# STABILITY OF AN ELECTRON CURRENT WITH A VELOCITY GRADIENT

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We study phenomena arising when plasma layers slip over one another, using the simplest example of an electron current with a linear velocity profile in a strong longitudinal magnetic field. We show that in this case the plasma is stable and initial perturbations are damped in time according to a power law $t^{-\alpha}$ ($\alpha > 0$).

## INTRODUCTION

In this paper we consider the vibrations of a plane electron current moving along a magnetic field with a velocity changing linearly in the transverse direction. We assume that the magnetic field is sufficiently large and thus inhibits transverse motion of the plasma in the vibrations. Harrison[1] has studied the stability of such a flow in its simplest form. More complicated systems were considered in a number of other papers (see, e.g., [2-4]). The authors of these papers arrived at the conclusion that the slipping of plasma layers in flow with a variable velocity is a destabilizing factor. The instability observed by them for sufficiently large velocity gradients was called the slipping-instability.

At the same time the influence of slipping of layers of a continuous medium on its vibrations was studied in the theory of plane-parallel flow of an ideal liquid (see, e.g., [5]) and also in connection with "flute" vibrations of a rarefied plasma in an electric field.[6, 7] It was shown in those cases that for the simplest velocity profiles such slipping leads to the elimination of all undamped vibrations, i.e., to the stabilization of the current. (In hydrodynamics this statement of the content of Rayleigh's theorem.) The relatively general proof of this statement given in [7] can be transferred to the system considered here. However, it refers only to the short-wavelength (quasi-classical) vibrations, localized in the interior part of the plasma.

In the present paper we study vibrations with arbitrary wavelength. We find that for each value of $k_z$ ($k_z$ is the component of the wave vector along the magnetic field) there are up to two neutral eigenvibrations, the phase velocity of which is the same as the current velocity at its left-hand and right-hand boundaries, respectively, while the imaginary part of the frequency vanishes in the hydrodynamic approximation. We note that in the case of a plasma at rest there is a complete set of eigenvibrations for each value of $k_z$ and one can expand a perturbation with an arbitrary profile across the magnetic field in terms of them.

In the problem considered the eigenfunctions are not a complete set. We shall therefore follow [6, 8, 9] where a similar situation was studied and use a Laplace transform method to track directly the fate of the initial perturbations. As in the papers quoted above it turns out that the elementary excitations which form a complete set are the so-called modulated beams (similar to the van Kampen waves) moving with the local plasma velocity; they are described by the solutions with a discontinuous derivative. An arbitrary perturbation consisting of such beams is deformed in the course of time and spread out along the magnetic field which leads to a damping which asymptotically follows a power law $t^{-\alpha}$ ($0 < \alpha < 3/2$). Harrison's conclusion about the instability of flow with a linear velocity profile is thus incorrect. We indicate the mathematical errors which led to his conclusion.

When solving time-dependent problems by the Laplace transform method one assumes that the perturbation arises instantaneously at $t = 0$. We show that a current moving with a variable velocity "resonates" when a perturbation is suddenly switched on. As a result the damping law (the quantity $\alpha$) is changed in some cases. This effect was not taken into account in earlier papers (see [6, 8]).

## 1. BASIC EQUATIONS

We consider the vibrations of an electron current with a velocity changing in the transverse direction in the simplest case. In particular, we assume that the current is uniform in density and moves along a uniform and sufficiently large magnetic field. One shows easily that if the conditions $\omega_H \gg \omega_p$, $\omega_H \gg V_0'$ are satisfied, we need not take into account the displacement of the electrons across the magnetic field. Here $\omega_H$ is the electron cyclotron frequency, $\omega_p$ the plasma frequency, and $V_0'$ the velocity gradient of the current. We shall assume that the current is included between two conducting surfaces at a distance $2a$ from one another and that its velocity changes linearly, $V_0(x) = V_0'(x)(-a < x < a)$. The 0x axis is at right angles to the boundary surfaces, the 0z axis parallel to them along the magnetic field. It is clear that as the velocity gradient of the current decreases the vibrations of this system change to the usual Langmuir vibrations.

We want to trace the evolution in time of an arbitrary initial perturbation. To do this we apply as usual the Laplace transform method. The behavior of the system will be described by means of a kinetic equation linearized with respect to small perturbations and the Poisson equation. One finds easily from these equations

$$\left\{\frac{\partial^2}{\partial x^2} - k_z^2 - k_y^2 - \frac{\omega_p^2}{v_T^2}(1 + i\sqrt{\pi}\, sW(s))\right\}\varphi_{p,\mathbf{k}} = -4\pi e \int dv \frac{f_{\mathbf{k}}(x, v, 0)}{p + ik_z V_0' x + ik_z v}. \tag{1}$$

Here $\varphi_{p,\mathbf{k}}$ is the Laplace transform of the perturbed potential $\varphi_{\mathbf{k}}(t)$:

$$\varphi_{p,\mathbf{k}} = \int_0^\infty dt\, e^{-pt}\varphi_{\mathbf{k}}(t),$$

$f_k(x, v, 0)$ is the initial perturbation of the electron distribution function. (We assume the unperturbed distribution to be Maxwellian.) The spatial dependence of the perturbed quantities is chosen in accordance with the symmetry of the problem to be of the form $[\exp\{ik_z z + ik_y y\}]\ \varphi(x)$; W(s) is the probability integral of a complex argument:

$$W(s) = e^{-s^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^s e^{t^2}\,dt\right), \qquad s = \frac{ip - k_z V_0' x}{k_z v_T}$$

Owing to the Stokes phenomenon, the function W(s) has different asymptotic behavior in different regions of the complex argument s (see, e.g., [10]):

$$W(s) \underset{|s|\to\infty}{\to} e^{-s^2} + \frac{i}{\sqrt{\pi}}\frac{1}{s}\left(1 + \frac{1}{2s^2} + \ldots\right), \quad |\operatorname{Im} s| < |\operatorname{Re} s|;$$

$$W(s) \underset{|s|\to\infty}{\to} e^{-s^2}(1 - \operatorname{sign}\operatorname{Im} s) + \frac{i}{\sqrt{\pi}}\frac{1}{s}\left(1 + \frac{1}{2s^2} + \ldots\right), \quad |\operatorname{Im} s| > |\operatorname{Re} s|. \tag{2}$$

It follows from these expressions that effects connected with the finite temperature of the electrons are important not only when $|s| \lesssim 1$, but also when $|s| \gg 1$, if $-3\pi/4 < \arg s < -\pi/4$. In the other regions of the complex variable s the transition to the asymptotic behavior leads to the usual hydrodynamic equations

$$\left\{\frac{\partial^2}{\partial x^2} - k_z^2 - k_y^2 - \frac{k_z^2\omega_p^2}{(p + ik_z V_0' x)^2}\right\}\varphi_{p,\mathbf{k}} = -4\pi e\left\{\frac{n_\mathbf{k}(x,0)}{p + ik_z V_0' x} + \frac{ik_z V_\mathbf{k}(x,0)\, n_0}{(p + ik_z V_0' x)^2}\right\}. \tag{3}$$

Here $n_k(x, 0)$ and $V_k(x, 0)$ are, respectively, the initial perturbations of the plasma density and velocity.

To find the solution of the inhomogeneous Eqs. (1) and (3) we use a Green function in the following representation

$$G_{p,\mathbf{k},x_0}(x) = \frac{1}{W}\begin{cases} g^-_{p,\mathbf{k}}(x)\, g^+_{p,\mathbf{k}}(x_0), & (x < x_0) \\ g^+_{p,\mathbf{k}}(x)\, g^-_{p,\mathbf{k}}(x_0), & (x > x_0) \end{cases} \tag{4}$$

Here $g^{\pm}_{p,\mathbf{k}}(x)$ are the solutions of the homogeneous equations which satisfy the boundary conditions at the right-hand (left-hand) end of the interval (−a, a), respectively. These functions can be expressed in terms of the linearly independent solutions of the homogeneous equations

$$g^{\pm}_{p,\mathbf{k}}(x) = \varphi_{1,p,\mathbf{k}}(x)\,\varphi_{2,p,\mathbf{k}}(\pm a) - \varphi_{2,p,\mathbf{k}}(x)\,\varphi_{1,p,\mathbf{k}}(\pm a). \tag{5}$$

We denoted by W in Eq. (3) the functional determinant $W(g^+, g^-)$

$$W = [\varphi_{1,p,\mathbf{k}}(a)\,\varphi_{2,p,\mathbf{k}}(-a) - \varphi_{2,p,\mathbf{k}}(a)\,\varphi_{1,p,\mathbf{k}}(-a)] \times\left[\frac{\partial\varphi_{1,p,\mathbf{k}}}{\partial x}\varphi_{2,p,\mathbf{k}} - \frac{\partial\varphi_{2,p,\mathbf{k}}}{\partial x}\varphi_{1,p,\mathbf{k}}\right] = \text{const.} \tag{6}$$

If there are eigenfunctions of the homogeneous equations satisfying the boundary conditions at both ends, W vanishes. For such values of p the Green function has a pole. Using the Green function we can write the solution of Eq. (1) in the form

$$\varphi_{p,\mathbf{k}}(x) = \int_{-a}^{a} dx_0\, G_{p,\mathbf{k},x_0}(x)\, 4\pi e \int dv \frac{f_\mathbf{k}(x, v, 0)}{p + ik_z V_0' x + ik_z v}. \tag{7}$$

When $|s| \gg 1$, $3\pi/4 > \arg s > -\pi/4$ we get from Eq. (7)

$$\varphi_{p,\mathbf{k}}(x) = \int_{-a}^{a} dx_0\, G_{p,\mathbf{k},x_0}(x)\, 4\pi e\left\{\frac{n_\mathbf{k}(x_0,0)}{p + ik_z V_0' x_0} - \frac{ik_z V_\mathbf{k}(x_0,0)\, n_0}{(p + ik_z V_0' x_0)^2}\right\}. \tag{8}$$

The time dependence of the perturbed potential is determined by performing the inverse Laplace transformation:

$$\varphi_\mathbf{k}(x,t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} dp\, e^{pt}\varphi_{p,\mathbf{k}}(x). \tag{9}$$

## 2. THE EIGENVIBRATION PROBLEM

We consider first of all the eigenvibration problem in the hydrodynamic approximation. Following Harrison[1] we introduce in the homogeneous equation which corresponds to Eq. (3) a new function $\psi(r) = r^{-1/2}\varphi_{p,\mathbf{k}}(x)$, where $r = x - ip/k_z V_0'$,

$$\psi'' + \frac{1}{r}\psi' - k_\perp^2\psi - \frac{\nu^2}{r^2}\psi = 0. \tag{10}$$

Here $\nu^2 = 1/4 - \omega_p^2/V_0'^2$, $k_\perp^2 = k_z^2 + k_y^2$. We multiply (10) by $r\psi^*$ and integrate over the layer containing the plasma. Taking the boundary conditions into account and separating the real and imaginary parts of the integral relation obtained we find

$$\int_{-a}^{a} dx\, \operatorname{Re} r\left\{|\psi'|^2 + k_\perp^2|\psi|^2 + \frac{\nu^2}{|r|^2}|\psi|^2\right\} = 0, \tag{11}$$

$$\operatorname{Im}\omega\int_{-a}^{a} dx\left\{|\psi'|^2 + k_\perp^2|\psi|^2 - \frac{\nu^2}{|r|^2}|\psi|^2\right\} = 0. \tag{12}$$

If the plasma velocity changes sufficiently slowly $\omega_p^2 > \frac{1}{4}V_0'^2$ ($\nu^2 < 0$), (12) can be satisfied only by neutral vibrations with $\operatorname{Im}\omega = 0$. One sees easily that in the limiting case $\omega_p^2 \gg V_0'^2$ such vibrations change to the usual plasma vibrations with frequency

$$\omega^2 = k_z^2\omega_p^2 / k^2, \qquad k^2 = k_z^2 + k_y^2 + n^2(\pi / 2a)^2.$$

More interesting is the problem of the plasma vibrations when $\omega_p^2 < V_0'^2$ when $\nu^2 > 0$. (We note that $\nu^2 = \frac{1}{4} - \omega_p^2/V_0'^2 < \frac{1}{4}$.) In that case (12) allows the possibility of vibrations with $\operatorname{Im}\omega \neq 0$. However, in order to show that such vibrations exist indeed it is necessary to construct the eigenfunctions of Eq. (10) with $\operatorname{Im}\omega \neq 0$. Such an attempt was made in [1]. The Bessel functions were there expressed in the form $J_{\pm\nu}(r) = A_\pm r^{\pm\nu}F(r)$, where the function F(r) was chosen to be the same for $J_{\pm\nu}$ which is incorrect. Moreover, expressions in the dispersion relations were raised to an integral power, which led to the appearance of fictitious roots. All this was the cause of the incorrect conclusion about the instability of the plasma when $\omega_p^2 < \frac{1}{4}V_0'^2$.

We shall show in the following that when $\omega_p^2 < \frac{1}{4}V_0'^2$ there are no eigenfunctions of Eq. (10) with $\operatorname{Im}\omega > 0$. It follows from (11) that for eigenvibrations there is necessarily a point $x_c$ ($-a < x_c < a$) such that in it the phase velocity of the wave is the same as the local plasma velocity $\operatorname{Re}(\omega/k_z) = V_0' x_c$ ($\operatorname{Re} r = 0$). It follows from this that the eigenfunctions, if they exist, cannot vanish in more than two points. To prove this assertion it is sufficient to substitute as the limits of integration in (11) successively the coordinates of two adjacent points in which $\psi(r)$ vanishes according to the assumptions. However, owing to the monotonic behavior of the velocity profile, the relation $\operatorname{Re}(\omega/k_z) = V_0' x_c$

and with it also Eq. (11) can be satisfied only once.

We consider now the position of the zeroes of the function $\psi$ in more detail. An arbitrary solution of (10) can be written in the form

$$\psi = CJ_\nu(z_1) + J_{-\nu}(z_1), \tag{13}$$

where $z_1 = ik_1(x - ip/k_Z V_0')$. It has a branch point $z_1 = 0$. We must give the rule for going around this point, or what amounts to the same, the position of the cut in the plane of the complex variable $z_1$. To do this we note that it follows from (2) that the hydrodynamic Eqs. (3) and (10) are already invalid in the region

$$-3\pi/4 < \arg s < -\pi/4 \qquad (|\arg z_1| > 3\pi/4).$$

when $|s| \gg 1$. This region is shaded in the figure. Therefore, if we wish to use only hydrodynamic equations we must go around the point $z_1 = 0$ from the side of the positive values of Re $z_1$.

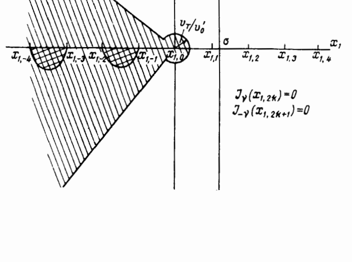
The position of the zeroes of the function $\psi$ in the complex plane of the variable $z_1$ is determined by the constant C in Eq. (13). The trajectory of the zeroes is for changing C given by the equations

$$CJ_\nu(z_1) + J_{-\nu}(z_1) = 0, \tag{14}$$

$$\frac{dz_1}{dC} = \frac{\pi}{2\sin(\nu\pi)} z_1 J_\nu^2(z_1). \tag{15}$$

If C = 0, the zeroes of the function $\psi$ are the same as those of $J_{-\nu}(z_1)$ and they are all on the real axis. The zeroes of $J_\nu$ lie also on the real axis while the zeroes of $J_\nu(z_1)$ and $J_{-\nu}(z_1)$ alternate.[11] Since we have for Im $z_1 = 0$, Re $z_1 > 0$ that Im $J_{\pm\nu}(z_1) = 0$, the zeroes of the function $\psi$ lying on the real positive semi-axis correspond to real values of the function C. The Bessel functions satisfy the equation $J_{\pm\nu}(z_1^*) = J_{\pm\nu}^*(z_1)$ so that complex conjugate values of C according to (14) correspond to $z_1^*$ and it is hence sufficient to consider the case Im C > 0. Equation (15) shows that when d Im C > 0 all the zeroes move from the real semi-axis Re $z_1 > 0$ into the upper half-plane. They only return to the real axis when Im C = 0. Therefore, if Im C ≥ 0 all zeroes starting from the semiaxis Im $z_1$, Re $z_1 > 0$ lie in the upper half-plane. However, to satisfy the boundary conditions two zeroes of the function $\psi$ must lie on the line Re $z_1$ = Re $k_1(ix + p/k_Z V_0')$ = const and, as was shown earlier, on different sides of the real axis.

We now trace the trajectories of the zeroes which for C = 0 are on the lower side of the cut Im $z_1 = 0$, Re $z_1 < 0$. These zeroes cannot go into the right-hand half-plane when Im C > 0. Indeed, to do that they must cross the imaginary semi-axis Im $z_1 < 0$, but on the axis we can write (14) as $C = -e^{i\nu\pi} I_{-\nu}(y_1)/I_\nu(y_1)$, whence follows Im C < 0. A more detailed study shows that when Im C > 0 the trajectories of the zeroes on the lower edge of the cut are distributed inside the shaded region (in the figure this region is indicated by double hatching), i.e., they are in the region where the hydrodynamic equations themselves are inapplicable.




We have thus shown that in the right-hand half-plane Re $z_1 > 0$ all zeroes lie on one side of the real axis and that there are therefore no eigenfunctions for Eq. (10) corresponding to increasing or stationary vibrations (Re p ≥ 0). We now recall that Eq. (10) was obtained from (3) by the substitution $\varphi = r^{1/2}\psi(r)$ where $r = x - ip/k_Z V_0'$. If $p = -ik_Z V_0' a$ at $x = a$, where $r = 0$, $\varphi$ vanishes even if $\psi = CJ_\nu + J_{-\nu} \neq 0$. If $p = ik_Z V_0' a$, then the situation is similar at $x = -a$. In order that for $p = -ik_Z V_0' a$ the function $\psi$ from (13) be an eigenfunction it is necessary to put the constant $C = -J_{-\nu}(-2iak_1)/J_\nu(2iak_1)$.

We found thus for each value of $k_Z$ up to two eigenfrequencies $\omega = \pm k_Z V_0' a$ with the corresponding eigenfunctions. Usually, for instance, in the problem of Langmuir vibrations in a plasma at rest there corresponds to each value of $k_Z$ a denumerable set of eigenfunctions in terms of which we can expand a perturbation with a given $k_Z$ and which depends arbitrarily on the other coordinates. We note that when $k_1 a \gg 1$ the eigenfunctions found by us decrease from the boundaries of the plasma exponentially $\varphi \approx e^{k_1(x \pm a)}$. This expression is obtained by using the asymptotic expressions for the Bessel functions (see, e.g., [11]).

It follows from the figure that when $p = \pm ik_Z V_0' a$ the boundary points fall in the shaded region where we must take into account effects connected with the fact that the electron temperature is finite (see Eq. (1)). Under the influence of these effects the zeroes of the eigenfunctions can shift in the complex plane over distances of the order $v_T/V_0'$. To this displacement there corresponds a change of the order of $k_Z v_T$ in the eigenfrequencies. It is thus possible that the eigenvibrations found by us are in fact increasing (damped) with small increments (decrements) $|\text{Im}\,\omega| \lesssim k_Z v_T$.

An exact use of (1) in the shaded region is difficult since the interior "potential" $\nu^2/x^2$ occurring in (3) is changed to the very complicated expression $\omega_p^2 v_T^{-2}(1 + i\sqrt{\pi} sW(s))$. Inside the shaded region W(s) has the asymptotic behavior $2e^{-s^2} \gg 1$ ($s = (ip - k_Z V_0' x)/k_Z v_T$). In this region the solution oscillates very fast and, possibly, vanishes. In this case vibrations with such values of Re $\omega$ and Im $\omega < 0$ ($|\text{Im}\,\omega| \gg k_Z v_T$) that at least one of the end points falls in the shaded area are damped. This condition is necessary since we showed earlier that if both points $x = \pm a$ lie outside the shaded region, damping of the vibration is also impossible. However, damped vibrations with $|\text{Im}\,\omega| \gg k_Z v_T$ even if they exist, must give a small contribution to the asymptotic time dependence of the initial perturbation (see next section).

## 3. EVOLUTION OF THE INITIAL PERTURBATIONS

The evolution in time of the perturbations is determined by Eqs. (7)–(9). We are interested in the asymptotic value of $\varphi_{\mathbf{k}}(x, t)$ as $t \to \infty$, which is well known to determine the singularities of the integrand in (9). Usually these are the poles of the Green function corresponding to the frequency eigenvalues. In our case, as in a number of other problems on plasma (liquid) vibrations with a variable velocity, the eigenfunctions do not form a complete set and there appears a new kind of elementary solutions—similar to van Kampen waves which determine the evolution of the initial perturbations.[6, 8, 9]

The integrand in (9) depends on the combinations $p + ik_zV_0'$, $p + ik_zV_0'x_0$, $p \pm ik_zV_0'a$ (see Eqs. (4)–(6)). It is thus convenient when studying (9) to use a figure in which we give the plane of the complex variable $ik_1x + k_1p/k_zV_0'$. In particular, the integration contour is depicted in that figure by a line parallel to the ordinate axis and lying in the right-hand half-plane. The Green function in (7) and (8) is constructed from the solutions of the homogeneous equations. The hydrodynamic Eq. (3) has a singularity at $x = ip/k_zV_0'$. Therefore, for given $x$ and $x_0$ the singularities of the Green function are in the points $p = -ik_zV_0'x$, $-ik_zV_0'x_0$, $\pm ik_zV_0'a$. These singularities are removed when one takes the thermal smearing into account, but it then turns out that the hydrodynamic approximation is valid only outside the shaded areas

$$|\arg(p + ik_zV_0'x)| < 3\pi/4,$$
$$|\arg(p + ik_zV_0'x_0)| < 3\pi/4$$

and so on. Simple considerations show that for not too large values of the time when $k_zv_Tt \ll 1$ we can use for the integrand the hydrodynamic approximation taking into account the rule for going around the branch point by a cut in the left-hand half-plane.

The complete evaluation of the asymptotic expressions is rather complicated. We demonstrate the calculation technique by estimating the contribution to the asymptotic behavior from the singularities at $p = -ik_zV_0'x$ and $p = -ik_zV_0'x_0$. We shall assume also that at $t = 0$ only the plasma density in the region $x_0 < x$ is perturbed. As the problem is linear the contributions to the asymptotic behavior from different perturbations can be evaluated separately. In the case considered we have

$$\varphi_{\mathbf{k}}(x,t) = \int_{-a}^{x} dx_0 \frac{4\pi e n_{\mathbf{k}}(x_0, 0)}{W} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt} \frac{\overset{+}{g}_{p,\mathbf{k}}(x)\overset{-}{g}_{p,\mathbf{k}}(x_0)}{p + ik_zV_0'x_0}. \quad (16)$$

We have changed here the order of integration as the integrals are absolutely converging for sufficiently large $\sigma$. The functions $g^{\pm}$ are constructed from the solutions of the homogeneous equation (see (5)) which we have chosen in the form

$$\varphi_{1,2}(q) = \frac{2^{\mp\nu}q^{1/2}}{\Gamma(\pm\nu + 1)} J_{\pm\nu}(q) \underset{q\to 0}{\approx} q^{1/2\pm\nu}.$$

It follows from the last expression that the largest contribution to the asymptotic behavior is given by the functions $\varphi_2(k_1p/k_zV_0' + ik_1x)$ and $\varphi_2(k_1p/k_zV_0' + ik_1x_0)$. When $|x - x_0| \ll a$, we have (see, e.g., [12])

$$\Phi_{\mathbf{k},x_0}(x,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt} \frac{\varphi_{2,p,\mathbf{k}}(x)\varphi_{2,p,\mathbf{k}}(x_0)}{p + ik_zV_0'x_0}$$

$$\approx \frac{\sqrt{\pi}}{2\Gamma(\nu + 1/2)} t^{\nu}(k_zV_0'(x - x_0))^{-\nu+1} \exp\left\{-\frac{1}{2}k_zV_0'(x + x_0)t\right\}$$
$$\times \left\{J_{\nu-1}\left(\frac{1}{2}k_zV_0'(x - x_0)t\right) + iJ_{\nu}\left(\frac{1}{2}k_zV_0'(x - x_0)t\right)\right\}. \quad (17)$$

If $|k_zV_0'(x - x_0)t| \ll 1$, we may assume the singularities to be confluent, and to evaluate the asymptotic behavior we can put $x = x_0$. Indeed, in that case we have from (17)

$$\Phi_{\mathbf{k},x_0}(x,t) \approx \frac{t^{2\nu-1}}{\Gamma(2\nu)} \exp\{-ik_zV_0'xt\}. \quad (18)$$

In the opposite limiting case $|k_zV_0'(x - x_0)t| \gg 1$ the contributions from the singularities in the points $p = -ik_zV_0'x$ and $p = -ik_zV_0'x_0$ separate and we can calculate them separately:

$$\Phi_{\mathbf{k},x_0}(x,t) \approx \frac{1}{\Gamma(\nu + 1/2)} \left(\frac{t}{ik_zV_0'(x_0 - x)}\right)^{\nu-1/2} \exp\{-ik_zV_0'x_0t\}$$
$$+ \frac{1}{\Gamma(\nu - 1/2)} (ik_zV_0'(x_0 - x))^{-\nu+1/2} t^{\nu-3/2} \exp\{-ik_zV_0'xt\}. \quad (19)$$

Equations (17)–(19) are valid for $|x_0 - x| \ll a$ and it is clear that expressions such as (19) can be obtained also when $|x - x_0| \sim a$.

The asymptotic behavior of $\varphi_{\mathbf{k}}(x, t)$ in the point $x$ is thus made up of perturbations of two types with frequencies $\omega = k_zV_0'x_0$ and $\omega = k_zV_0'x$, respectively. Here, as before, $x_0$ is the "source" position (see (3), (8)) and $x$ the observation point. The contributions of these perturbations are separated when $|k_zV_0'(x - x_0)t| \gg 1$. A perturbation of the first kind is caused by a plane layer of electrons localized at the point $x_0$. This layer is modulated with a wavevector $\mathbf{k} = \{0, k_y, k_z\}$ and moves with a velocity $V_{0z}(x_0) = V_0'x_0$. It excites a wave with a laboratory-system frequency $\omega = k_zV_0'x_0$. The field of such a layer is described by the Green function $G_{p,\mathbf{k},x_0}(x)$.

Perturbations of the second kind with a frequency $\omega = k_zV_0'x$ are to a well-defined degree connected with the use of the Laplace transformation. Indeed, solving the problem with the initial data, using the Laplace transforms, we assume that the perturbations arise instantaneously at $t = 0$. The expansion of a discontinuous function of the time in a Fourier integral contains all frequencies so that when suddenly a modulated layer of electrons with coordinate $x = x_0$ occurs the whole spectrum of frequencies must, generally speaking, be excited and not only the frequency corresponding to the local drift velocity $\omega = k_zV_0'x_0$. In accordance with this the right-hand side of (3) will be non-vanishing, even for perturbations localized at $x_0$, for arbitrary values of $p$, although perturbations with $p = -ik_zV_0'x_0$ occur with the largest weight. At a point $x \neq x_0$ the mutual interference of the field with $\omega \neq k_zV_0'x$ leads to their destruction after a time $t \approx (k_zV_0'(x - x_0))^{-1}$. However, perturbations with a frequency $\omega \approx k_zV_0'x$ resonate with the motion of the plasma at the point $x$ and as a result this point is a branch point for the field of a perturbation with $\omega = k_zV_0'x$ (see (10), (13)). The branch point also contributes to the asymptotic behavior (19).

We note that in [6, 8], where similar problems were considered, only perturbations of the first kind were taken into account. However, perturbations of the second kind give, generally speaking, similar and in some cases even larger contributions to the asymptotic behavior. Its exact magnitude must, in view of what we have said earlier, be determined by the concrete details of the process of switching on the initial perturbation. It is thus inexpedient to evaluate exactly the asymptotic expressions in the present case.

Perturbations of the first kind enter into the asymptotic behavior through a Fourier integral of the kind

$$\int\limits_{-a} f(x, x_0)\, e^{-ik_z V_0' x_0 t} dx_0.$$

In accordance with [13] its asymptotic behavior is for $k_z V_0' at \gg 1$ also determined by the branch point of the function $f(x, x_0)$ when $x = x_0$ (see (19)). Evaluating the Fourier integral we find that the contribution from perturbations of the first kind are damped in time as $t^{2\nu-2}$, where $0 < \nu < 1/2$ (see the preceding section). The contribution from perturbations of the second type can also easily be estimated. It turns out that in the given case it just determines the asymptotic behavior of $\varphi_{\mathbf{k}}(x, t)$:

$$\varphi_{\mathbf{k}}(x, t) \approx \frac{4\pi e}{k_1^2} n_{\mathbf{k}}(0) \left(\frac{k_z}{k_1} V_0' t\right)^{\nu-3/2} \exp\{-ik_z V_0' x t\}. \quad (20)$$

One finds similarly the part of the asymptotic behavior of $\varphi_{\mathbf{k}}(x, t)$ which is connected with a perturbation of the velocity $V_{\mathbf{k}}(x_0, 0)$. In that case the contribution from perturbations of the first and second kind are equal in order of magnitude:

$$\varphi_{\mathbf{k}}(x, t) \approx \frac{4\pi e}{k_1} \frac{V_{\mathbf{k}}(0) n_0}{V_0'} \left(\frac{k_z}{k_1} V_0' t\right)^{2\nu-1} \exp\{-ik_z V_0' x t\}. \quad (21)$$

In Eqs. (20) and (21) $n_{\mathbf{k}}(0)$ and $V_{\mathbf{k}}(0)$ denote certain average values of the initial perturbations of the plasma density and its velocity.

We found in the preceding section that to each value of k there correspond two eigenfunctions with frequencies $\omega = \pm k_z V_0' a$. For those frequencies the functional determinant W vanishes and the Green function becomes infinite. However, although usually the poles of the Green function correspond to the eigenfrequencies, here we have a branch point. This singularity also makes a contribution to the asymptotic behavior of $\varphi_{\mathbf{k}}(x, t)$ and when evaluating it it is necessary to take into account that the functions $g^{\pm}$ occurring in the numerator of the Green function also have branch points for $p = \pm ik_z V_0' a$. The total contribution of these singularities to the asymptotic behavior is to order of magnitude equal to

$$\varphi_{\mathbf{k}}(x, t) \approx \frac{4\pi e}{k_1^2} \left[n_{\mathbf{k}}(0) + \frac{k_1 V_{\mathbf{k}}(0) n_0}{V_0'}\right] \left(\frac{k_z}{k_1} V_0' t\right)^{-2\nu-1} \exp\{\mp ik_z V_0' a t\}. \quad (22)$$

If initially the plasma velocity is perturbed near the boundaries $|k_z V_0'(x_0 \pm a)t| \lesssim 1$, the damping becomes much slower. In that case the combination $k_z k_1^{-1} V_0' t$ occurs in the asymptotic behavior with the power $-\nu - 1/2$. Finally, if the perturbations are non-vanishing near the boundaries and the observation occurs in the same region $|k_z V_0'(x \pm a)t| \ll 1$, we have for $\varphi_{\mathbf{k}}(x, t)$

$$\varphi_{\mathbf{k}}(x, t) \approx \frac{4\pi e}{k_1} (x \pm a) \left[n_{\mathbf{k}}(0) \left(\frac{k_z V_0' t}{k_1}\right)^{-1} + \frac{k_1 V_{\mathbf{k}}(0) n_0}{V_0'}\right] e^{\mp ik_z V_0' a t}. \quad (23)$$

In conclusion we note that very short wavelength perturbations with $k_1 a \gg 1$ are localized near the region of the initial emergence over distances of the order of $\delta x \approx k_1^{-1} \ll a$.

Over large time intervals, when $k_z v_T t \gg 1$ small ranges of p ($\delta p \approx k_z v_T$) and x ($\delta x \approx v_T/V_0'$) become important. In that case it is necessary to use the kinetic expressions (see (1), (7)). When taking the thermal spread into account the integrand in (9) becomes analytic and therefore the asymptotic behavior of $\varphi_{\mathbf{k}}(x, t)$ is damped faster than any power of $t^{-1}$. Thus, for instance,

$$\frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt} \int dv \frac{e^{-mv^2/2T}}{p + ik_z v} = \sqrt{\frac{2\pi T}{m}} \exp\left\{-k_z^2 \frac{2T}{m} t^2\right\}.$$

The eigenvibrations found in the preceding section had a frequency $\omega = \pm k_z V_0' a$. When kinetic effects are considered the values of the frequencies may change by quantities of the order of $k_z v_T$. Not excluded is the possibility that these vibrations become increasing with $\mathrm{Im}\,\omega \lesssim k_z v_T$. We must note that when $k_1 a \gg 1$ such vibrations, if they exist, will be localized near the boundaries of the plasma at distances of the order of $\delta x \approx k_1^{-1}$ and thus will not influence the evolution of perturbations inside the plasma.

## CONCLUSION

We have thus studied the stability of an electron current with a linear velocity profile in a strong longitudinal magnetic field. We showed in Sec. 2 that in the hydrodynamic approximation the imaginary part of the frequency of the eigenvibrations equals zero in the case considered and that the corresponding eigenfunctions do not give a complete set. In Sec. 3 we used the Laplace transform method to solve the problem of the evolution of arbitrary initial perturbations. We found that initial perturbations are damped asymptotically as $t^{-\alpha}$ ($0 < \alpha < 3/2$) and that thus the current considered is stable.

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