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GENERATION OF LIGHT BY A SCATTERING MEDIUM WITH NEGATIVE RESONANCE ABSORPTION

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Generation of light by a scattering medium with negative resonance absorption is considered theoretically for the case when the photon mean free path is much smaller than the dimensions of the scattering region. The negative feedback in such a quantum generator is not resonant. The generation threshold of the quantum generator is determined and the dynamics of the establishment of stationary conditions and narrowing of the radiation spectrum are considered. The limiting width of the radiation spectrum under generation conditions, due to fluctuation motion of the scattering particles, is found. The use of such a quantum generator as a source of stable frequency light oscillations is discussed.

1. INTRODUCTION

THE presently known lasers comprise an optically homogeneous medium with negative absorption and a configuration of elements which return part of the radiation back into the medium in order to obtain feedback. If the radiation is returned with the aid of a system of mirrors of the Fabry-Perrot resonator type [1], then the feedback is resonant, and if it is returned by means of back scattering, then the feedback is nonresonant [2]. In an earlier paper [3] we considered the case of generation in the interstellar medium, when the feedback is realized by scattering particles (electrons, dust particles, etc.) distributed in an active medium, but here, too, the negative-absorption medium was essentially "optically" homogeneous, since the mean free path of the quantum with respect to scattering was much larger than the dimensions of the generator.

In the present paper we consider the generation of light by a scattering medium with negative resonant absorption in the case when the mean free path of the photon due to scattering is much smaller than the dimensions of the region, that is, when the photon motion is diffuse. The possibility of realizing generation of light in this case was demonstrated in our paper [4]. The quantum generator under consideration, which actually constitutes only a scattering medium with negative absorption, is a modification of the quantum generator with nonresonant feedback. The reason for the interest in its properties is the prospect of its use for the construction of a highly stable optical frequency standard [2,4]. The present paper is devoted to a theoretical investigation of such a quantum generator. We find the generation threshold of the scattering region with the negative absorption. We consider the dynamics of the establishment of the stationary generation regime and the narrowing of the radiation spectrum. We obtain the limiting width of the emission spectrum of the generation regime, brought about by the fluctuation motion of the scattering particles.

2. FUNDAMENTAL EQUATIONS

Consider an ensemble of identical dielectric particles with density N_0 and a complex dielectric constant $\epsilon = \epsilon_0 + i\epsilon''$, where $\epsilon'' > 0$ in the vicinity of frequency ω_0. Let Q_s be the scattering cross section and Q_ω the cross section for negative absorption of light of frequency per particle. We consider the case when the mean free path $\Lambda_s = 1/N_0 Q_s$ of the photon with respect to scattering, the average dimension R of the region occupied by the ensemble, and the length λ of the radiation wave satisfy the relation

$$R \gg \Lambda_s \gg \lambda, \tag{1}$$

and $N_0^{-1/3} \gg \lambda$. Then the change of flux density $\Phi_\omega(\mathbf{r}, t)$ of photons of frequency ω at the point $\mathbf{r}$ can be described in the diffusion approximation[1)]:

$$\frac{1}{c}\frac{\partial \Phi_\omega(\mathbf{r},t)}{\partial t} = D\Delta\Phi_\omega(\mathbf{r},t) + Q_\omega(\mathbf{r},t)N_0\Phi_\omega(\mathbf{r},t); \tag{2}$$

here D is the diffusion coefficient, Δ the Laplace operator, and c the average speed of light in the region occupied by the ensemble. The frequency dependence of the photon density is connected with the resonant character of the negative absorption. The cross section $Q_\omega(\mathbf{r}, t)$ does not remain constant, since the imaginary part of the dielectric constant $\epsilon''_\omega(\mathbf{r}, t)$ depends on the photon flux density (the saturation effect [6]). The process of diffusion multiplication of photons, described by Eq. (2), recalls in many respects the diffusion of neutrons in a homogeneous reactor. We shall therefore use below a number of results that are known for neutron diffusion (for example, the criticality condition), with reference to the monograph [5]. At the same time, our case has several distinguishing features connected with the resonant character and the nonstationary nature of the absorption cross section $Q_\omega(\mathbf{r}, t)$.

The diffusion coefficient in an absorbing medium with anisotropic scattering is [5]

$$D = \Sigma_s / 3\Sigma(\Sigma - \bar{\mu}\Sigma_s), \tag{3}$$

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1) Strictly speaking, the necessary condition for the diffusion approximation is also the requirement that the mean free path due to scattering be much smaller than the mean free path due to absorption (in this case, creation) of the photons [5]. As will be shown later, in the stationary generation regime this condition is a consequence of condition (1). In the nonstationary case this may not be so and it is necessary to stipulate $\Lambda_s \ll \Lambda_a$ from the very beginning.

where $\Sigma_s = Q_s N_0$ and $\Sigma_a = Q_\omega N_0$ are the macroscopic cross sections for scattering and negative absorption; $\Sigma = \Sigma_s + \Sigma_a$; $\bar{\mu}$ is the average cosine of the scattering angle. In our case $\Sigma_s \gg \Sigma_a$, and consequently

$$D \approx \frac{1}{3\Sigma_s(1-\bar{\mu})} = \frac{\Lambda_s}{3(1-\bar{\mu})}. \tag{4}$$

For axially-symmetrical scattering (of the Rayleigh type), $\bar{\mu} = 0$. In the case of predominant forward scattering ($\bar{\mu} > 0$) the diffusion coefficient increases.

In the case of homogeneous line broadening, the imaginary part of the dielectric constant of the particles $\epsilon''(\mathbf{r}, t)$ can be represented in the form

$$\varepsilon_\omega''(\mathbf{r}, t) = a(\omega)\varepsilon''(\mathbf{r}, t), \tag{5}$$

where $a(\omega)$ is the absorption line shape normalized to unity at the maximum. In the case of a Lorentz shape, for example,

$$a(\omega) = \frac{(\Delta\omega_0/2)^2}{(\omega-\omega_0)^2 + (\Delta\omega_0/2)^2}. \tag{6}$$

The quantity $\epsilon''(\mathbf{r}, t)$ is determined by the pump power and depends on the intensity of the photon flux (saturation effect). The equation for $\epsilon''(\mathbf{r}, t)$ can be written in the form

$$\frac{\partial\varepsilon''(\mathbf{r},t)}{\partial t} + \frac{1}{T_1}\varepsilon''(\mathbf{r},t) = -2\sigma_0\varepsilon''(\mathbf{r},t)\int a(\omega)\Phi_\omega(\mathbf{r},t)d\omega + \frac{1}{T_1}\tilde{\varepsilon}''(\mathbf{r}), \tag{7}$$

where $\sigma_0 = \sigma(\omega_0)$ is the cross section for the radiative transition of the "atoms" of the dielectric particles that are responsible for the negative resonance absorption; T_1 is the time of longitudinal relaxation, describing the spontaneous decay of the negative absorption; $\tilde{\epsilon}''(\mathbf{r})$ is a term proportional to the pump power. In the stationary case in the absence of a field (below threshold) we have $\epsilon''(\mathbf{r}, t) \equiv \tilde{\epsilon}''(\mathbf{r})$.

The system of equations (2) and (7) describes completely the stimulated emission from an ensemble of scattering particles with negative absorption. To solve this system it is necessary to specify boundary conditions. The photon density is very small on the outer boundary of the medium. However, it cannot be equal to zero, since the photons diffuse from the medium and can pass through the surface. We can introduce a distance d at which the photon flux, linearly extrapolated to the outside from the boundary of the medium, will be equal to zero. This distance, called the length of linear extrapolation, is given by the expression [5]:

$$d = 2/3\Sigma \approx {}^2/_3\Lambda_s. \tag{8}$$

3. CONDITION FOR THE GENERATION THRESHOLD

The right side of (2) describes the attenuation of the radiation due to diffusion "spreading" and the amplification due to the resonant negative absorption. There obviously exists a threshold at which the radiation losses are compensated by the negative absorption. At the threshold, the saturation of the gain can be neglected and we can assume that $\epsilon(\mathbf{r}, t) = \tilde{\epsilon}''(\mathbf{r})$. We confine ourselves to the case of homogeneous pumping $\tilde{\epsilon}''(\mathbf{r}) = \epsilon_0''$, when the complex dielectric constant is the same for all the scattering particles. Then the solution of the problem reduces to solving (2) with a cross section $Q_\omega(\mathbf{r}, t) \equiv Q_\omega$ which is constant over the ensemble and in time.

The general solution of (2) is

$$\Phi_\omega(\mathbf{r}, t) = \sum_n a_n\psi_n(\mathbf{r})\exp[-(DB_n^2 - Q_\omega N_0)ct], \tag{9}$$

where $\psi_n(\mathbf{r})$ and B_n are the eigenfunctions and eigenvalues of the equation

$$\Delta\psi_n(\mathbf{r}) + B_n^2\psi_n(\mathbf{r}) = 0 \tag{10}$$

with boundary condition $\psi_n = 0$ at a distance d from the boundary of the region, and a_n are arbitrary constants determined by the initial distribution $\Phi(\mathbf{r}, t)$ at $t = 0$.

From (9) there follows immediately the threshold condition

$$DB^2 - N_0 Q_0 = 0, \tag{11}$$

where B is the smallest eigenvalue B_n (usually $B = B_1$), $Q_0 = Q_{\omega_0}$, and ω_0 is the frequency at maximum gain.

If the region occupied by the ensemble of scattering particles has the form of a sphere of radius R, then[2)] [5]

$$\psi_n = \frac{1}{r}\sin\frac{n\pi r}{R}, \quad B_n = \frac{n\pi}{R}, \quad B = \frac{\pi}{R}. \tag{12}$$

for a cylinder of height H and radius R, the smallest eigenvalue B is

$$B = \left[\left(\frac{2{,}4}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right]^{1/2}, \tag{13}$$

and for a parallelepiped with sides a, b, and c it is equal to

$$B = \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]^{1/2}. \tag{14}$$

Strictly speaking, all the dimensions of the regions must be increased by the length of linear extrapolation (8), but, owing to condition (1), this results in a small correction which we shall neglect.

The cross sections Q_s and Q_0 are determined by the geometry and by the value of the complex dielectric constant $\epsilon = \epsilon_0 + i\epsilon_0''(\omega_0)$ of the scattering particles. Their calculation, for an arbitrary relation between the dimension of the particles and the wavelength, is a very complicated problem even for a definite particle shape.[7]. We therefore confine ourselves to examples of spherical particles with radius a much smaller or much larger than the wavelength λ.

The case $ka \ll 1$ can be calculated exactly[7]:

$$Q_s = \frac{8}{3}(ka)^4\left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 G, \quad Q_0 = 4ka\,\mathrm{Im}\frac{\varepsilon-1}{\varepsilon+2}G, \quad \bar{\mu} = 0 \tag{15}$$

($G = \pi a^2$ is the geometric cross section). The imaginary part of the dielectric constant is $\epsilon_0'' = \alpha_0/k \ll 1$, where α_0 is the gain per unit length in the material of scattering particles, and k is the wave vector. We then have

$$Q_s = \frac{8}{3}(ka)^4\left(\frac{\varepsilon_0-1}{\varepsilon_0+2}\right)^2 G, \quad Q_0 = \frac{4}{\varepsilon_0+2}a\alpha_0 G. \tag{16}$$

Relations (11) and (16) determine the threshold (critical) dimension of the generation region or the threshold gain α_0. For example, the expression for the

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[2)] It follows from (12) and (11) that the ratio of the mean free paths in scattering and in amplification, $\Lambda_s/\Lambda_a = \pi^2(\Lambda_s/R)^2/3(1-\bar{\mu})$ be much smaller than unity when condition (1) is satisfied.

threshold dimension π/B_{thr} (in the case of sphere—the critical radius R_{thr}) is

$$\frac{\pi}{B_{thr}}=\frac{g^3}{(ka)^2}\left(\frac{\varepsilon_0+2}{\varepsilon_0-1}\right)\sqrt{\frac{2(\varepsilon_0+2)a}{\alpha_0}}, \tag{17}$$

where $g = N_0^{-1/3}/2a$ is the ratio of the average distances between particles to their diameter.

In the case when $ka \gg 1$, the calculation of the exact values of the cross sections and of the diagram is a complicated matter. However, for approximate estimates in the region $\epsilon_0 - 1 \approx 1$ we can assume that

$$Q_s \approx G, \quad Q_0 \approx 2\eta a\alpha_0 G, \quad \bar{\mu} \approx 0, \tag{18}$$

where η is the average transmission of the boundaries of the dielectric sphere. Then the expression for the critical dimension takes the form

$$\frac{\pi}{B_{thr}} \approx g^3\sqrt{\frac{32a}{3\eta\alpha_0}}. \tag{19}$$

For example, for a spherical distribution of ruby particles ($\lambda = 7 \times 10^{-5}$ cm) with radius $a = 2 \times 10^{-4}$ cm, $\eta \approx 1$, $g = 2$, and a gain $\alpha_0 \approx 1\ \text{cm}^{-1}$ (at 77°K), the critical radius of the region is $R_{thr} \approx 4$ mm.

We have considered above the case of homogeneous distribution of the pumping in the absence of any photon reflection from the boundary of the region. In many cases this is not so.

In particular, for external optical pumping of the scattering particles by diffusion and absorption of the pump light, the $\epsilon_0''(\mathbf{r})$ distribution has a maximum on the boundary of the region and a minimum inside. Solving the diffusion equation with absorption and with influx of pump photons on the boundary, we can obtain the distribution of the pump intensity and consequently the function $\tilde{\epsilon}_0''(\mathbf{r})$.

For a spherical region the solution is

$$\tilde{\varepsilon}''(\mathbf{r})=\tilde{\varepsilon}_m''\frac{R}{r}\frac{e^{\varkappa r}-e^{-\varkappa r}}{e^{\varkappa R}-e^{-\varkappa R}}, \tag{20}$$

where $\tilde{\epsilon}_m''$ is the value of $\epsilon_0''(\mathbf{r})$ on the boundary of the region, R is the radius of the region, $\kappa = \sqrt{N_0Q_a/D_p}$, D_p and Q_a are the diffusion coefficient and the cross section for the absorption of the pump light. The quantities D_p and Q_a are determined by the previous equations (4), (16), and (18), with allowance for the difference between the wavelength of the pump light and the wavelength of the generated radiation, and also for the difference between the absorption coefficient of the pump light α_p and the gain α_0. Allowance for the inhomogeneity of $\tilde{\epsilon}_0''(\mathbf{r})$ is important only when $\kappa R \lesssim 1$ or, putting $R = R_{thr}$, when $|\alpha_p| \gtrsim |\alpha_0|$. However, the threshold condition (11) remains the same as before even in this case, but the quantity B is the smallest eigenvalue of the following equation:

$$\Delta\psi_n(\mathbf{r})+B_n^2\frac{\tilde{\varepsilon}''(\mathbf{r})}{\tilde{\varepsilon}_m''}\psi_n(\mathbf{r})=0. \tag{21}$$

Obviously, allowance for the inhomogeneity increases the value of B and consequently raises the generation threshold.

The presence of reflection (specular or diffuse) from the boundary of the region, to the contrary, lowers the threshold. In this case the threshold can be calculated either by replacing the zero boundary conditions, or by moving the boundary to an effective distance determined by the reflection coefficient.

4. SCATTERING PARTICLES IN AN AMPLIFYING GASEOUS MEDIUM

Particular interest attaches to the case when the scattering particles are distributed in a gaseous medium with negative absorption (for example in an He-Ne or He-Xe gas mixture excited by an electric discharge[8]). In this case the scattering particles effect a nonresonant feedback, and the gaseous active medium effects resonant amplification. The use of a gaseous active medium ensures a higher stability of the frequency of the center of the amplification line and is of interest for the development of an optical frequency standard.

An important property of an amplifying gaseous medium is in our case the "homogeneous broadening" of the emission line. As is well known, in gas lasers with Fabry-Perrot resonators, within the limits of a line homogeneously broadened by the Doppler effect in a strong monochromatic field, "dips" are produced as a result of the saturation effect[9,10]. Physically, the formation of the "dips" is connected with generation of a light field in the form of guided waves. In our case the mean free path of the photon is much shorter than the dimensions of the generated region, and therefore the radiation is essentially isotropic. In a monochromatic isotropic field, the probability of stimulated emission does not depend on the direction of the atom velocity, and the formation of "dips" is impossible. In this sense, the amplification line is "homogeneously broadened," and the frequency of the maximum gain does not depend on the field intensity. This is a very important factor for the creation of an optical frequency standard.

The threshold condition (11) remains valid also in the case when the gain per unit length in the scattering medium $Q_\omega N_0$ is replaced by the effective gain in the gaseous medium $\alpha_{eff}(\omega)$. Strictly speaking, the effective gain $\alpha_{eff}(\omega)$ in the presence of scattering particles differs somewhat from the gain $\alpha_0(\omega)$ in a pure gaseous medium, owing to the possible penetration of the field inside the particles, which may be absorbing. For an estimate we can assume

$$\alpha_{eff}(\omega)=f\alpha_0(\omega)+(1-f)\varkappa_0, \tag{22}$$

where $(1 - f)$ is a parameter describing the relative magnitude of the mean free path of the photon inside the scattering particles in the case of random wandering inside the region; κ_0 is the absorption coefficient per unit length in the material of the scattering particles. If the field penetrates freely inside the scattering particles, one can assume approximately that $(1 - f) \approx (V_s/V_0)^{1/3}$, where V_s is the volume of all the scattering particles and V_0 is the volume of the generation region.

The critical radius of the generation region is

$$R_{thr}=\pi[\Lambda_s/3(1-\bar{\mu})\alpha_{eff}(\omega_0)]^{1/2}. \tag{23}$$

For example, it is realistic to obtain in a He-Xe mixture at $\lambda = 3.51\,\mu$ a value $\alpha_{eff} \approx 0.1\ \text{cm}^{-1}$ [11]. When $\Lambda_s \approx 0.1$ cm and $\bar{\mu} \approx 0$ the critical radius is $R_{thr} \approx 1.8$ cm.

5. GENERATION DYNAMICS

The generation process, including the transient process, the establishment of the stationary generation regime, the narrowing of the emission spectrum, etc., is described by the nonlinear equations (2) and (7). Before solving these equations, let us qualitatively discuss the behavior of the solution.

After going through the threshold, an exponential increase of the emission intensity takes place, starting from the spontaneous-noise level. This process is still described by the linear equation (9). Owing to the resonant character of the negative absorption, the rate of growth is maximal for frequencies near the center of the line ω_0, which leads to simultaneous narrowing of the spectrum. The exponential increase of the integral intensity $\Phi(\mathbf{r}, t) = \int \Phi_\omega(\mathbf{r}, t)\, d\omega$ continues until the negative absorption $\epsilon''(\mathbf{r}, t)$ is decreased by the saturation effect. This process is described by Eq. (7). As a result of the abrupt saturation, the system is below threshold and the radiation intensity decreases, but then, the threshold is exceeded, as a result of the pumping $\tilde{\epsilon}''(\mathbf{r})$, and the pulsations repeat. Such pulsations (spikes) are the usual transient process of a quantum generator [12]. They gradually attenuate and the generation goes over into the stationary regime.

Let us investigate first the nonstationary solution of Eqs. (2) and (7). To this end we integrate (2) with respect to the frequency and take into account the fact that

$$\int \Phi_\omega(\mathbf{r}, t) Q_\omega(\mathbf{r}, t)\, d\omega \simeq Q_0(\mathbf{r}, t) \int \Phi_\omega(\mathbf{r}, t)\, d\omega.$$

The latter is valid on the basis that the width of the radiation spectrum becomes much lower than the width of the resonance of negative absorption $\Delta\omega$ already at the start of the first spike. Then Eqs. (2) and (7) take on the following form:

$$\frac{1}{c}\frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = D\Delta\Phi(\mathbf{r}, t) + Q_0(\mathbf{r}, t) N_0 \Phi(\mathbf{r}, t), \tag{24}$$

$$\frac{\partial \epsilon''(\mathbf{r}, t)}{\partial t} = -2\sigma_0 \epsilon''(\mathbf{r}, t)\Phi(\mathbf{r}, t) + \frac{\tilde{\epsilon}''(\mathbf{r}) - \epsilon''(\mathbf{r}, t)}{T_1}. \tag{25}$$

The cross section $Q_0(\mathbf{r}, t)$ is proportional to $\epsilon''(\mathbf{r}, t)$ (see Sec. 3).

The solution of the partial differential equation (24) will be sought in the form of a series in the eigenfunctions of the equation (10):

$$\Phi(\mathbf{r}, t) = \sum_{k=1}^{\infty} a_k(t)\psi_k(\mathbf{r}). \tag{26}$$

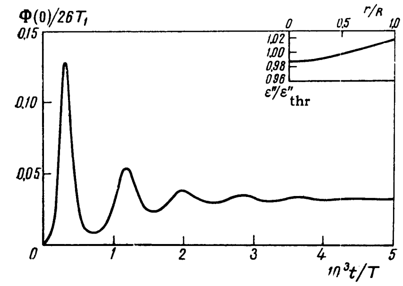
Substituting (26) in (24) and recognizing that the functions $\psi_k(\mathbf{r})$ form a complete orthonormal system, we obtain after making the standard transformations,

$$\frac{da_m(t)}{dt} = -cDB_m^2 a_m(t) + cN_0 \sum_{k=1}^{\infty} a_k(t) Q_{km}(t), \tag{27}$$

where

$$Q_{km}(t) = \langle \psi_k(\mathbf{r}) | Q_0(\mathbf{r}, t) | \psi_m(\mathbf{r}) \rangle, \tag{28}$$

and $Q_0(\mathbf{r}, t)$ satisfies Eq. (25). The solution of the system of equations (25) and (27) defines completely the development of the generation, but it cannot be obtained analytically. Therefore we integrated the equations numerically with an electronic computer.



Dynamics of establishment of stationary generation regime in a spherical scattering region with negative absorption. The initial excess over threshold is $\eta = 1.02$; $T/T_1 = 3 \times 10^{-3}$. In the upper corner – radial distribution of ϵ'' in the stationary regime.

We considered the case of a spherical region of radius R, when the eigenfunctions and the eigenvalues are determined by (12). The pump distribution $\tilde{\epsilon}''(\mathbf{r})$ was homogeneous. The parameter T/T_1, where T is the average lifetime of the photon in the region where $\epsilon \equiv 0$, determined by the expression

$$T = 1 / DB^2 c, \tag{29}$$

and T_1 is the time of longitudinal relaxation, was much smaller than unity. The figure shows the results of the solution for the case $T/T_1 = 3 \times 10^{-3}$ with the pump-to-threshold ratio $\eta = 1.02$. One can see clearly the occurrence of pulsations of the photon density and a transition to the stationary generation regime. In the upper part of the figure we show the radial distribution of $\epsilon''(\mathbf{r})$ in the stationary regime. The inhomogeneity of the distribution of $\epsilon''(\mathbf{r})$ is due to the predominant saturation of the negative absorption at the center of the region.

The stationary solution of (24) and (25), which corresponds to the steady-state generation, satisfies the equation

$$\Delta\Phi(\mathbf{r}) + \frac{N_0}{D}\frac{\tilde{Q}(\mathbf{r})}{1 + 2\sigma_0 T_1 \Phi(\mathbf{r})}\Phi(\mathbf{r}) = 0, \tag{30}$$

where $\tilde{Q}(\mathbf{r})$ is the stationary value of $Q_0(\mathbf{r}, t)$ in the absence of the field, and is determined by the value $\tilde{\epsilon}''(\mathbf{r})$. In the case of a small excess of pump over threshold, when the saturation effect can be neglected ($2\sigma_0 T_1 \Phi \ll 1$), the solution (30) is the eigenfunction of Eq. (10) (we confine ourselves to the case of homogeneous pumping), corresponding to the smallest eigenvalue, and the nonlinearity of (30) determines the amplitude of the solution. If B_n is the smallest eigenvalue, then the solution takes the form

$$\Phi(\mathbf{r}) = \frac{\eta - 1}{2\sigma_0 T_1 \eta}\left(\int \psi_n^3(\mathbf{r})\, d\mathbf{r}\right)^{-1} \psi_n(\mathbf{r}), \quad \eta - 1 \ll 1, \tag{31}$$

where $\eta = \tilde{Q}N/DB_n^2$ is the coefficient of the excess of the pump over the threshold value. When the pump increases above the threshold, the distortion of the potential in (30) as a result of saturation becomes appreciable. As a result, an admixture of other eigenfunctions appears in the solution. In the case of weak saturation ($2\sigma_0 T_1 \Phi \lesssim 0.1$), the solution can be obtained from perturbation theory, by regarding the "deformation" of the potential in (30) as the perturbation. In the first approximation, the solution takes the form

$$\Phi(\mathbf{r}) = \frac{\eta - 1}{2\sigma_0 T_1 \eta U_{nn}}\left[\psi_n(\mathbf{r}) + (\eta - 1)\frac{B_n^2}{U_{nn}}\sum_{p \neq n} \frac{U_{np}}{\eta B_n^2 - B_p^2}\psi_p(\mathbf{r})\right], \tag{32}$$

where

$$U_{np} = \int \psi_n{}^2 \psi_p \, dr. \tag{33}$$

The intensity of the output radiation of a quantum generator is determined by the diffusion of the photons through the boundary of the region. This process is analogous to the "leakage" of neutrons from a reactor, and the intensity of the output radiation in the stationary case is determined by the expression[5]

$$P = \frac{\Lambda_s}{6} \, |\mathrm{grad}_n \, \Phi(\mathbf{r})|, \tag{34}$$

where n is normal to the boundary of the region. The intensity of the output radiation is proportional to the ratio of the length of the mean free path Λ_S to the dimensions of the region. Physically this is connected with the fact that the photons diffuse to the outside predominantly from a surface layer of thickness $\sim\Lambda_S$. Therefore the emission of such a generator, like "black body" radiation, is isotropic.

6. EMISSION SPECTRUM

At a stationary value of the integral radiation intensity $\Phi(\mathbf{r}, t)$, the spectral density $\Phi_\omega(\mathbf{r}, t)$ is, generally speaking, not stationary. The non-stationary nature of the spectrum consists in the fact that a continuous narrowing of the emission spectra occurs after the start of the generation. This effect is a feature of lasers with nonresonant feedback, and was first investigated theoretically and experimentally in[14]. In our case this can be easily verified by considering Eq. (2) for the spectral density at a stationary value of the negative-absorption cross section Q_0 and a stationary spatial distribution of the photon density. For simplicity we shall neglect the distortion of Q_0 due to saturation, so that $\Delta\Phi_\omega = -B^2\Phi_\omega$. Then Eq. (2) reduces to the following:

$$\frac{1}{c}\frac{\partial \Phi_\omega}{\partial t} = -DB^2\Phi_\omega + a(\omega) Q_0 N_0 \Phi_\omega. \tag{35}$$

Inasmuch as $a(\omega_0) Q_0 N_0 = DB^2$, we get

$$\Phi_\omega(t) \sim \exp\{-[a(\omega) - a(\omega_0)] Q_0 N_0 ct\}. \tag{36}$$

Expanding the line-shape function $a(\omega)$ (Eq. (6)) near the center ω_0 of the line, we obtain the law governing the narrowing of the spectrum in the stationary generation regime:

$$\Delta\omega(t) = (Q_0 N_0 ct / \ln 2)^{-1/2} \Delta\omega_0, \tag{37}$$

which is analogous to that obtained in[14].

The narrowing of the spectrum, as usual, continues up to a certain fluctuation limit. In the ideal case, the limiting width of the line is determined by the spontaneous emission. In practice, however, an important role is played by the inevitable fluctuation (Brownian) motion of the scattering particles, which leads to a random variation (wandering) of the photon frequency as a result of the Doppler effect on the scattering particles. Assume that in each scattering the photon frequency changes by an average amount $\delta\omega \ll \Delta\omega_0$. Then the change of the spectrum can be obtained by considering only the frequency diffusion of the photons:

$$\frac{1}{c}\frac{\partial \Phi_\omega}{\partial t} = \Gamma \frac{\partial^2 \Phi_\omega}{\partial \omega^2} + Q(\omega) N_0 \Phi_\omega, \tag{38}$$

where Γ is the frequency diffusion coefficient, defined in our model by the expression

$$\Gamma = (\delta\omega)^2 / 2\Lambda_s. \tag{39}$$

Let us find the limiting emission line width in this case.

The equation of the stationary state Φ_ω is of the form

$$\frac{\partial^2 \Phi_\omega}{\partial \omega^2} + Q_\omega \frac{N_0}{\Gamma} \Phi_\omega = 0. \tag{40}$$

Since $\delta\omega$ is much smaller than $\Delta\omega_0$ (the width of the resonance width Q_ω), the width of Φ_ω should also be much smaller than $\Delta\omega_0$. Consequently, the function Q_ω in (40) can be expanded in the center ω_0 of the line accurate to a quadratic term. For a Lorentz line shape (6), Eq. (40) then takes the form

$$\frac{\partial^2 \Phi_\omega}{\partial \omega^2} + \frac{N_0 Q_0}{\Gamma}\left[1 - \left(2\frac{\omega - \omega_0}{\Delta\omega_0}\right)^2\right] \Phi_\omega = 0, \tag{41}$$

which is perfectly analogous to the Schrödinger equation for the harmonic oscillator[15], where $Q_0 = Q_{\omega_0}$. The Φ_ω line shape can be determined by the eigenfunction of the ground state of (41), since this is the only function satisfying the condition $\Phi_\omega > 0$. Consequently,

$$\Phi_\omega = A \exp\left[-\left(\frac{\omega - \omega_0}{\Delta\omega_0}\right)^2 \sqrt{\frac{N_0 Q_0}{\Gamma}}\right], \tag{42}$$

or, substituting the values Γ and Λ_S:

$$\Phi_\omega = A \exp\left[-\frac{(\omega - \omega_0)^2}{\Delta\omega_0 \delta\omega} \sqrt{2\frac{Q_0}{Q_s}}\right], \tag{43}$$

where A is the normalization constant. The expression for the limiting width of the spectrum at the half-maximum $\Delta\omega_{min}$ is[3)]

$$\Delta\omega_{min} = (4 \ln 2 \, \Delta\omega_0 \delta\omega \sqrt{Q_s / 2Q_0})^{1/2}. \tag{44}$$

In the considered approximation, we did not take into account the spontaneous emission and the line broadening of the generation. It can be assumed that this contribution, as in ordinary lasers, is much smaller than the width (44) connected with the fluctuation motion of the feedback elements. The calculation of the generation line broadening due to the spontaneous emission should take into account the statistical properties of the emission of such a laser, which are distinctly different from the statistical properties of the emission of ordinary lasers.

7. CONCLUSION

From the point of view of the modes, a "stochastic resonator" in the form of a scattering medium constitutes a system with a large number of modes (waves of different directions), which are strongly coupled by scattering and which have large radiation losses. The large radiation losses and the strong interaction of the modes lead to a complete overlap of their frequency spectrum. The concept "mode" loses here its usual meaning and the spectrum becomes continuous. If the number N of the interacting "modes" is sufficiently large, the feedback becomes nonresonant[2]. The number of the "modes" N which are coupled by scattering is given by the expression

[3)]The condition for the validity of separately considering diffusion with respect to frequency and with respect to space is $\Delta\omega_{min} \ll \Delta\omega_0$.

$$N \simeq \Omega_{gen} / (\lambda / D)^2, \quad (45)$$

where Ω_{gen} is the generation solid angle and D is the diameter of the generation region. In the lasers described in[2,14] ($\Omega_{gen} \approx 10^{-3}$–$10^{-4}$ sr) we have $N \simeq 10^5$ and the feedback can be regarded as nonresonant[4)]. The larger the number of interacting "modes," the more effective the disintegration of the resonant properties. To obtain maximum stability of the generation frequency we must have lasers with maximum Ω_{gen}. Therefore the laser considered in the present article has a feedback which is nonresonant to the highest degree, since the solid angle of its generation is maximal ($\Omega_{gen} = 4\pi$). For example, when $D \approx 1$ cm at $\lambda \approx 10^{-4}$ cm the number of interacting "modes" reaches 10^8–10^9.

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[4)] However, at small generation angles, when N is small (for example, when $\Omega_{gen} \simeq (\lambda/D)^2$), the mode overlap is small and random "resonances" can appear at certain frequencies (C. H. Townes, private communication), but in this case the concept of nonresonant feedback becomes meaningless.

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