the interval (30).

According to (9), the growth increment v_2 is the reciprocal runaway time of closely-lying liquid particles.

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