DYNAMICS OF THE INTERMEDIATE STATE IN SUPERCONDUCTORS

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A macroscopic description is proposed for the dynamic lamellar structure of the intermediate state of superconductors under nonstationary external conditions or in the presence of a constant current. The case of a current-carrying plane-parallel plate located in a perpendicular magnetic field is considered in detail.

TYPE I superconductors in the intermediate state located in an external magnetic field are mixtures of normal and superconducting phase. In the state of thermodynamic equilibrium, the boundaries of separation between the phases are fixed and their location, which is determined from the condition of minimum energy, corresponds to a set of alternating layers.[1] Such regular structures of the intermediate state can be realized experimentally in a number of cases[2] with high accuracy.

Upon the superposition of an external field, the boundaries of separation begin to move about. Furthermore, motion of the layers is shown to be possible when direct current flows through the specimen, i.e., under stationary external conditions.[3,4] If the velocity and layer configuration here changed little over distances of the order of their thickness, then the intermediate state admits of a macroscopic description, into which enter the intensities of the electric and magnetic fields **E** and **H**, averaged over a large number of layers.[5-7]

However, inasmuch as completely different layer configurations are possible for specific average fields, such a description does not allow us to answer the question of the location and velocity of motion of the boundaries of separation between the phases. The aim of the present research is to show that a more complete macroscopic description of the intermediate state is possible, a description which permits the calculation of the quantities mentioned.

We shall consider the case of sufficiently low temperatures, which allows us to neglect the effects connected with the liberation of heat in the motion of the phase boundaries.

1. The location of the boundaries between phases can be characterized by specifying, at each point of the volume occupied by the intermediate state, a unit vector **n** directed normal to the boundary. Moreover, if the intensities **E** and **H** are known, that is, the macroscopic electrodynamic equations are solved, then it is possible to determine the rate of motion of the layers. Actually, from the conditions of continuity of the tangential components of the electric field and the normal components of the magnetic field in a system of coordinates attached to the moving boundaries, and also from the fact that both the electric and magnetic fields are absent in superconducting regions, it follows that

$$\left[\mathbf{e}+\frac{V}{c}[\mathbf{nh}],\mathbf{n}\right]=0,\quad \mathbf{nh}=0,\qquad (1)^*$$

where **e** and **h** are the values of the electric and magnetic fields in the normal layers; V is the velocity of the layers; positive V corresponds to motion in the direction **n**. By taking it into account that $\mathbf{e} = \mathbf{E}/x_n$ (x_n is the concentration of the normal phase) and $\mathbf{h} = \mathbf{H}$, where $|\mathbf{H}| = H_c$ (H_c is the critical magnetic field), we get from (1)

$$V=\frac{c}{H_c^2x_n}\mathbf{n}[\mathbf{EH}],\quad \mathbf{nH}=0.\qquad (2)$$

The second of the given equations shows that the vector **H** is directed everywhere tangentially to the boundary between the phases.

At a given instant of time t, let some boundary of phase separation be the surface given by the parametric equation $\mathbf{r} = \mathbf{r}(\xi, \eta)$, where ξ, η are parameters so chosen that at the point $\mathbf{r}_0 = \mathbf{r}(0, 0)$, the derivatives $\mathbf{r}_\xi \equiv \partial\mathbf{r}/\partial\xi$ and $\mathbf{r}_\eta \equiv \partial\mathbf{r}/\partial\eta$ are mutually perpendicular and both are of unit magnitude. The vectors $\mathbf{r}_\xi$, $\mathbf{r}_\eta$, and $\mathbf{n}(\mathbf{r}_0, t)$ form a system of three mutually perpendicular unit vectors.

At the time $t + \delta t$, the equation of our boundary of separation can obviously be written in the form

$$\mathbf{r}=\mathbf{r}'(\xi,\eta)\equiv\mathbf{r}(\xi,\eta)+V(\mathbf{r}(\xi,\eta))\mathbf{n}(\mathbf{r}(\xi,\eta))\delta t.\qquad (3)$$

The vectors

$$\mathbf{r}_\xi'=\mathbf{r}_\xi+V_\xi\mathbf{n}\delta t+V\mathbf{n}_\xi\delta t$$

and

$$\mathbf{r}_\eta'=\mathbf{r}_\eta+V_\eta\mathbf{n}\delta t+V\mathbf{n}_\eta\delta t,$$

where the indices denote derivatives with respect to the corresponding variable, tangent to the new surface. Therefore, the vector

$$[\mathbf{r}_\xi'\mathbf{r}_\eta']=[\mathbf{r}_\xi\mathbf{r}_\eta]+\delta t\{V_\xi[\mathbf{n}\mathbf{r}_\eta]+V_\eta[\mathbf{r}_\xi\mathbf{n}]+V[\mathbf{n}_\xi\mathbf{r}_\eta]+V[\mathbf{r}_\xi\mathbf{n}_\eta]\}$$

is directed along the normal to it. Taking it into account that for $\xi = \eta = 0$ we have $\mathbf{r}_\xi \times \mathbf{r}_\eta = \mathbf{n}$ and, moreover, $\mathbf{n}_\xi \times \mathbf{r}_\eta$ and $\mathbf{r}_\xi \times \mathbf{n}_\eta$ are directed along **n**, we find the unit vector normal to the surface at the point $\mathbf{r}_0 + V\mathbf{n}\delta t$ and at the time $t + \delta t$:

$$\mathbf{n}'=\frac{[\mathbf{r}_\xi'\mathbf{r}_\eta']}{|[\mathbf{r}_\xi'\mathbf{r}_\eta']|}=\mathbf{n}(\mathbf{r}_0,t)-\delta t(V_\xi\mathbf{r}_\xi+V_\eta\mathbf{r}_\eta)=\mathbf{n}(\mathbf{r}_0,t)-\delta t\{\nabla V-\mathbf{n}(\mathbf{n}\nabla V)\}.\qquad (4)$$

On the other hand,

$$\mathbf{n}'-\mathbf{n}(\mathbf{r}_0,t)\equiv\mathbf{n}(\mathbf{r}_0+V\mathbf{n}\delta t,t+\delta t)-\mathbf{n}(\mathbf{r}_0,t)=\left\{\frac{\partial\mathbf{n}}{\partial t}+V(\mathbf{n}\nabla)\mathbf{n}\right\}\delta t.\qquad (5)$$

*$[\mathbf{nh}] \equiv \mathbf{n} \times \mathbf{h}$.

Comparing (4) and (5), we find

$$\frac{\partial \mathbf{n}}{\partial t}+V(\mathbf{n}\nabla)\mathbf{n}=-\nabla V+\mathbf{n}(\mathbf{n}\nabla V). \quad (6)$$

If E, H, and x_n are given as functions of the coordinates and time, and the vector n(r) at the initial instant of time, then Eq. (6), together with the conditions (2), determine at any instant of time the vector **n**, i.e., the location of the layers.

2. We apply these equations to the study of the structure of the intermediate state which arises in a plane parallel plate under the action of a perpendicular magnetic field $\mathcal{H}$ and a direct current. We shall consider here two limiting cases, $l \ll R$ and $l \gg R$, where l is the mean free path of the electrons, R is the Larmor radius in the critical magnetic field.

In the case $l \ll R$, one can neglect the Hall effect and, if we consider, for simplicity, that the crystalline lattice of the metal is cubic, we can write the conductivity tensor of the normal phase in the form $\sigma_{ik} = \sigma\delta_{ik}$. We then have (see[7]):

$$H_x=0,\quad H_y=-H_c\sin\varkappa z,\quad H_z=H_c\cos\varkappa z, \quad (7)$$

$$E_x\equiv E=\text{const},\quad E_y=E_z=0;$$

$$x_n=\mathcal{H}/H_c\cos\varkappa z. \quad (8)$$

Here $\kappa = 4\pi\sigma E/c\mathcal{H}$, the z axis is directed perpendicular to the plane of the plate, the x axis along the direction of flow. With the aid of (7), (8), and (2), we find the velocity of the layers

$$V=-cEn_y/\mathcal{H}. \quad (9)$$

We are most interested in the stationary distribution of the layers, when all the quantities depend only on the z coordinate. Here Eq. (6) can be written in the form

$$V\frac{d\mathbf{n}_t}{dz}=\mathbf{n}_t\frac{dV}{dz},$$

$$Vn_z\frac{dn_z}{dz}=-\frac{dV}{dz}+n_z^2\frac{dV}{dz}, \quad (10)$$

where the index t denotes the component of n tangential to the plane of the plate. Equations (9) and (10) have the following general solution, which satisfies the condition $\mathbf{n}^2 = 1$:

$$n_x=\frac{\alpha\cos\varkappa z}{\sqrt{1+\alpha^2\cos^2\varkappa z}},\quad n_y=\frac{\cos\varkappa z}{\sqrt{1+\alpha^2\cos^2\varkappa z}},\quad n_z=\frac{\sin\varkappa z}{\sqrt{1+\alpha^2\cos^2\varkappa z}}. \quad (11)$$

α is an arbitrary constant. Its calculation is not possible within the framework of the macroscopic theory and would require consideration of the effects of surface tension on the boundary between the phases. However, one can think that one of the two limiting possibilities exists: $\alpha \to \infty$ or $\alpha = 0$.

In the first case, the layers are located perpendicular to the current and are fixed. The form of the layers can be determined if we note that the system possesses a period d which does not depend on x, because of the homogeneity in the direction of the x axis. Use of Eq. (8) then leads to the following dependence of the thickness of the superconducting layers on z:

$$\frac{a_s(z)}{d}=\frac{\cos\varkappa z-\mathcal{H}/H_c}{\cos\varkappa z}. \quad (12)$$

For $\cos\kappa a > \mathcal{H}/H_c$, where a is the half thickness of the plate, i.e., for not very large values of the current, the superconducting layers appear on the surface. In the opposite case, a_s vanishes inside the plate.

If $\alpha = 0$, then the layers are located along the flow and move in the direction of the y axis with the velocity $V = -cE/\mathcal{H}$ or, if we take the dependence of the electric field on the current into account (see[7]),

$$V=-\frac{c^2}{4\pi\sigma a}\arcsin\frac{4\pi ja}{cH_c}, \quad (13)$$

where j is the average current density over the cross section, and it is assumed that $\cos\kappa a > \mathcal{H}/H_c$.[1)] The lines of intersection of the boundaries of phase separation with the plane yz, which correspond to the equalities $n_x = 0$, $n_y = \cos\kappa z$, $n_z = \sin\kappa z$, are given by the equation

$$y=\frac{c\mathcal{H}}{4\pi\sigma E}\ln\cos\varkappa z+\text{const}, \quad (14)$$

where the different values of const correspond to different boundaries. The thicknesses of the superconducting layers are determined from (12) as before.

In the case $l \ll R$, we limit ourselves to a consideration of metals with unequal numbers of electrons and holes, where, in contrast to the case considered above, the Hall effect plays an important role. Here (see[7]):

$$E_x=0,\quad E_y=\frac{\mathcal{H}j}{Nec},\quad E_z=-\frac{H_y}{H_z}E_y, \quad (15)$$

$$H_x=0,\quad H_y=-\frac{4\pi j}{c}z,\quad H_z=\sqrt{H_\kappa^2-H_y^2};$$

$$x_n=\mathcal{H}/\sqrt{H_c^2-H_y^2}, \quad (16)$$

where N is the difference between the number of electrons and the number of holes per unit volume.

In place of (9) and (11), we have now, in analogous fashion,

$$V=\frac{j}{Ne}n_x; \quad (17)$$

$$n_x=\frac{\alpha H_z}{\sqrt{\alpha^2H_z^2+H_c^2}},\quad n_y=\frac{H_z}{\sqrt{\alpha^2H_z^2+H_c^2}},\quad n_z=-\frac{H_y}{\sqrt{\alpha_2H_z^2+H_c^2}}. \quad (18)$$

When $\alpha \to \infty$, the layers are perpendicular to the current as in the case $l \ll R$; however, now they move with the velocity j/Ne, i.e., with the mean drift velocity of the current carriers.[2)] The form of the layers also changes. In place of (12), we now have

$$\frac{a_s(z)}{d}=1-\frac{c\mathcal{H}}{4\pi j}\left[\left(\frac{cH_c}{4\pi j}\right)^2-z^2\right]^{-1/2}. \quad (19)$$

For $\alpha = 0$, the layers are located along the flow and are at rest. The equation of the phase separation boundary in the yz plane here has the form

$$y=\sqrt{(cH_c/4\pi j)^2-z^2}+\text{const}. \quad (20)$$

3. Let us consider the problem of the propagation of disturbances in the intermediate state, which are described by Eqs. (6) and (2). Here we are dealing with disturbances in the distribution of layers without a change in the mean electromagnetic field, i.e., for constant **E**, **H**, and x_n.

1) For a conductivity of the specimen of the order of $10^9\,\text{ohm}^{-1}-\text{cm}^{-1}$, H_c = 100 Oe, and current density j = 1 A/cm^2, we get $V = 10^{-3}$ cm/sec.

2) For indium, for example, with j = 1 A/cm^2, we get $V = 1.6 \times 10^{-4}$ cm/sec.

Let some stationary solution be given, corresponding to the flow of a direct current in the specimen. In the general case, such a solution is inhomogeneous, i.e., $\mathbf{E}_0$, $\mathbf{H}_0$, $x_n^{(0)}$, and n_0 depend on the coordinates. However, this dependence can be neglected if we consider disturbances with wavelengths much shorter than the distance at which the inhomogeneity becomes significant. We set $\mathbf{n} = \mathbf{n}_0 + \mathbf{n}'$, where $\mathbf{n}'$ is a small addition, proportional to $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$. By linearizing Eq. (6) relative to it, we obtain

$$\left\{\frac{H_c^2 x_n}{c}\omega - (\mathbf{nk})([\mathbf{EH}]\mathbf{n})\right\}\mathbf{n}' = \{\mathbf{k} - \mathbf{n}(\mathbf{nk})\}([\mathbf{EH}]\mathbf{n}'), \qquad (21)$$

where we have omitted the index zero for brevity on quantities referring to the unperturbed solution. Since $\mathbf{n}\cdot\mathbf{H} = \mathbf{n}'\cdot\mathbf{H}' = 0$, it is seen from (21) that $\mathbf{k}\cdot\mathbf{H} = 0$.

We multiply both sides of Eq. (21) in scalar fashion by the vector $\mathbf{E}\times\mathbf{H}$. We then have

$$\left\{\omega - \frac{c}{H_c^2 x_n}[\mathbf{EH}]\mathbf{k}\right\}([\mathbf{EH}]\mathbf{n}') = 0,$$

whence we find

$$\omega = \frac{c}{H_c^2 x_n}[\mathbf{EH}]\mathbf{k}. \qquad (22)$$

In the case of the plane parallel plate considered above, the latter formula can be rewritten in the form

$$\omega = -\frac{cE}{\mathcal{H}}k_y, \quad l \ll R$$

and

$$\omega = \frac{j}{Ne}k_x, \quad l \gg R.$$

The rate of propagation of the disturbances is equal to

$$\mathbf{v} = \frac{\partial\omega}{\partial\mathbf{k}} = \begin{cases} c[\mathbf{E}\vec{\mathcal{H}}]/\mathcal{H}^2 & (l \ll R) \\ \mathbf{j}/Ne & (l \gg R) \end{cases}. \qquad (23)$$

We note that Eq. (23) actually determines the rate of propagation of the arbitrary disturbances (and not only short-wave ones with small amplitude). Actually, substituting (9) and (17) in (6), we get

$$\frac{\partial n_y}{\partial t} - \frac{cE}{\mathcal{H}}\frac{\partial n_y}{\partial y} = 0, \quad l \ll R$$

$$\frac{\partial n_x}{\partial t} + \frac{j}{Ne}\frac{\partial n_x}{\partial x} = 0, \quad l \gg R,$$

whence it is seen that an arbitrary (but one agreeing with (7), (8) and (15), (16)) initial picture of the distribution of the layers drifts in the plane of the plate with velocity v.

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