COULOMB COLLISIONS OF PARTICLES IN A TURBULENT PLASMA

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Submitted June 22, 1967

Zh. Eksp. Teor. Fiz. 53, 1789–1805 (November, 1967)

The effect of Coulomb collisions on the intensity and direction of the spectral shift in a turbulent plasma is considered. A mixed kinetic-hydrodynamic approach is developed which permits the study of those cases when the virtual (difference) waves produced in nonlinear scattering are in the region of frequent collisions. It is shown that in this case Coulomb collisions significantly affect the intensity as well as the direction of the spectral transfer.

INTRODUCTION

THE theory of nonlinear wave interaction in a turbulent plasma has been rapidly advancing lately (see [1-3]). The influence of the Coulomb collisions of the particles on the nonlinear interaction was not investigated in these references. The purpose of the present paper is to fill this gap.

It must be noted from the very beginning the Coulomb collisions in a turbulent plasma can be significant even if their influence is negligibly small in the linear approximation. The reason for it is that in nonlinear scattering there takes part a virtual wave whose frequency is the difference between frequencies of two interacting waves, and can be much smaller than the effective collision frequency ν_{eff}, whereas the frequency of each of the waves is much larger than ν_{eff}. For Langmuir oscillations, such a difference is particularly small in the case of large phase velocities:

$$\omega_- = \omega_1 - \omega_2 \approx \frac{3}{2}\omega_{0e}\left(\frac{v_{Te}}{v_{\text{ph}}}\right)^2, \quad v_{\text{ph}} = \frac{\omega_{0e}}{k}, \quad v_{Te} = \sqrt{\frac{T_e}{m_e}}.$$

At the same time, the process of nonlinear scattering leads to a decrease of ω_- by increasing v_{ph}. One can expect in this connection that, regardless of the initial spectrum of the oscillations, the spectral transfer will bring them into that region of wave numbers where the collisions are significant.[1)] This naturally raises the question whether the collisions can change the direction of the spectral transfer. As is well known,[4] in a turbulent liquid, the scale of the pulsations decreases, which is diametrically opposite to the situation that takes place in a collisionless turbulent plasma. It must be noted here, however, that in the nonlinear interaction in a plasma, which was described above, only a virtual wave falls into the region of the frequent collisions, whereas in a liquid all the interacting turbulent pulsations fall in that region. Therefore the question of the direction and intensity of the spectral transfer calls for a special investigation. An analysis performed by us, based on model collision integrals shows that the intensity and the direction of the transfer can change, but the result depends on the chosen model for the collision integral.[5] It was also noted in [5] that in the case of isotropic turbulence the collisions can change the dispersion properties of the interacting waves.

In the present paper we consider the problem of nonlinear interaction in a fully ionized plasma on the basis of the collision integral in the Landau form.[6] It should be noted that the obtained results can also be applied to the interaction of waves in a dense plasma, for example a solid-state plasma or a spark plasma produced in the focus of a laser.[7]

1. GENERAL RELATIONS

For weakly-damped waves, the nonlinear effects are determined, in the weak-turbulence approximation, by the components of the nonlinear current

$$j_i^{(2)}(k) = \int S_{ijl}(k, k_1, k_2) E_{k_1 j} E_{k_2 l}\, \delta(k - k_1 - k_2)\, dk_1\, dk_2, \tag{1.1}$$

$$j_i^{(3)}(k) = \int \Sigma_{ijls}(k, k_1, k_2, k_3) E_{k_1 j} E_{k_2 l} E_{k_3 s}\, \delta(k - k_1 - k_2 - k_3)\, dk_1\, dk_2\, dk_3.$$

We shall assume, without loss of generality, that the functions S_{ijl} and Σ_{ijls} satisfy the following symmetry condition:

$$S_{ijl}(k, k_1, k_2) = S_{ilj}(k, k_2, k_1),$$
$$\Sigma_{ijls}(k, k_1, k_2, k_3) = \Sigma_{ijsl}(k, k_1, k_3, k_2). \tag{1.2}$$

Maxwell's equations with allowance for (1.1) and also with allowance for the ordinary linear current lead, after averaging over the statistical ensemble of the turbulent pulsations, to a nonlinear equation for the squares of the amplitudes of the fields and of the longitudinal waves.[2)] Defining

$$\langle E_i^0(k_1) E_j^0(k_2)\rangle = |E_{k_1}|^2 \delta(k_1 + k_2) \frac{k_{1i} k_{1j}}{k_1^2},$$

(here E^0 are the first-approximation fields), we write the aforementioned equation in the form

$$\varepsilon(k)|E_k|^2 = |E_k|^2 \int \alpha_{kk_1} |E_{k_1}|^2 dk_1 + \int \beta_{kk_1k_2} |E_{k_1}|^2 |E_{k_2}|^2 \delta(k - k_1 - k_2)\, dk_1\, dk_2, \tag{1.3}$$

where

$$\alpha_{kk_1} = \frac{8\pi i}{\omega}\left[\Sigma(k, k_1, k, -k_1) - \frac{4\pi i}{\omega_- \varepsilon(k_-)} S(k_-, k, -k_1) S(k, k_1, k_-)\right]. \tag{1.4}$$

[1)]However, as shown by the subsequent analysis, in a number of cases there arises, besides the criterion $\omega_- \ll \nu_{\text{eff}}$, also the criterion $k_- v_T < \nu_{\text{eff}}$, i.e., $\nu_{\text{eff}} > \omega_{0e}(v_{Te}/v_{\text{ph}})$. It is of importance in what follows that both inequalities begin to be satified for large values of v_{hp}.

[2)]We neglect here effects of scattering via a virtual transverse wave, since usually they are important for a plasma of almost relativistic temperature[2].

Neglecting the collisions, α_{kk_1} describes the effects of induced decays and induced scattering, and $\beta_{kk_1k_2}$ describes effects of spontaneous decays. When collisions are taken into account, such a subdivision is not strictly valid. Equation (1.3) assumes the simplest form in the case when the decays are forbidden by the conservation laws, for example for Langmuir waves. Then Eq. (1.3) has the form of a nonlinear dispersion equation. The corrections to the frequency of the Langmuir waves are small by virtue of the weak nonlinearity, and therefore

$$\varepsilon(\mathbf{k},\omega)=\varepsilon(\mathbf{k},\omega_{\mathbf{k}}+\omega')\approx\omega'\frac{\partial\varepsilon}{\partial\omega}\Big|_{\omega=\omega_{\mathbf{k}}}, \qquad \left|\frac{\omega'}{\omega_{\mathbf{k}}}\right|\ll 1.$$

The real and imaginary parts of ω' determine respectively the dispersion properties and the intensity of the spectral transfer of the interacting waves. We note here that, for sufficiently large $v_{ph} \gg v_{Te}\sqrt{m_i/m_e}$, the dispersion of the waves is determined by the first term of (1.4), and the contribution of the second term is small as a result of the large values of ε. As to the imaginary part of ω', the most effective, without allowance for collisions, is scattering by ions, determined by the second term of (1.4), which describes nonlinear scattering, and $1/\varepsilon$, which enters in (1.4), describes the virtual longitudinal waves referred to above. It is easy to show that in the absence of collisions the second term of (1.4) can be written in the form

$$\sim S_1\frac{\operatorname{Im}\varepsilon_i}{|\varepsilon|^2}S_2. \tag{1.5}$$

It follows therefore that since $\operatorname{Im}\varepsilon_i \sim \delta(\omega_- - \mathbf{k}\cdot\mathbf{v}_i)$, Eq. (1.4) actually describes scattering by ions. It should be noted that the contribution of the ions to $S_{1,2}$ and Σ is negligibly small, since these functions contain in the denominator the ion mass raised to a large power. Allowance for the collisions modifies the picture as follows: The ion–ion collisions make a contribution to ε_1 and $|\varepsilon|^2$, and the electron–ion and electron–electron collisions change the functions Σ and S.

2. GENERAL EXPRESSIONS FOR NONLINEAR PLASMA CURRENTS

1. Since the frequency and wave vector of only the virtual wave fall into the region of the frequent collisions, it is necessary to use a new kinetic-hydrodynamic approach to determine the nonlinear polarizabilities $S(k_-, k_1, -k_2)$, $S(k_1, k_2, k_-)$, $\Sigma(k_1, k_2, k_1, -k_2)$, and $\Sigma(k_1, k_2, -k_2, k_1)$, ($k_- = k_1 - k_2$). If all the frequencies are in the frequent-collision region, then we can use the well known hydrodynamic equations (see [8]) to determine the components of the nonlinear currents. The case when the frequencies of all the waves are larger than the effective collision frequency has by now been thoroughly studied (see [1,2]). However, neither method is suitable for our purposes. We develop here a kinetic-hydrodynamic approach which makes it possible to calculate the components of the nonlinear currents in the case when

$$|\omega_- - k_- v_{T\alpha}| \ll \nu_{\text{eff}} \ll |\omega_{1,2} - k_{1,2}v_{T\alpha}|, \quad \alpha = e, i. \tag{2.1}$$

Let us illustrate this method by using as an example the calculation of the nonlinear polarizability $S(k_-, k_1, -k_2)$. We expand the distribution function in powers of the electric field: $f = f_0 + f^{(1)} + f^{(2)} + f^{(3)} + \cdots$. The kinetic equation for the Fourier components $f_k^{(i)}$ with allowance for the collision integral is of the form

$$-i(\omega-\mathbf{kv})f_{\alpha k}^{(1)}+\frac{e_\alpha}{m_\alpha}E_k\frac{(\mathbf{k}\partial f_0/\partial\mathbf{v})}{k}=I_{\alpha k}(1,0)+I_{\alpha k}(0,1), \tag{2.2}$$

$$-i(\omega-\mathbf{kv})f_{\alpha k}^{(2)}+\frac{e_\alpha}{m_\alpha}\int E_{k_1}\left(\frac{\mathbf{k}_1}{k_1}\frac{\partial f_{\alpha k_2}^{(1)}}{\partial\mathbf{v}}\right)\delta(k-k_1-k_2)\,dk_1\,dk_2$$
$$=I_{\alpha k}(2,0)+I_{\alpha k}(1,1)+I_{\alpha k}(0,2), \tag{2.3}$$

$$-i(\omega-\mathbf{kv})f_{\alpha k}^{(3)}=-\frac{e_\alpha}{m_\alpha}\int E_{k_1}\left(\frac{\mathbf{k}_1}{k_1}\frac{\partial f_{\alpha k_2}^{(2)}}{\partial\mathbf{v}}\right)\delta(k-k_1-k_2)\,dk_1\,dk_2$$
$$+I_{\alpha k}(3,0)+I_{\alpha k}(2,1)+I_{\alpha k}(1,2)+I_{\alpha k}(0,3). \tag{2.4}$$

Here

$$I_\alpha=\sum_{\alpha'}I_{\alpha\alpha'}, \quad \alpha=e,i,$$

and $I_{\alpha\alpha'}$ is taken in the Landau form:[6]

$$I_{\alpha\alpha'}=-\frac{2\pi Le^4}{m_\alpha}\frac{\partial}{\partial v_i}\int\left\{\frac{f_\alpha(\mathbf{v})}{m_{\alpha'}}\frac{\partial f_{\alpha'}(\mathbf{v}')}{\partial v_j'}-\frac{f_{\alpha'}(\mathbf{v}')}{m_\alpha}\frac{\partial f_\alpha(\mathbf{v})}{\partial v_j}\right\}U_{ij}d\mathbf{v}, \tag{2.5}$$

$$U_{ij}=\frac{1}{w^3}(w^2\delta_{ij}-w_iw_j), \quad \mathbf{w}=\mathbf{v}-\mathbf{v}'; \tag{2.6}$$

L is the Coulomb logarithm; I(m, n) denotes that it is necessary to take in the collision integral $f^{(m)}(v)$ in lieu of f(v) and $f^{(n)}(v')$ in lieu of f(v′); $I_k(m, n)$ is the Fourier component of the function I(m, n).

In the case when ω coincides with the frequency of the turbulent oscillations, the collision integral can be accounted for by ordinary perturbation theory. The equation in which ω is equal to the difference in the frequencies of the turbulent pulsations must be solved by a method similar to that of Enskog.[9] Equation (2.2) for the determination of $S(k_-, k_1, -k_2)$ has in first approximation the solution

$$f_{ek}^{(1)}=-\frac{ieE_k}{m_e(\omega-\mathbf{kv})}\left(\mathbf{k}\frac{\partial f_{0e}}{\partial\mathbf{v}}\right). \tag{2.7}$$

By substituting this expression into the right side of (2.2), we take into account the corrections of order $\nu/\omega \ll 1$.[10] Allowance for these corrections is essential, since according to (1.4) the nonlinear interaction is determined by the symmetrical combination $S(k_-, k_1, -k_2) + S(k_-, -k_2, k_1)$, in which the contribution of (2.7) is of relative order ω_-/ω_{0e} and the contribution of the corrections is of order $\nu_{\text{eff}}/\omega_{0e} \gg \omega_-/\omega_{0e}$. Integrating (2.7) we obtain, with allowance for the corrections terms, the following expression for the first-order electron current:

$$j_k^{(1)}=E_k\frac{e^2n_0i}{m_e\omega}\left(1-i\frac{\nu_e}{\omega}\right)\equiv en_0v_k^{(1)},$$

$$\nu_e=\frac{4}{3}\sqrt{2\pi}\frac{Ln_0e^4}{m_e^2v_{Te}^3}, \tag{2.8}$$

L is the Coulomb logarithm.

The solution of (2.3) should cause the collision integral to vanish in first approximation. We separate in the collision integral the largest terms, and move the remainder to the left side of (2.3), which we shall take into account by perturbation theory. The main terms in the collision integral are

$$I_{ee}(0,2)+I_{ee}(2,0)+I_{ei}(2,0).$$

Neglecting terms of order $(m_e/m_i) \ll 1$ we get

$$I_{kei}(2,0)=\frac{2\pi n_0 e^4}{m_e^2}\frac{\partial}{\partial v_i}U_{ij}(\mathbf{v})\frac{\partial f_k^{(2)}}{\partial v_j},\qquad(2.9)$$

and $I_{kei}(0,2)$ is negligibly small because it contains $f_{ik}^{(2)}\sim 1/m_i^2$. It is convenient to separate from (2.9) the small term

$$\delta I_{kei}(2,0)=\frac{4\pi Le^4 f_0(\mathbf{v})}{m_e^2 v^2_{Te} v^3}\int(\mathbf{v}\mathbf{v}')f_{ek}^{(2)}(\mathbf{v}')d\mathbf{v}'\qquad(2.10)$$

and transfer it to the left side.[3] Then the zeroth-approximation equation takes the form

$$I_{kee}(2,0)+I_{kee}(0,2)+I_{kei}(2,0)-\delta I_{kei}(2,0)=0.\qquad(2.11)$$

It is easy to verify by simple substitution that Eq. (2.11) is satisfied by the function

$$f_{ek}^{(2)0}=f_{00}\left\{\frac{n_k^{(2)}}{n_0}+\frac{\mathbf{v}\mathbf{V}_k^{(2)}}{v^2_{Te}}-\frac{3}{2}\left(1-\frac{v^2}{3v^2_{Te}}\right)\frac{T_k^{(2)}}{T_e}\right\},\qquad(2.12)$$

where

$$n_k^{(2)}=\int f_k^{(2)0}d\mathbf{v},\quad \mathbf{V}_k^{(2)}=\frac{1}{n_0}\int \mathbf{v}f_k^{(2)0}d\mathbf{v},$$
$$T_k^{(2)}=\frac{1}{3n_0}\int\frac{m_0v^2}{2}f_k^{(2)0}d\mathbf{v}-\frac{n_e^{(2)}}{n_0}T_e,\quad f_{00}=\frac{n_0}{v^3_{Te}(2\pi)^{3/2}}\exp\left(-\frac{v^2}{2v^2_{Te}}\right);\qquad(2.13)$$

n_0 and T_e are the unperturbed density and temperature of the plasma. The function (2.12) is the first term of the expansion of the difference of two Maxwellian functions

$$f_e^{(2)0}=n\left(\frac{m_e}{2\pi T}\right)^{3/2}\exp\left(-\frac{(\mathbf{v}-\mathbf{V})^2}{2T}m_e\right)-n_0\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m_ev^2}{2T_e}\right),\qquad(2.14)$$

$$n=\int(f_{00}+f^{(2)0})d\mathbf{v},\quad \mathbf{V}=\int(f_{00}+f^{(2)0})\mathbf{v}d\mathbf{v},$$
$$T=\frac{1}{3}\int(f_{00}+f^{(2)0})\frac{m_ev^2}{2}d\mathbf{v}.\qquad(2.15)$$

Since (2.12) does not contain the moments of the first-order distribution function,[4] the system of equations obtained with allowance for the small left-hand side of (2.3) differs greatly from the hydrodynamic equations. Integrating (2.3), we obtain the equations for the moments of the function $f_k^{(2)}$ (which coincide, just as in the Enskog method,[9] with the moments of the function $f_k^{(2)0}$):

$$-i\omega n_k^{(2)}=-n_0(\mathbf{k}\mathbf{V}_k^{(2)}),\qquad(2.16)$$

$$-m_en_0i\omega V_{k\alpha}^{(2)}+ik_\alpha(n_0T_k^{(2)}+T_en_k^{(2)})+ik_\beta\pi_{\alpha\beta,k}^{(2)}$$
$$-e\int E_{k_1}\frac{k_{1\alpha}n_{k_2}^{(1)}}{k_1}\delta(k-k_1-k_2)=R_{k,\alpha},\qquad(2.17)$$

$$-{}^3/_2n_0i\omega T_k^{(2)}+n_0T_ei(\mathbf{k}\mathbf{V}_k^{(2)})+i(\mathbf{k}\mathbf{q}_{ek})$$
$$-en_0\int E_{k_1}\frac{\mathbf{k}_1\mathbf{V}_{k_2}^{(1)}}{k_1}\delta(k-k_1-k_2)dk_1\,dk_2=0,\qquad(2.18)$$

where

$$\mathbf{R}_k=\int m_e\mathbf{v}I_{eik}(2,0)d\mathbf{v}\qquad(2.19)$$

is the analog of the friction force and the thermal force in ordinary hydrodynamics,

$$\mathbf{q}_k=\int\frac{m_ev^2}{2}\mathbf{v}f_{ek}^{(2)}d\mathbf{v}\qquad(2.20)$$

is the analog of the electronic heat flow due to the collisions, and

$$k_\beta\pi_{\alpha\beta k}^{(2)}=m_e\int\left[v_\alpha(\mathbf{k}\mathbf{v})-\frac{1}{3}k_\alpha v^2\right]f_{ek}^{(2)}d\mathbf{v}\qquad(2.21)$$

is the analog of the electronic viscosity.

Usually the left side of (2.18) contains besides q and k also a quantity

$$Q=\int\frac{m_ev^2}{2}I_{ei}(2)d\mathbf{v},$$

which represents the heat released by the electrons as a result of their collisions with the ions. Accurate to terms of first order in $f_{ek}^{(2)}$, we have $Q=-3(m_e/m_i)\times\nu_eT_k^{(2)}n_0$

In order to close the system of equations (2.16)–(2.18), we need to express $f_k^{(2)}$ in (2.20) and (2.21) in terms of the sought quantities $n_k^{(2)}$, $V_k^{(2)}$, and $T_k^{(2)}$. Let us examine the complete equation for $f_k^{(2)}$ with allowance for the small terms transferred to the left side of (2.3). Then we put in the left side, approximately, $f_k^{(2)}=f_k^{(2)0}$ in accordance with (2.12), and in the right side we take into account the small correction to $f_k^{(2)0}$, which is conveniently represented in the form $f_k^{(2)}=f^{(2)0}+f_{00}\Phi$ ($f_{00}\Phi\ll f_k^{(2)0}$). As a result, the left side of the sought equation becomes equal to

$$f_{00}\left\{\left(\frac{v^2}{3v^2_{Te}}-\frac{5}{2}\right)\frac{T_k^{(2)}}{T_e}i(\mathbf{k}\mathbf{v})+\left(3\sqrt{\frac{\pi}{2}}\frac{v_{Te}^3}{v^3}-1\right)\nu_e\frac{\mathbf{V}_k^{(2)}\mathbf{v}}{v^2_{Te}}\right.$$
$$\left.+\frac{1}{n_0T_e}(\mathbf{R}'\mathbf{v})+\frac{U_{ij}}{2v^2_{Te}}(W_{ij}-A_{ij}{}^0)-\frac{3}{4}\frac{\nu_e}{\omega}A_{ij}'\left(a\delta_{ij}+b\frac{v_iv_j}{v^2_{Te}}\right)\right\},\qquad(2.22)$$

$$A_{ij}=\frac{ie^2}{m_e^2v^2_{Te}}\int E_{k_1}E_{k_2}\frac{k_{1i}k_{2j}}{k_1k_2}\frac{1}{\omega_2}\left(1-i\frac{\nu_{\mathrm{eff}}}{\omega_2}\right)\delta(k-k_1-k_2)dk_1\,dk_2,\qquad(2.23)$$

$$A_{ij}^0=A_{ij}+A_{ji}-{}^2/_3\delta_{ij}A_{ii},\qquad(2.24)$$

$$W_{ij}=ik_iV_{kj}^{(2)}+ik_jV_{ki}^{(2)}-{}^2/_3i(\mathbf{k}V_k^{(2)})\delta^i{}_j,\qquad(2.25)$$

$$A_{ij}'=\frac{ie^2}{m_e^2v^2_{Te}}\int E_{k_1}E_{k_2}\frac{k_{1i}k_{2j}}{k_1k_2}\frac{\omega_{0e}}{\omega_2^2}\delta(k-k_1-k_2)dk_1\,dk_2.\qquad(2.26)$$

The last term of (2.22) is obtained from $I_{ee}(1,1)$, and a and b in this term are complicated functions of $y=(v/\sqrt{2}\,v_{Te})$, which are of the order of unity when $y\sim1$. The expressions for a and b are not presented here since, by virtue of the fact that $a\sim b\sim1$, the entire terms in question is of the relative order

$$\frac{1}{\nu_e}\max\left(\omega_-,\frac{k^2_-v_{Te}^2}{\nu_e}\right)\ll1$$

(see below) and is neglected.

By virtue of the symmetrization of A_{ij} with respect to k_2 and k_2, the first term of (2.23) is small compared with the second ($\sim\omega_-/\nu_{\mathrm{eff}}$), which is of the order $(\nu_e/\omega_{0e})A_{ij}'$ and can be discarded under the same assumptions ($\nu_{\mathrm{eff}}(y\sim1)\sim\nu_e$). We note that the terms containing A_{ij} and A_{ij}' have the character of corrections to the tensor W_{ij} of the rates of displacements connected with the presence of the fields E_{k_1} and E_{k_2}. In this approximation, both the left and the right sides

[3]The relative order of (2.10) is (ω_-/kv_{Te}), as can be verified by using the results of a solution of (2.3).

[4]It is easy to show that if $f=f_{00}+f^{(1)}+f^{(2)}$ in (2.15), then (2.14) does not satisfy (2.11). The more general statement can also be made. Any function $f^{(2)0}$ containing moments of first order order cannot satisfy Eq. (2.11).

of (2.3) assume the standard form which was used, for example, by Braginskiĭ.[8] Using the results of that paper, we can write in lieu of (2.19)–(2.21):

$$\mathbf{R}_k = -m_e n_0 \nu_e \cdot 0.51 \mathbf{V}_k^{(2)} - 0.71 i n_0 \mathbf{k} T_k^{(2)}, \quad (2.27)$$

$$\mathbf{q}_k = 0.71 n_0 T_e \mathbf{V}_k^{(2)} - 3.16 i\mathbf{k} T_k^{(2)} \frac{n_0 T_e}{m_e \nu_e}, \quad (2.28)$$

$$k_\beta \pi_{\alpha\beta}^{(2)} = -0.73 \frac{n_0 T_e}{\nu_e} k_\beta W_{\alpha\beta}. \quad (2.29)$$

Using (2.27)–(2.29) as well as (2.8) in the system (2.16)–(2.19), we obtain the sought-for longitudinal current

$$j_{k_-}^{(2)} = e n_0 \frac{\mathbf{V}_{k_-}^{(2)} \mathbf{k}_-}{|k_-|}$$

$$= -\frac{1.71 i n_0 e^3 \nu_e}{m_e^2 \omega_{0e}^2} \int E_{k_1} E_{k_2} |k_-| \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1 k_2} \frac{\delta(k_- - k_1 - k_2)}{\Omega \Omega_e} dk_1\, dk_2. \quad (2.30)$$

This result has been written out with accuracy $\nu_e^{-1} \max(\omega_-, k_-^2 v_{Te}^2/\nu_e)$,

$$\Omega = -i\omega_- + 0.51\nu_e + i \frac{k_-^2 v_{Te}^2}{\omega_-}\left(1 - 2.96 \frac{i\omega_-}{\Omega_e}\right), \quad (2.31)$$

$$\Omega_e = -\frac{3}{2} i\omega_- + 3.16 \frac{k_-^2 v_{Te}^2}{\nu_e}. \quad (2.32)$$

To estimate the order of the discarded terms, it is sufficient to calculate, say, $A^0_{\alpha\beta}$, since the remaining terms are of the same order. By virtue of the fact that the tensor properties of $A^0_{\alpha\beta}$ coincide with the tensor properties of $W_{\alpha\beta}$, allowance for $A^0_{\alpha\beta}$ leads only to the fact that (2.29) will contain $W_{\alpha\beta} - A^0_{\alpha\beta}$ in lieu of $W_{\alpha\beta}$. From this we get in lieu of (2.30)

$$j_{k_-}^{(2)} = \int E_{k_1} E_{k_2} \delta(k_- - k_1 - k_2)\, dk_1\, dk_2 \frac{n_0 |k_-| e^3}{m_e^2 \omega_{0e}^2 \Omega}$$

$$\times \left\{\left(1.71 \frac{\nu_e}{\Omega_e} - 0.49\right) \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1 k_2} + 1.46 \frac{(\mathbf{k}_- \mathbf{k}_1)(\mathbf{k}_- \mathbf{k}_2)}{k_1 k_2 k_-^2}\right\}. \quad (2.33)$$

It follows from (2.33) that the result (2.30) is valid if the following inequalities are taken into account

$$\left| -\frac{3}{2} i\omega_- + 3.16 \frac{k_-^2 v_{Te}^2}{\nu_e} \right| \ll \nu_e. \quad (2.34)$$

From (2.30) follows the sought-for expression for $S(\mathbf{k}_-, \mathbf{k}_1, -\mathbf{k}_2)$:

$$S(k_-, k_1, -k_2) = i \frac{|k_-| n_0 e^3 \cdot 1.71 \nu_e}{m_e^2 \omega_{0e}^2 \Omega \Omega_e} \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1 k_2}. \quad (2.35)$$

2. Let us proceed to find $S(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_-)$. In this case, the collision integral is decisive in (2.2). Using Enskog's method in lieu of (2)2), and putting $k = k_-$, we obtain a system of hydrodynamic equations, which have in the Fourier representation the form

$$-i\omega_- n_{ek_-}^{(1)} + i\mathbf{k}_- \mathbf{V}_{k_-}^{(1)e} n_0 = 0, \quad -i\omega_- n_{ik_-}^{(1)} + i\mathbf{k}_- \mathbf{V}_{k_-}^{(1)i} n_0 = 0,$$

$$-i\omega_- m_e n_0 V_{k_-\alpha}^{(1)e} = -ik_{-\alpha}(n_0 T_{ek_-}^{(1)} + T_e n_{ek_-}^{(1)}) - ik_{-\beta} \pi_{\alpha\beta}^{(1)e}(k_-)$$

$$+ e n_0 E_{k_-} \frac{k_{-\alpha}}{|k_-|} + R_{k_-\alpha}^{(1)},$$

$$-i\omega_- m_i n_0 V_{k_-\alpha}^{(1)i} = -ik_{-\alpha}(n_0 T_{ik_-}^{(1)} + T_i n_{ik_-}^{(1)}) - ik_{-\beta} \pi_{\alpha\beta}^{(1)i}(k_-)$$

$$- e n_0 E_{k_-} \frac{k_{-\alpha}}{|k_-|} - R_{k_-\alpha}^{(1)},$$

$$-{}^3/_2\, i\omega_- n_0 T_{ek_-}^{(1)} + n_0 T_e i(\mathbf{k}_- \mathbf{V}_{k_-}^{(1)e}) = -i\mathbf{k}_- \mathbf{q}_{k_-}^{(1)e},$$

$$-{}^3/_2\, i\omega_- n_0 T_{ik_-}^{(1)} + n_0 T_i i(\mathbf{k}_- \mathbf{V}_{k_-}^{(1)i}) = -i\mathbf{k}_- \mathbf{q}_{k_-}^{(1)i}, \quad (2.36)$$

where

$$\mathbf{R}_{k_-}^{(1)} = -0.51 m_e n_0 \nu_e (\mathbf{V}_{k_-}^{(1)e} - \mathbf{V}_{k_-}^{(1)i}) - 0.71 n_0 i\mathbf{k}_- T_{ek_-}^{(1)},$$

$$\mathbf{q}_{k_-}^{(1)e} = 0.71 n_0 T_e (\mathbf{V}_{k_-}^{(1)e} - \mathbf{V}_{k_-}^{(1)i}) - 3.16 \frac{n_0 T_e}{m_e \nu_e} i\mathbf{k}_- T_{ek_-}^{(1)},$$

$$\mathbf{q}_{k_-}^{(1)i} = -3.9 \cdot \frac{n_0 T_i}{m_i \nu_i} i\mathbf{k}_- T_{ik_-}^{(1)},$$

$$\pi_{\alpha\beta}^{(1)e} = -0.73 \cdot \frac{n_0 T_e}{\nu_e}\left(ik_{-\alpha} V_{k_-\beta}^{(1)e} + ik_{-\beta} V_{k_-\alpha}^{(1)e} - \frac{2}{3} i\mathbf{k}_- \mathbf{V}_{k_-}^{(1)e} \delta_{\alpha\beta}\right),$$

$$\pi_{\alpha\beta}^{(1)i} = -0.96 \frac{n_0 T_i}{\nu_i} i\left(k_{-\alpha} V_{k_-\beta}^{(1)i} + k_{-\beta} V_{k_-\alpha}^{(1)i} - \frac{2}{3} \mathbf{k}_- \mathbf{V}_{k_-}^{(1)i} \delta_{\alpha\beta}\right). \quad (2.37)$$

In order for Eqs. (2.36) with constant T_e and T_i to be valid, it is necessary either that the frequency ω_- be larger than the reciprocal temperature relaxation time $(m_e/m_i)\nu_e$, or that the plasma be isothermal, $T_e = T_i$.

Solving the system (2.36) and (2.37), we obtain

$$V_{k_-}^{(1)e} = \frac{eE_{k_-}}{m_e \varkappa \omega_e}, \quad V_{k_-}^{(1)i} = -\frac{eE_{k_-}}{m_i \varkappa \omega_i}, \quad V_{k_-}^{(1)\alpha} = \frac{\mathbf{k}_- \mathbf{V}_{k_-}^{(1)\alpha}}{|\mathbf{k}_-|}, \quad (2.38)$$

where

$$\varkappa = 1 + \left(0.51\nu_e + 1.22 \frac{k_-^2 v_{Te}^2}{\Omega_e}\right)\left(\frac{1}{\omega_e} + \frac{m_e}{m_i} \frac{1}{\omega_i}\right), \quad (2.39)$$

$$\omega_e = -i\omega_- + i \frac{k_-^2 v_{Te}^2}{\omega_-}\left(1 - 1.71 \frac{i\omega_-}{\Omega_e}\right), \quad (2.40)$$

$$\omega_i = -i\omega_- + i \frac{k_-^2 v_{Ti}^2}{\omega_-}\left(1 - \frac{i\omega_-}{\Omega_i} - 1.28 \frac{i\omega_-}{\nu_i} + 0.71 i \frac{T_e}{T_i} \frac{\omega_-}{\Omega_e}\right), \quad (2.41)$$

$$\Omega_i = -\frac{3}{2} i\omega_- + 3.9 \frac{k_-^2 v_{Ti}^2}{\nu_i}, \quad \nu_i = \frac{4}{3} \sqrt{\pi} \frac{L n_0 e^4}{m_i^2 v_{Ti}^3}. \quad (2.42)$$

With the aid of (2.38) we can find both $S(k_1, k_2, k_-)$ and the linear dielectric constant $\varepsilon(k_-)$ of the plasma. We have

$$\varepsilon(k_-) = 1 + i \frac{4\pi n_0 e (V_{k_-}^{(1)e} - V_{k_-}^{(1)i})}{\omega_- E_{k_-}} = 1 + i \frac{\omega_{0e}^2}{\varkappa \omega_- \omega_e} + i \frac{\omega_{0i}^2}{\varkappa \omega_- \omega_i}. \quad (2.43)$$

On the other hand, neglecting the collision integral in (2.3)[5] we get

$$j_{k_1}^{(2)} = e \int f_{k_1 e}^{(2)} \frac{\mathbf{v}\mathbf{k}_1}{k_1} d\mathbf{v}$$

$$= \frac{e^2}{m_e} \int \frac{\mathbf{k}_1 \mathbf{v}}{k_1} \frac{d\mathbf{v} E_{k_2}}{i(\omega_1 - \mathbf{k}_1 \mathbf{v})} \left(\frac{\mathbf{k}_2}{k_2} \frac{\partial f_{ek'}^{(1)}}{\partial \mathbf{v}}\right) \delta(k_1 - k' - k_2)\, dk'\, dk_2$$

$$\approx \frac{ie^2}{m_e} \int \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1 k_2} \frac{1}{\omega_1} E_{k_2} n_{k_-e}^{(1)} \delta(k_1 - k_- - k_2)\, dk_- dk_2. \quad (2.44)$$

In (2.44) we neglected the Doppler corrections to ω_1. Their inclusion results in small corrections of relative order ω_-/ω_1.

From the first equations of (2.36) and (2.33) we get

$$j_{k_1}^{(2)} = \frac{ie^3 n_0}{m_e^2} \int \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1 k_2} |\mathbf{k}_-| \frac{E_{k_-} E_{k_2}}{\omega_1 \omega_e \varkappa \omega_-} \delta(k_1 - k_2 - k_-)\, dk_2\, dk_-, \quad (2.45)$$

i.e., the sought S_2 is

$$S_2(k_1, k_2, k_-) = \frac{ie^3 n_0 |\mathbf{k}_-| (\mathbf{k}_1 \mathbf{k}_2)}{k_1 k_2 m_e^2 \omega_{0e} \varkappa \omega_e \omega_-}. \quad (2.46)$$

3. We turn to calculate the sum

$$\Sigma(k_1, k_2, k_1, -k_2) + \Sigma(k_1, k_2, -k_2, k_1). \quad (2.47)$$

[5]In this case it is sufficient to confine oneself to this approximation, unlike the calculation of $S(k_-, k_1, -k_2)$ when the collision integral must be taken into account in first order of perturbation theory. This is the consequence of the fact that in the nonlinear interaction (1.4) there is no symmetrization with respect to k_1 and k_2 in $S(k_1, k_2, k_-)$.

Neglecting the Doppler corrections and the collision integral in (2.4) we have

$$j_k^{(3)} = e\int \frac{\mathbf{kv}}{k} f_k^{(3)} dv$$

$$\approx \frac{e^2}{m_e}\int E_{k_1} \frac{\mathbf{kv}}{i\omega k}\left(\frac{\mathbf{k}_1}{k_1}\frac{\partial f_{k_-}^{(2)}}{\partial \mathbf{v}}\right)\delta(k-k_1-k_-)\,dk_1\,dk_-\,d\mathbf{v}$$

$$\approx \frac{ie^2}{m_e\omega}\int \frac{\mathbf{kk}_1}{kk_1} E_{k_1} n_{k_-}^{(2)} \delta(k-k_1-k_-)\,dk_-\,dk_1$$

$$= \frac{ie^2}{m_e\omega}\int \frac{|\mathbf{k}_-|(\mathbf{kk}_1)}{kk_1\omega_-} n_0 V_{k_-}^{(2)} E_{k_1}\delta(k-k_1-k_-)\,dk_1\,dk_-$$

$$= \frac{ie}{m_e\omega}\int \frac{|\mathbf{k}_-|(\mathbf{kk}_1)E_{k_1}}{kk_1\omega_-} j_{k_-}^{(2)}\delta(k-k_1-k_-)\,dk_1\,dk_-, \tag{2.48}$$

or, using (2.30),

$$j_k^{(3)} = \frac{1.71\nu_e n_0 e^4}{m_e^3\omega_{0e}}\int dk_1\,dk_2\,dk_3\,E_{k_1}E_{k_2}E_{k_3}\cdot$$
$$\cdot\frac{(\mathbf{k}_2\mathbf{k}_3)\,|\mathbf{k}_2+\mathbf{k}_3|\,|\mathbf{k}-\mathbf{k}_1|\,(\mathbf{kk}_1)\,\delta(k-k_1-k_2-k_3)}{\omega_{0e}^2 kk_1(\omega-\omega_1)k_2k_3\Omega(k_2+k_3)\Omega_e(k_2+k_3)}. \tag{2.49}$$

We note that the sought sum (2.47) is symmetrical with respect to the indices 2 and 3, thus justifying the use of (2.30), in the derivation of which we used essentially this symmetry property. From (2.49) we get

$$\frac{1}{2}(\Sigma(k_1,k_2,k_1,-k_2)+\Sigma(k_1,k_2,-k_2,k_1))$$
$$= -\frac{1.71\nu_e n_0 e^4 k_-^{-2}(\mathbf{k}_1\mathbf{k}_2)^2}{\omega_- k_1^2 k_2^2 m_e^3\omega_{0e}^3\Omega\Omega_e}. \tag{2.50}$$

3. SPECTRAL TRANSFER OF WEAK LANGMUIR TURBULENT PULSATIONS

We shall assume that the intensity of the Langmuir pulsations is so small that the change of their dispersion properties due to the nonlinear interactions can be neglected.[6] The term "weak Langmuir pulsations" will be used from now on in this sense. For weak waves $\omega_- \equiv {}^3\!/_2 v_{Te}^2(k^2-k_2^2)/\omega_{0e}$.

The spectral transfer is determined by the imaginary part (1.4):

$$\gamma_{k_1} = \operatorname{Im}\int \Sigma'(\mathbf{k}_1,\mathbf{k}_2)\,|E_{\mathbf{k}_2}|^2 d\mathbf{k}_2, \quad |E_{k_2}|^2 = |E_{\mathbf{k}_2}|^2\delta(\omega-\omega_{\mathbf{k}_2}), \tag{3.1}$$

where

$$\Sigma'(\mathbf{k}_1,\mathbf{k}_2) = 2\pi i\{\Sigma(k_1,k_2,k_1,-k_2)+\Sigma(k_1,k_2,-k_2,k_1)$$
$$-\frac{8\pi i}{\omega_-\varepsilon(k_-)}S_1(k_-,k_1,-k_2)S_2(k_1,k_2,k_-)\}. \tag{3.2}$$

Using (2.35), (2.46), (2.43), and (2.49) we can obtain

$$\Sigma'(\mathbf{k}_1,\mathbf{k}_2) = -i\frac{1.71\nu_e k_-^{-2}e^2(\mathbf{k}_1\mathbf{k}_2)^2}{\omega_{0e}k_1^2k_2^2\omega_-\Omega\Omega_e m_e^2}\left(1-\frac{\varepsilon_e(k_-)-1}{\varepsilon(k_-)}\right). \tag{3.3}$$

The first term of (3.3) corresponds to the contribution of the current of the third power in the field, and the second term corresponds to the second power, and $\varepsilon(k_-) = \varepsilon_e(k_-) + \varepsilon_i(k_-) + 1$. If we neglect the ionic term in ε, then these contributions cancel each other completely when $\varepsilon_e \gg 1$. The resultant compensation effect is similar to the compensation of the nonlinear and Compton scattering in the collisionless case.[2]

[6] Landau absorption of Langmuir waves is assumed to be exponentially small, whereas absorption due to collisions is of the order of ν_e.

Since $\omega_- \ll k_- v_{T\alpha}$, the region of applicability of (3.3) has, in accordance with (2.1), the form

$$k_- v_{Te} \ll \nu_e, \qquad k_- v_{Ti} \ll \nu_i. \tag{3.4}$$

By virtue of these inequalities, the phase velocities of the waves are sufficiently large, and the Doppler corrections in (3.3) are negligibly small compared with the ionic contribution. Then

$$\Sigma'(\mathbf{k}_1,\mathbf{k}_2) = -i\frac{1.71\nu_e k_-^{-2}e^2\varepsilon_i(k_-)(\mathbf{k}_1\mathbf{k}_2)^2}{\omega_{0e}k_1^2k_2^2 m_e^2\omega_-\Omega\Omega_e\varepsilon(k_-)}. \tag{3.5}$$

Under the conditions $\omega_- \ll k_- v_{Ti}$, i.e., of sufficiently large phase velocities, if

$$\frac{\nu_i}{\omega_{0e}} \gg \frac{T_i}{T_e}\frac{m_e}{m_i}, \tag{3.6}$$

we get in the case of

$$\max(\nu_e^2, \nu_i\omega_{0e}) \gg k_-^2 v_{Te}^2 \gg \nu_e\nu_i T_e/T_i, \tag{3.7}$$

the value

$$\gamma_{\mathbf{k}_1} = \frac{1}{|E_{\mathbf{k}_1}|^2}\frac{\partial}{\partial t}|E_{\mathbf{k}_1}|^2$$
$$= -\omega_{0e}\int \frac{0.69\nu_e^2\omega_- T_i(\mathbf{k}_1\mathbf{k}_2)^2}{\nu_i|\mathbf{k}_-|^2 T_e(1+{}^5\!/_3 T_i/T_e)^2 k_1^2k_2^2}\frac{|E_{\mathbf{k}_2}|^2 d\mathbf{k}_2}{n_0 m_e v_{Te}^2}. \tag{3.8}$$

The inequality (3.7) is satisfied if $(T_e/T_i)^5 \ll m_i/m_e$, which always takes place for an isothermal plasma ($T_e = T_i$). We emphasize that, just as in the absence of collisions,[2] the redistribution is such as to decrease the frequencies of the turbulent pulsations. We note that the condition for the appearance of the nonlinear interaction (3.8) is (see footnote [6]) $\gamma_k \gg \nu_e$. Recognizing that T_e cannot differ greatly from T_i, we obtain for $T_e \lesssim T_i$

$$\gamma \sim \nu_e\frac{\nu_e W}{\nu_i n_0 T_e} \sim \nu_e\sqrt{\frac{m_i}{m_e}}\frac{W}{n_0 T_e},$$

i.e., it is necessary to have $W/n_0T_e > \sqrt{m_e/m_i}$. The question whether the change of dispersion under such intensities can be small calls for a separate analysis (Sec. 5).

In the other limiting case, when (3.6) is satisfied but (3.7) is violated, namely

$$\nu_e\omega_{0e}\frac{m_e}{m_i} \ll k_-^2 v_{Te}^2 \ll \nu_e\nu_i\frac{T_e}{T_i}, \tag{3.9}$$

we have

$$\gamma_{\mathbf{k}_1} = \omega_{0e}\int \frac{(\mathbf{k}_1\mathbf{k}_2)^2}{k_1^2k_2^2}\frac{0.12\omega_-\nu_e^3}{k_-^4 v_{Te}^4}\frac{|E_{\mathbf{k}_2}|^2 d\mathbf{k}_2}{4\pi n_0 T_e(1+{}^5\!/_3 T_i/T_e)^2}. \tag{3.10}$$

In this case the direction of the spectral transfer corresponds to an increase of the frequencies of the turbulent pulsations. The region in which (3.9) takes place vanishes if (3.6) is violated. Thus, in order for (3.10) to take place it is necessary to satisfy (3.6) with a large margin.

To estimate the degree to which (3.6) is satisfied, it is convenient to rewrite this inequality in a different form, introducing the number of electrons in the Debye sphere

$$N_D \approx \omega_{0e}/\nu_e. \tag{3.11}$$

We then have in lieu of (3.6)

$$1 \ll N_D \ll \left(\frac{T_e}{T_i}\right)^{5/2}\left(\frac{m_i}{m_e}\right)^{1/2}. \tag{3.12}$$

For an isothermal plasma $T_e \approx T_i$, the plasma density should be sufficiently large, and its temperature should be small. When $T_e \gg T_i$, condition (3.12) is satisfied at much lower temperatures and larger densities. The spectral transfer prevails over linear damping if the following inequality is satisfied:

$$W / n_0T_e > k_-^2 v_{Te}^2 / \nu_e^2. \tag{3.13}$$

Let us consider now the nonlinear interaction under the conditions of an inequality that is the inverse of (3.6),

$$\frac{\nu_e}{\omega_{0e}} \ll \frac{T_i}{T_e}\frac{m_e}{m_i}. \tag{3.14}$$

We have

$$\gamma_{\mathbf{k}_1} = -\omega_{0e}\int \frac{0{,}13\nu_e^2\nu_i\omega_- m_i(\mathbf{k}_1\mathbf{k}_2)^2}{4|k_-|m_e v_{Te}^4(1+T_i/T_e)^2k_1^2k_2^2}\,\frac{|E_{\mathbf{k}_2}|^2\,d\mathbf{k}_2}{4\pi n_0T_e}. \tag{3.15}$$

The order of magnitude of the increment (3.15) is

$$\gamma \sim \frac{W}{n_0T_e}\nu_e\frac{\nu_e\nu_i}{|k_-|^2v_{Te}^2}\frac{m_i}{m_e}.$$

Consequently $\gamma > \nu_e$ when

$$\frac{W}{n_0T_e} > \frac{v_{Te}^2}{v_\Phi^2}\frac{m_e}{m_i}\frac{\omega_{0e}^2}{\nu_e\nu_i} \gg N_D\frac{T_e}{T_i}\frac{v_{Te}^2}{v_\Phi^2}. \tag{3.16}$$

We note that the formulas obtained in the present section are valid also in the case when account is taken of the change in the wave dispersion due to the nonlinear interactions, provided only $\omega_- \ll k_- v_{Ti}$.

4. CHANGE OF DISPERSION OF LANGMUIR WAVES WHEN $\omega_- \ll k_-^2 v_{Te}^2/\nu_e$

Besides changing the spectral transfer, the collisions can greatly alter the spectra of the Langmuir waves if the phase velocities are sufficiently large. The correc- $\delta\omega_k$ of the Langmuir-wave frequency is

$$\delta\omega_{\mathbf{k}_1} = \mathrm{Re}\int \Sigma'(\mathbf{k}_1,\mathbf{k}_2)\,|E_{\mathbf{k}_2}|^2\,d\mathbf{k}_2. \tag{4.1}$$

Since

$$\omega_- \ll k_-^2 v_{Te}^2/\nu_e, \tag{4.2}$$

where ω_- is the frequency difference of the Langmuir waves with allowance for (4.1), we get

$$\delta\omega_{\mathbf{k}_1} = -\omega_{0e}\int\frac{1.71\nu_e^2(\mathbf{k}_1\mathbf{k}_2)^2|E_{\mathbf{k}_2}|^2\,d\mathbf{k}_2}{4\pi\, n_0T_e k_-^2 v_{Te}^2(1+\varepsilon T_i/T_e)\,k_1^2k_2^2}. \tag{4.3}$$

Here $\varepsilon = 5/3$ when (3.6) is satisfied and $\varepsilon = 1$ when the inequality inverse to (3.6) is satisfied. The order of $\delta\omega_{k_1}$ is

$$\delta\omega_k \sim \omega_{0e}\frac{\nu_e^2}{k_-^2v_{Te}^2}\frac{W}{n_0T_e}.$$

When account is taken of $\delta\omega_k$, an appreciable change can take place in the frequency difference ω_- of the turbulent pulsations, from which the large term ω_{0e} drops out:

$$\tilde{\omega}_- = \omega_{\mathbf{k}_1} + \delta\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \delta\omega_{\mathbf{k}_2} \approx \frac{k_-^2v_{Te}^2}{\omega_{0e}} + \omega_{0e}\frac{\nu_e^2}{k_-^2v_{Te}^2}\frac{W}{n_0T_e}.$$

It follows therefore that the change of the dispersion due to the nonlinearity is significant when

$$W / n_0T_e > k_-^4 v_{Te}^4 / \omega_{0e}^2\nu_e^2. \tag{4.4}$$

On the other hand, by virtue of (4.2)

$$W / n_0T_e \ll k_-^4 v_{Te}^4 / \omega_{0e}\nu_e^3. \tag{4.5}$$

It is obvious that in an unbounded plasma one can always find small k such that (4.4) is satisfied. In a bounded plasma $k_{min} \sim 1/a$, and by virtue of $W/n_0T_e \ll 1$ we have

$$v_{Te}^4 \ll a^4\omega_{0e}^2\nu_e^2,$$

which, generally speaking, can also be satisfied.

We note that the results obtained in Sec. 3 are valid when

$$W / n_0T_e \ll k_-^4 v_{Te}^4 / \omega_{0e}^2\nu_e^2. \tag{4.6}$$

This inequality is in contradiction with (3.13), indicating that the breaking up of the turbulence scale, described by (3.10), can occur only against the background of the more intense process wherein they are absorbed as a result of the collisions. Let us take further account of the change of the wave dispersion in (3.10). By virtue of the fact that $\delta\omega_k$ is inversely proportional to k, the sign of $\tilde{\omega}_-$ coincides with the sign of $k_1 - k_2$. This shows that (3.10) describes a spectral triangle which likewise leads to a breaking up of the turbulence scales. When (4.5) is satisfied we get

$$\gamma \sim \omega_{0e}^2\left(\frac{W}{n_0T_e}\right)^2\frac{\nu_e^5}{k^6v_{Te}^6}. \tag{4.7}$$

This increment is larger than ν_e if

$$\frac{W}{n_0T_e} \gg \frac{k^2v_{Te}^2}{\nu_e^2}\frac{kv_{Te}}{\omega_{0e}},$$

which, together with (4.5) yields $\nu_e \ll k_- v_{Te}$ and contradicts (3.4). This again indicates that the breaking up of the turbulence scale occurs against the background of the more intense process of their dissipation.

We note that (4.6) does not contradict the condition of the applicability of (3.8), but if the dispersion is determined by the nonlinear interaction, i.e., if (4.4) and (4.5) are satisfied, then to estimate the intensity of the spectral transfer it is necessary to replace ω_- in (3.8) by $\tilde{\omega}_-$:

$$\gamma \approx \left(\frac{W}{n_0T_e}\right)^2\omega_{0e}^2\frac{\nu_e^4}{k_-^4v_{Te}^4\nu_i}\left(1+\frac{T_i}{T_e}\right)^{-4}\frac{T_i}{T_e}, \tag{4.8}$$

which is larger than ν_e when

$$\frac{W}{n_0T_e} > \frac{k_-^2v_{Te}^2}{\omega_{0e}\nu_e}\left(\frac{m_e}{m_i}\right)^{1/4}\left(\frac{T_e}{T_i}\right)^{5/4}\left(1+\frac{T_i}{T_e}\right)^2. \tag{4.9}$$

In this case the spectral redistribution is such that the turbulence scales increase. However, comparing the conditions (4.9) and (4.5) we see that the interaction (4.8) is possible only against the background of intense damping. Similarly, under the same assumptions as for (4.8), we obtain an estimate for the nonlinear increment (3.15):

$$\gamma \sim \left(\frac{W}{n_0T_e}\right)^2\omega_{0e}^2\frac{\nu_e^4\nu_i}{k_-^6v_{Te}^6}\left(1+\frac{T_i}{T_e}\right)^{-4}, \tag{4.10}$$

which exceeds ν_e if

$$\frac{W}{n_0T_e} > \frac{k_-^3v_{Te}^3}{\omega_{0e}\nu_e^2}\left(\frac{m_i}{m_e}\right)^{1/4}\left(\frac{T_i}{T_e}\right)^{3/4}\left(1+\frac{T_i}{T_e}\right)^2. \tag{4.11}$$

Comparing the conditions (3.16), (4.5), and (3.14) we find that the estimate (4.11) is valid if the following inequalities are satisfied

$$\frac{k_-^4 v_{Te}^4}{\omega_{0e}\nu_e^3} > \frac{W}{n_0 T_e} > \frac{m_e}{m_i}\frac{k_-^2 v_{Te}^2}{\nu_e \nu_i}, \quad \frac{T_i}{T_e}\frac{m_e}{m_i} \gg \frac{\nu_i}{\omega_{0e}} > \frac{m_e}{m_i}, \quad (4.12)$$

which are contradictory. Thus, the interactions which occur under conditions $\omega_- \ll k_-^2 v_{Te}^2/\nu_e$ cannot lead, in general, to a noticeable distortion of the energy distribution over the spectrum.

5. DISPERSION AND SPECTRAL TRANSFER IN THE PRESENCE OF INTENSE TURBULENCE ($\omega_- \gg k_-^2 v_{Te}^2/\nu_e$)

We note that this case is of greatest interest. If

$$\frac{m_e}{m_i} \ll \left(\frac{T_e}{T_i}\right)^5, \quad \frac{k_-^2 v_{Te}^2}{\nu_e} \ll \omega_- \ll k_- v_{Te},\ k_- v_{Ti}, \quad (5.1)$$

then

$$\gamma_{\mathbf{k}_1} = -\int \frac{31|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2\omega_{0e}k_-^4 v_{Te}^4(1.85 + T_i/T_e)}{4\pi n_0 T_e k_1^2 k_2^2 \omega_-^3 \nu_e (2.14 + T_i/T_e)^2} d\mathbf{k}_2. \quad (5.2)$$

The change in the pulsation spectrum is determined by the equation[7)]

$$\delta\omega_{\mathbf{k}_1} = \omega_{0e}\int \frac{3.84|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 k_-^2 v_{Te}^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2 \omega_-^2(2.14 + T_i/T_e)}. \quad (5.3)$$

It must be noted that the right sides of (5.2) and (5.3) represent the imaginary and real parts of the dispersion equation for the corrections to the frequency of the Langmuir oscillations due to nonlinear interactions. It is seen from (5.1) that the imaginary part of this equation is small compared with the real part, and the solutions of the equations obtained from (5.3),

$$\delta\omega_{\mathbf{k}_1} = \omega_{0e}\int \frac{3.84|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 k_-^2 v_{Te}^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2 (2.14 + T_i/T_e)(\delta\omega_{\mathbf{k}_1} - \delta\omega_{\mathbf{k}_2})^2}, \quad (5.4)$$

are in general complex and have imaginary parts, as can be seen from the very form of (5.4) that are of the same order as the real parts. In this case the nonlinear instability that leads to the spectral transfer is due to the solutions of (5.4). In connection with the fact that the solution of (5.4) is difficult, we confine ourselves to a qualitative investigation of this solution, which enables us to estimate the characteristic times and to determine the direction of the redistribution process.

Let us assume that the noise spectrum is concentrated in some wave-number region near $k_2 \approx k_{20}$. We consider first the limiting case $k_1 \gg k_{20}$. Then (5.4), assuming $\delta\omega_{k_1} \gg \delta\omega_{k_2}$ yields

$$(\delta\omega_{\mathbf{k}_1})^3 = -3.84\omega_{0e}k_1^2 v_{Te}^2 \int \frac{|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2 (2.14 + T_i/T_e)} \quad (5.5)$$

For the unstable root we have

$$\gamma_{\mathbf{k}_1} = \mathrm{Im}\,\delta\omega_{\mathbf{k}_1} = \frac{1.36\omega_{0e}^{1/3}}{(2.14 + T_i/T_e)^{1/3}} k_1^{2/3} v_{Te}^{2/3} \left[\int \frac{|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2}\right]^{1/3}. \quad (5.6)$$

As seen from (5.6), $\delta\omega_{k_1}$ increases with increasing k_1, thus justifying the assumption that $\delta\omega_{k_1} \gg \delta\omega_{k_2}$ when $k_1 \gg k_2$. An estimate of the increment (5.6) when $T_e \sim T_i$ is of the form

$$\gamma \sim \omega_{0e}\left(\frac{k_1 v_{Te}}{\omega_{0e}}\right)^{2/3}\left(\frac{W}{n_0 T_e}\right)^{1/3}.$$

We note that the real part of $\delta\omega_{k_1}$ is of the same order as the imaginary part, and this, the spatial dispersion of the Langmuir oscillations is almost completely connected with their nonlinear instability. We note also that (5.2) yields an estimate of the nonlinear increment due to the imaginary part in the dispersion relations (which is analogous to the kinetic instability in the linear theory), $\gamma' = k^2 v_{Te}^2/\nu_e$. On the other hand, by virtue of (5.1) and of $\omega_- \sim \gamma$, and consequently of $\gamma' \ll \gamma$, i.e., within the framework of the initial premises (5.1), the "kinetic" instability can be neglected. Condition (5.1) is satisfied if

$$\frac{v_{Ti}^2}{v_{Te}^2}\frac{k v_{Ti}}{\omega_{0e}} \gg \frac{W}{n_0 T_e} \gg \frac{\nu_e^3}{k^2 v_{Te}^2 \omega_{0e}}. \quad (5.7)$$

It must be specially emphasized that the condition for the smallness of the collisions is in this case not limiting, since when $\gamma \ll \nu_e$ there also occurs a "nonlinear dissipative instability." Indeed, to take into account the absorption of the Langmuir waves due to the collisions, it is sufficient to replace the left side of (5.3) by $\delta\omega_{k_1} + i\nu_e$ and when $\delta\omega_{k_1} \gg \delta\omega_{k_2}$ we obtain for the increasing root

$$\gamma_{\mathbf{k}_1} = \mathrm{Im}\,\delta\omega_{\mathbf{k}_1} = \frac{1.39 k_1 v_{Te}}{(2.14 + T_i/T_e)^{1/2}}\left[\frac{\omega_{0e}}{\nu_e}\int \frac{|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2}\right]^{1/2}. \quad (5.8)$$

We then get in lieu of (5.7)

$$\frac{k^2 v_{Te}^2}{\nu_e \omega_{0e}} \ll \frac{W}{n_0 T_e} \ll \frac{\nu_e}{\omega_{0e}}\frac{v_{Ti}^2}{v_{Te}^2}. \quad (5.9)$$

Let us consider now $\delta\omega_{k_1} \ll \delta\omega_{k_2}$. We then get from (5.4)

$$\delta\omega_{\mathbf{k}_1} = \omega_{0e}\int \frac{3.84|E_{\mathbf{k}_2}|^2(\mathbf{k}_1\mathbf{k}_2)^2 k_-^2 v_{Te}^2 d\mathbf{k}_2}{4\pi n_0 T_e k_1^2 k_2^2 (2.14 + T_i/T_e)(\delta\omega_{\mathbf{k}_2})^2}. \quad (5.10)$$

Estimating $\delta\omega_k$ from (5.10) and substituting in (5.2), we obtain a contradiction to the initial premises. Thus, the spectral transfer in the entire investigated region is such that the turbulence scales are broken up.

6. DISCUSSION OF RESULTS

Summarizing our analysis, we note that the investigated region was bounded by the conditions $\nu_e \gg k_- v_{Te}$, $\nu_i \gg k_- v_{Ti}$, and also $\omega_- \ll k_- v_{Ti}$, $\omega_- \ll k_- v_{Te}$. It is precisely in the region $\omega_- \ll k_- v_{Ti}$ where the nonlinear interactions of a collisionless plasma are the strongest. This is the reason for the interest in this region in the presence of collisions. At the same time, the employed condition $\omega_- \ll k_- v_{Ti}$ is not fundamental and it is easy to obtain also from the derived general formulas concrete expressions for the nonlinear interactions when $\omega_- \gg k_- v_{Ti}$.

Let us summarize the results briefly.

1. We have observed that the spectral transfer, heretofore considered in a collisionless plasma, is a

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[7)] An equation coinciding with (5.3) can be obtained as a dispersion equation for the electric field of weak waves excited by intense turbulence (the phases of the weak waves are arbitrary). This remark pertains to all the dispersion relations considered here.

particular case of a more general nonlinear instability. Such an instability can have both a kinetic and a hydrodynamic character. In the former case it is determined by the imaginary part of the nonlinear dispersion equation, and in the latter case by its real part.

2. Just as in the linear theory, the increments of the kinetic instabilities are as a rule smaller than the increments of the hydrodynamic instabilities. In particular, as shown by the analysis, the nonlinear kinetic instabilities under frequent-collision conditions are therefore usually suppressed by the linear damping for virtual waves.

3. The nonlinear hydrodynamic instability becomes manifest in a broad region of the plasma parameters and leads to a qualitatively new effect: a change takes place in the direction of the spectral transfer. This takes place in the phase velocity region

$$v_{\text{ph}} / v_{Te} \gg N_D. \qquad (6.1)$$

In practically the entire investigated region, the spectral transfer leads to a breaking up of the scales of the turbulent pulsations.

4. A nonlinear hydrodynamic instability develops also in the case when its increments are much smaller than the collision frequencies. This leads to a new important conclusion, consisting in the fact that there is no damping of the Langmuir waves due to collisions in the region of applicability of (5.10), and nonlinear dissipative instability takes place even in the case of very frequent collisions.

5. The usual subdivision of nonlinear interactions into decay interactions and induced-scattering processes becomes meaningless. At the same time, characteristic resonance effects, corresponding to the vanishing of the denominator of (3.5), can appear in the spectral-transfer effects. If

$$\operatorname{Im}\Sigma(k_1, k_2) \sim \operatorname{Im}\frac{1}{\varepsilon_e(k_-) + \varepsilon_i(k_-)} = i\pi\frac{\omega_-}{|\omega_-|}\delta(\varepsilon_e(k_-) + \varepsilon_i(k_-)), \qquad (6.2)$$

then such processes are connected with the vanishing of the Green's function of the virtual wave and, consequently, are analogous to processes in which the Langmuir waves break up into low-frequency ones. The corresponding term (6.2) consequently describes the spectral transfer of the Langmuir waves due to their decays into sonic waves that are located in the region of the frequent collision $\omega_s \ll \nu_e$, ν_i and are determined by the dispersion equation

$$\varepsilon_e(k_-) + \varepsilon_i(k_-) = 0. \qquad (6.3)$$

We note that both the spectrum of the "collision" sound of the plasma and the spectral transfer due to the decay of the Langmuir waves into such a sound can be readily obtained with the aid of the results (2.43) and (3.5). In such a redistribution process $\omega_- = kv_s$ and in order of magnitude $\omega_- = k_-v_{Ti}$. The condition $\omega_- \ll k_-v_{Ti}$ which was used above is not of fundamental character. It is also easy to write out formulas for the change in the dispersion and the spectral transfer when $\omega_- = kv_s$ and $\omega_- \gg k_-v_{Ti}$. However, the condition $\omega_- \ll k_-v_{Te}$ is quite important, i.e., the entire calculation must be repeated anew by the method developed above if it is violated; it is then necessary to solve (2.11) without separating the term (2.10), which is no longer small under these conditions. The violation of the condition $\omega_- \ll k_-v_{Te}$ is possible, naturally, only if the nonlinear change of the dispersion of the Langmuir waves is very large.

Finally, when $T_e \gg T_i$, there exists a broad region of values of the wave numbers of the turbulent pulsations, for which

$$\nu_e \ll k_-v_{Te}, \qquad \nu_i \gg k_-v_{Ti}. \qquad (6.4)$$

In this case we can use the known expressions for S_1 and S_2 of a collisionless plasma, and use the quantity (2.43) for $\varepsilon_i(k)$.

The obtained effect of the change in the direction of the spectral transfer is of great significant from the point of view of many problems, particularly in the problem of effective turbulent plasma heating, the efficiency of interaction between beams and a plasma, etc. Besides these questions, which are connected with various applications of the observed change in the spectral distribution, attention must be called also to the fact that an increase of the density of the redistribution at small values of ω_1, for which collisions must be taken into account, can change the overall estimates of the efficiency of the nonlinear interactions.

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Translated by J. G. Adashko
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