RADIATION TRANSPORT EQUATION FOR CORRELATED SCATTERERS

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A derivation of the equation for radiation transport in scattering media is presented. Underlying the derivation are the general equations for the average over an ensemble of values of the Green function for a scalar wave field (Dyson equation) and the bilinear combination of two Green functions (Bethe-Salpeter equation). The mass operator and intensity operator (kernels of the Dyson and Bethe-Salpeter equations) are represented in the single-group approximation in which wave scattering effects by groups of correlated scatterers are taken into account. The extinction and scattering coefficients of the transport equation can be directly expressed in terms of the Fourier transform of the mass and intensity operators evaluated on the energy shell. The conditions of applicability of the equation are discussed.

1. INTRODUCTION

THE radiation transport equation plays an important role in the theory of multiple scattering of waves with its numerous applications, such as radiation equilibrium in stars, diffusion of thermal neutrons, resistivity of conductors, scattering of light by fluctuations of material media, etc. For scalar waves the transport equation is of the form

$$(\mathbf{s}\nabla_{\mathbf{x}})I(\mathbf{x},\mathbf{s}) = -\varkappa I(\mathbf{x},\mathbf{s}) + \int f(\mathbf{s},\mathbf{s}')I(\mathbf{x},\mathbf{s}')d^2\mathbf{s}', \qquad (1)$$

where $I(\mathbf{x}, \mathbf{s})$ is the ray intensity of the wave field at the point $\mathbf{x}$ and in the direction of the unit vector $\mathbf{s}$, κ is the extinction coefficient and $f(\mathbf{s}, \mathbf{s}')$ is the scattering coefficient.

The transport equation (1) was first formulated by Khvol'son, and then by Schwarzschild. In its present form the equation was independently obtained by Chandrasekhar,[1] Rozenberg,[2] and somewhat earlier by Sobolev[3] who considered the special case of Rayleigh scattering. The derivation of the equation by these authors was phenomenological or semi-phenomenological and was based on energy balance considerations. No explicit microscopic interpretation of the extinction and scattering coefficients that enter the equation was given. For nonabsorbing media it was only required that the relation

$$\varkappa = \int f(\mathbf{s},\mathbf{s}')d^2\mathbf{s}, \qquad (2)$$

be satisfied; this ensured that the principle of conservation of energy was satisfied for Eq. (1). The coefficients κ and f are sometimes related with the total and differential cross sections for scattering by an isolated scatterer. In this case relation (2) follows from the optical theorem for an isolated scatterer[4].

A number of works have appeared in recent years in which attempts have been made to provide a derivation of the transport equation and to evaluate the limits of its applicability. Dolin[5] considered the scattering of narrow beams of electromagnetic waves by large-scale fluctuations of the index of refraction of the medium, assuming the waves to be small and the fluctuations weak (κ and f are quadratic in the fluctuations). Even earlier Gnedin and Dolginov[6] considered the problem of deriving the transport equation in their study of quantum mechanical scattering by a system of independent scattering centers of a target. Later Borovoĭ[7] and one of the authors[8] derived Eq. (1) in the model of independent scatterers. An electromagnetic wave field was considered in[7] and a scalar one in[8].

Here we wish to consider a derivation of the transport equation which generalizes the results of the cited papers above all in that it does not assume independence of the scatterers or a normal distribution law of the fluctuations of the index of refraction of the medium. We shall obtain expressions for the coefficients of extinction κ and scattering f; these expressions include effects of correlation and multiple scattering of waves. The importance of taking into account cooperative effects in the transport equation has been noted by Rozenberg.[9] However, so far this problem has not been solved in general form.

Simultaneously with the derivation of the equation we refine the conditions for its applicability obtained in[5-8]. Some of these conditions must be replaced by more stringent ones, whereas with some it will be possible to dispense.

2. INITIAL EQUATIONS

We take as the initial equations the general equations which are satisfied by the Green's function averaged over the ensemble $\mathcal{G}(\mathbf{x}, \mathbf{x}') = \langle G(\mathbf{x}, \mathbf{x}')\rangle$ and by the average double Green's function $\mathcal{W}(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}') = \langle G(\mathbf{x}, \mathbf{x}')\overline{G}(\mathbf{y}, \mathbf{y}')\rangle$ of the scalar wave field $\psi(\mathbf{x})$. The unaveraged Green's function $G(\mathbf{x}, \mathbf{x}')$ satisfies the equation

$$(\Delta_{\mathbf{x}} + k_0^2 + V(\mathbf{x}))G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'). \qquad (3)$$

$V(\mathbf{x})$ denotes the random effective potential which characterizes the spatial distribution of the potential energy of the force centers in the quantum mechanical problem, and the distribution of the index of refraction in the acoustic problem. Analogously the constant k_0^2 represents the energy of the particle E ($\hbar^2/2m = 1$) or the square of the wave number of free space. The equations for the average Green's functions are by analogy with

quantum electrodynamics called the Dyson equation[1]:

$$\mathcal{G} = G_0 + G_0 M \mathcal{G} \tag{4}$$

and the Bethe-Salpeter equation

$$\mathcal{W} = (\mathcal{G} \otimes \overline{\mathcal{G}}) + (\mathcal{G} \otimes \overline{\mathcal{G}}) K \mathcal{W}. \tag{5}$$

Two new operators, M—the mass operator, and K—the intensity operator, enter in these equations.

In the theory of multiple scattering of waves Eqs. (4) and (5) were first formulated by Foldy[10] for a model of isotropic noncorrelated point scatterers, and by Bourret[11] for scattering by fluctuations of a random effective potential $V(\mathbf{x})$ with a normal distribution. The operators M and K were calculated by Foldy in the first approximation in the concentration n of the scatterers, and by Bourret also in the first approximation in the correlation function $B(\mathbf{x}, \mathbf{x}') = \langle V(\mathbf{x})V(\mathbf{x}')\rangle$, $\langle V\rangle = 0$. Subsequently Frisch extended Eqs. (4) and (5) to the case of a potential $V(\mathbf{x})$[12] with an arbitrary distribution and arbitrary correlations between the locations of discrete scatterers.[3] However, Frisch did not generalize the approximations for the operators M and K adopted by Foldy and Bourret to the case considered by him. Such a generalization along with a refinement of the conditions of applicability of the corresponding approximations has been carried out by one of the authors.[14]

According to the results obtained in[14], the mass operator M and the intensity operator K can under the conditions specified there be replaced by the first terms M_1 and K_1 of their expansions in the number of their correlation groups. In the case of discrete scatterers the operators M_1 and K_1 include effects of scattering of waves by groups of correlated scatterers. The number of scatterers in the group can be arbitrary. In the case of a continuous scattering medium these operators have the same meaning and the role of the scatterers is taken on by fluctuations of the potential. We call the operators M_1 and K_1 single-group operators, since they take into account only single scattering of waves by correlation groups. These operators for independent scatterers coincide with the approximation of Foldy, and Gnedin and Dolginov, and in the case of a potential with a normal distribution—with Bourret's approximation.

It should be noted that the single-group operators M_1 and K_1 have the important property of compactness. It consists in the fact that their kernels tend to zero when the distance between their arguments becomes large compared with the correlation length l. The rate of decrease of the kernels is determined by the behavior of the correlation functions for large distances between their arguments.

For the purpose of bringing the presentation below close to the terminology accepted for the case of independent scatterers, we introduce the concept of the specific operators M_0 and K_0. We shall specify these operators by the relations

$$M_1 = \int M_0(\mathbf{x}_1)\, d^3\mathbf{x}_1, \qquad K_1 = \int K_0(\mathbf{x}_1)\, d^3\mathbf{x}_1, \tag{6}$$

where $\mathbf{x}_1$ is the coordinate of the center of one of the scatterers of the correlation group. The operators $M_0(\mathbf{x}_1)$ and $K_0(\mathbf{x}_1)$ are determined with the aid of expressions obtained in[14] in which one must however consider the coordinate $\mathbf{x}_1$ fixed. In the case of independent scatterers the specific operators $M_0(\mathbf{x}_1)$ and $K_0(\mathbf{x}_1)$ go over into $nT(\mathbf{x}_1)$ and $nT(\mathbf{x}_1) \otimes \overline{T}(\mathbf{x}_1)$ where $T(\mathbf{x}_1)$ is the scattering operator of an isolated scatterer with its center at the point $\mathbf{x}_1$. This circumstance makes it possible to interpret the operators M_0 and K_0 as scattering operators of certain independent effective inhomogeneities of the medium.

3. CALCULATION OF THE AVERAGE GREEN'S FUNCTION

In the case of a statistically homogeneous medium the Dyson equation (4) is solved by a Fourier transformation. The Fourier transform of the average Green's function $\hat{\mathcal{G}}(\mathbf{p})$ turns out to be

$$\check{\mathcal{G}}(\mathbf{p}) = \frac{1}{k_0^2 - p^2 - \check{M}_1(\mathbf{p})}. \tag{7}$$

For an isotropic medium the Fourier transform of the mass operator $\check{M}_1(\mathbf{p}) = \check{M}_0(p, \mathbf{p})$ does not depend on the direction of the vector $\mathbf{p}$.

The conditions for the applicability of the single-group approximation for the mass operator[14] allow one to make without loss of accuracy an appreciable simplification in calculating the Green's function $\mathcal{G}$. It turns out to be possible to replace in expression (7) the Fourier transform $\check{M}_1(\mathbf{p})$ by its value on the energy shell $M_1(k_0)$. Formally this replacement is explained as follows. Seeking out the pole of expression (7) closest to $p = k_0$ and taking into account the inequality[14] $|\check{M}_1(k_0)| \ll k_0^2$, we solve the equation

$$p^2 = k_0^2 - \check{M}_1(p) \tag{8}$$

by an iterative method. Taking into account the second iteration

$$p^2 \cong k_0^2 - \check{M}_1(k_0) + (d\check{M}_1 / dp^2)_{p=k_0} \check{M}_1(k_0). \tag{9}$$

If the derivative is small

$$|d\check{M}_1 / dp^2|_{p=k_0} \ll 1, \tag{10}$$

then we can confine ourselves in (9) to the first two terms. At the same time the fact that the derivative (10) is small follows from the conditions of applicability of the single-group approximation.

For the following it is useful to explain the meaning of the described replacement of the Fourier transform of the mass operator by its value on the energy shell from the point of view of coordinate space. Calculating the Fourier integral, and omitting at the same time the third term in (9), we obtain

$$\mathcal{G}(\mathbf{x} - \mathbf{x}') \cong -\exp(ik_{eff}|\mathbf{x} - \mathbf{x}'|) / 4\pi|\mathbf{x} - \mathbf{x}'|. \tag{11}$$

The effective wave number is

$$k_{eff} \cong k_0 - \check{M}_1(k_0) / 2k_0. \tag{12}$$

Let us now turn to the Dyson equation (4) and represent its solution in the form of an iterative series. Let us consider, for example, the third term of the series $G_0M_1G_0M_1G_0$ which we shall, for the sake of clarity, depict by the diagram in Fig. 1, denoting the repeated arguments of the operators by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$. First

[1]We write these equations in symbolic operator form. $G_0(\mathbf{x}-\mathbf{x}') = -\exp(ik_0|\mathbf{x}-\mathbf{x}'|)/4\pi|\mathbf{x}-\mathbf{x}'|$ denotes the retarded Green's function of Eq. (3) for V = 0 (free space). The sign $\otimes$ denotes the direct or tensor product of two operators.

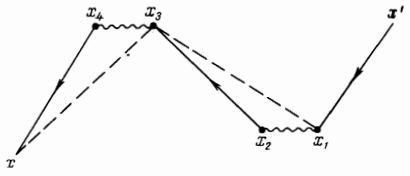

FIG. 1.

we carry out the integration over $\mathbf{x}_4$, assuming all other points to be fixed. By virtue of the compactness of the operator $M_1(\mathbf{x}_4 - \mathbf{x}_3)$ the positions of the point $\mathbf{x}_4$ and of the center of the corresponding inhomogeneity are localized in the neighborhood of the point $\mathbf{x}_3$ with a linear dimension of the order of the correlation length l. If the point of observation $\mathbf{x}$ lies in the region of Fraunhofer diffraction with respect to the above inhomogeneity, then the Green's function $G_0(\mathbf{x} - \mathbf{x}_4)$ can approximately be represented in the form

$$G_0(\mathbf{x} - \mathbf{x}_4) \sim G_0(\mathbf{x} - \mathbf{x}_3) \exp[-ik_0 \mathbf{s}_{\mathbf{x}\mathbf{x}_3}(\mathbf{x}_4 - \mathbf{x}_3)], \quad (13)$$

where the unit vector $\mathbf{s}_{\mathbf{x}\mathbf{x}}$ is directed along the vector $\mathbf{x} - \mathbf{x}_3$. Proceeding analogously in integrating over $\mathbf{x}_2$, we obtain for the iterative term under consideration the expression $\check{M}_1^2(k_0) G_0 G_0 G_0$. Summing all the terms of the iterative series transformed in such a way, we again arrive at the Green's function (11).

4. TRANSFORMATION OF THE BETHE-SALPETER EQUATION INTO THE TRANSPORT EQUATION

Equation (5) can be given a graphic physical meaning if one represents its solution in the form of an iterative series and introduces the specific intensity operator K_0. Let us consider, for example, the third term of the series

$$(\mathcal{G} \otimes \overline{\mathcal{G}}) K_0 (\mathcal{G} \otimes \overline{\mathcal{G}}) K_0 (\mathcal{G} \otimes \overline{\mathcal{G}}), \quad (14)$$

in which the integration is carried out both over all repeated arguments of the operators and over the centers of two inhomogeneities. We represent the given term by the diagram of Fig. 2, associating the upper lines with the operators $\mathcal{G}$, the lower ones with $\overline{\mathcal{G}}$, the circles with four points with the operators K_0, and the fifth point within the circles with an inhomogeneity center. The diagram describes the successive scattering of waves by two inhomogeneities.

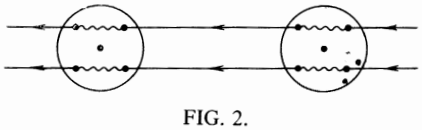

FIG. 2.

We substitute in (14) the Green's function (11). We shall make use of the graphic concept of the Fraunhofer diffraction region presented in the previous section, and write the Green's functions $\mathcal{G}$ in expression (14) approximately, in analogy with (13), expanding the exponents of their exponentials in a series relative to the inhomogeneity centers. Then, integrating over all repeated arguments of the operators, we arrive at a transformed expression of the form

$$\int d^3\mathbf{x}_2 \int d^3\mathbf{x}_1 \mathcal{G}(\mathbf{x} - \mathbf{x}_2) \mathcal{G}(\mathbf{y} - \mathbf{x}_2) \times \check{K}_0(k_0 \mathbf{s}_{\mathbf{x}\mathbf{x}_2}, k_0 \mathbf{s}_{\mathbf{x}_2\mathbf{x}_1}; k_0 \mathbf{s}_{y\mathbf{x}_2}, k_0 \mathbf{s}_{\mathbf{x}_2\mathbf{x}_1}) |\mathcal{G}(\mathbf{x}_2 - \mathbf{x}_1)|^2 \times \check{K}_0(k_0 \mathbf{s}_{\mathbf{x}_2\mathbf{x}_1}, k_0 \mathbf{s}_{\mathbf{x}_1\mathbf{x}'}; k_0 \mathbf{s}_{\mathbf{x}_2\mathbf{x}_1}, k_0 \mathbf{s}_{\mathbf{x}_1 y'}) \mathcal{G}(\mathbf{x}_1 - \mathbf{x}') \overline{\mathcal{G}}(\mathbf{x}_1 - \mathbf{y}'), \quad (15)$$

where $\check{K}_0$ is the Fourier transform of the specific intensity operator K_0. Expression (15) corresponds to the diagram of Fig. 3 which is obtained from the preceding by making each foursome of points coincide with the inhomogeneity center.

FIG. 3.

Further transformations of expression (15) can be carried out by a method presented in [8]. Here we modernize the method of that paper with the aid of symbolic operator language. With this in mind, we introduce the functional space of the point $\mathbf{x}$ and direction $\mathbf{s}$, and in this space—the operators

$$F_0(\mathbf{x}, \mathbf{s};\ \mathbf{x}', \mathbf{s}') = |\mathcal{G}(\mathbf{x} - \mathbf{x}')|^2 \delta\left(\mathbf{s} - \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}\right) \delta(\mathbf{s} - \mathbf{s}'), \quad (16)$$

$$\tau(\mathbf{x}, \mathbf{s};\ \mathbf{x}', \mathbf{s}') = \delta(\mathbf{x} - \mathbf{x}') \check{K}_0(k_0 \mathbf{s}, k_0 \mathbf{s}';\ k_0 \mathbf{s}, k_0 \mathbf{s}'). \quad (17)$$

In addition, we introduce the operator

$$R(\mathbf{z}, \mathbf{s};\ \mathbf{x}', \mathbf{y}') = \check{K}_0(k_0 \mathbf{s}, k_0 \mathbf{s}_{\mathbf{z}\mathbf{x}'};\ k_0 \mathbf{s}, k_0 \mathbf{s}_{\mathbf{z}\mathbf{y}'}) \mathcal{G}(\mathbf{z} - \mathbf{x}') \overline{\mathcal{G}}(\mathbf{z} - \mathbf{y}'), \quad (18)$$

which transforms a function of the points $\mathbf{x}'$ and $\mathbf{y}'$ into a function of the point $\mathbf{z}$ and direction $\mathbf{s}$. With the aid of the F_0 and R operators expression (15) can be written in abbreviated form as $\tilde{R} F_0 R$ where $\tilde{R}$ denotes a transposed operator that transforms a function of the point $\mathbf{z}$ and direction $\mathbf{s}$ into a function of the points $\mathbf{x}$ and $\mathbf{y}$.

Other terms of the iterative series of Eq. (5) are transformed in a perfectly analogous way. In doing this, in the second term $(\mathcal{G} \otimes \overline{\mathcal{G}}) K_0 (\mathcal{G} \otimes \overline{\mathcal{G}})$ the operator K_0 is replaced by its Fourier transform $\check{K}_0$ with arguments of the type $k_0 \mathbf{s}$. It corresponds to the diagram of Fig. 4 The fourth term will be transformed into $\tilde{R} F_0 \tau F_0 R$. The corresponding diagram is shown in Fig. 5.

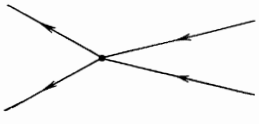

FIG. 4.

FIG. 5.

After the described transformation, we sum all the terms of the iterative series under consideration. As a result we obtain for the double Green's operator W the representation

$$\mathcal{W} = (\mathcal{G} \otimes \overline{\mathcal{G}}) + (\mathcal{G} \otimes \overline{\mathcal{G}}) \check{K}_0 (\mathcal{G} \otimes \overline{\mathcal{G}}) + \tilde{R} F R. \quad (19)$$

F denotes a new operator that appears in the summing. It satisfies the equation

$$F = F_0 + F_0 \tau F. \quad (20)$$

The operator F plays an important role in the derivation of the transport equation and has a simple physical meaning. Let us note its principal properties.

First, it satisfies the integro-differential equation

$$(\mathbf{s} \nabla_{\mathbf{x}}) F(\mathbf{x}, \mathbf{s};\ \mathbf{x}', \mathbf{s}') = -\varkappa F + \int d^2\mathbf{s}'' f(\mathbf{s}, \mathbf{s}'') F(\mathbf{x}, \mathbf{s}'';\ \mathbf{x}', \mathbf{s}') + \frac{1}{16\pi^2} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{s} - \mathbf{s}'). \quad (21)$$

The coefficients κ and f are

$$\varkappa = -\operatorname{Im} \check{M}_0(k_0, k_0) / k_0;$$

$$f(\mathbf{s}, \mathbf{s}') = \frac{\hat{}}{16\pi^2} \check{K}_0(k_0\mathbf{s}, k_0\mathbf{s}';\ k_0\mathbf{s}, k_0\mathbf{s}') \qquad (22)$$

Equation (21) is obtained from the integral equation (20) by differentiation.

Second, the field intensity operators

$$J(\mathbf{x}; \mathbf{x}', \mathbf{y}') = \mathscr{W}(\mathbf{x}, \mathbf{x}'; \mathbf{x}, \mathbf{y}')$$

and the flux densities

$$\mathbf{\Pi}(\mathbf{x}; \mathbf{x}', \mathbf{y}') = \frac{k_0}{2i}(\nabla_{\mathbf{x}} - \nabla_{\mathbf{y}})_{\mathbf{x}=\mathbf{y}} \mathscr{W}(\mathbf{x}_1, \mathbf{x}';\ \mathbf{y}_1, \mathbf{y}')$$

are expressed in terms of the operator F. The expressions are of the form

$$J(\mathbf{x}; \mathbf{x}', \mathbf{y}') = J_0(\mathbf{x}; \mathbf{x}', \mathbf{y}') + \int d^2\mathbf{s}\, I(\mathbf{x}, \mathbf{s}; \mathbf{x}', \mathbf{y}'), \qquad (23)$$

$$\mathbf{\Pi}(\mathbf{x}; \mathbf{x}', \mathbf{y}') = \mathbf{\Pi}_0(\mathbf{x}; \mathbf{x}', \mathbf{y}') + \int d^2\mathbf{s}\, \mathbf{s} I(\mathbf{x}, \mathbf{s}; \mathbf{x}', \mathbf{y}'), \qquad (24)$$

$$I(\mathbf{x}, \mathbf{s}; \mathbf{x}', \mathbf{y}') = \int F(\mathbf{x}, \mathbf{s}; \mathbf{x}'', \mathbf{s}'') d^3\mathbf{x}'' \int d^2\mathbf{s}'' R(\mathbf{x}'', \mathbf{s}''; \mathbf{x}', \mathbf{y}'), \qquad (25)$$

where J_0 and Π_0 are determined by the first term $(\mathscr{G} \otimes \overline{\mathscr{G}})$ of representation (19). Let us multiply Eqs. (23)–(25) by $j(\mathbf{x}')\overline{j}(\mathbf{y}')$ where j is the field source, and integrate over $\mathbf{x}'$ and $\mathbf{y}'$. We obtain the field intensity $J(\mathbf{x})$, the flux density $\Pi(\mathbf{x})$, and the function $I(\mathbf{x}, \mathbf{s})$ of the point $\mathbf{x}$ and direction $\mathbf{s}$. It is natural to take this function to be the ray intensity of the scattered radiation in the sense that it satisfies the transport equation (1) with a source

$$Q(\mathbf{x}, \mathbf{s}) = \frac{1}{16\pi^2} \int R(\mathbf{x}, \mathbf{s}; \mathbf{x}', \mathbf{y}') j(\mathbf{x}') \bar{j}(\mathbf{y}') d^3\mathbf{x}' d^3\mathbf{y}' \qquad (26)$$

and is related to the field intensity $J(\mathbf{x})$ and the flux density $\Pi(\mathbf{x})$ by the relations usual in the phenomenological transport theory.

5. THE TRANSPORT EQUATION AND THE OPTICAL THEOREM

We have derived the transport equation (1) with the extinction and scattering coefficients (22). These coefficients should be related by relation (2). This relation is a consequence of the general optical theorem in the theory of multiple scattering of waves[15] if it is expanded in the number of correlation groups and one confines oneself to the first term of the expansion. In addition, this relation can be verified directly.

Let us note yet another important meaning of the optical theorem, which consists in the following. We have obtained the transport equation by means of the described transformation of the Bethe-Salpeter equation. From the point of view of the optical theorem this transformation is in a definite sense the only one possible as soon as the operators M and K are chosen in the single-group approximation, and the Green's function is chosen in the form (11).

6. THE CORRELATION FUNCTION OF THE FIELD AND THE RAY INTENSITY

The correlation operator of the field $\mathscr{W} - (\mathscr{G} \otimes \overline{\mathscr{G}})$ is defined by formula (19). Multiplying it by $j(\mathbf{x}')\overline{j}(\mathbf{y}')$ and integrating over $\mathbf{x}'$ and $\mathbf{y}'$, we obtain the correlation function of the field $B_\psi(\mathbf{x}, \mathbf{y})$. Thus, the calculation of the correlation function of the field reduces to the solution of the transport equation (21) with a δ-like source. If the distance $|\mathbf{x} - \mathbf{y}|$ is smaller or of the order of the correlation length l of the effective potential $V(\mathbf{x})$, then we find from (19)

$$B_\psi(\mathbf{x}, \mathbf{y}) \cong \int \exp[-ik_0\mathbf{s}(\mathbf{x} - \mathbf{y})] I(\mathbf{x}, \mathbf{s}) d^2\mathbf{s}. \qquad (27)$$

According to (27) the ray intensity of the scattered radiation has the meaning of the spectrum of the correlation function of the field. A similar type of relation was first established by Dolin[5] in the special case of scattering of light beams by large-scale inhomogeneities of the index of refraction.

7. CONCLUSION

The derivation of the transport equation which we have presented assumes the statistical homogeneity of the effective potential $V(\mathbf{x})$. We have thereby formally excluded the case of a bounded medium. However, it is readily seen that the results obtained can be easily generalized to the case of a medium whose statistical characteristic change smoothly on a scale of the correlation length l and of the wavelength $\lambda_0 = 2\pi/k_0$.[8] Since the effective wave number k_{eff} differs little from k_0, then in the geometric-optics approximation the rays become straight lines. All the formulas of Sec. 4 retain their validity if in the exponents of the exponentials of the Green's functions $\mathscr{G}(\mathbf{x} - \mathbf{x}')$ we replace $k_{eff}|\mathbf{x} - \mathbf{x}'|$ by the integral

$$\int_0^{|\mathbf{x}-\mathbf{x}'|} k_{eff}(\mathbf{x}' + \mathbf{s}_{\mathbf{x}\mathbf{x}'}\rho) d\rho.$$

The generalization to the case of an electromagnetic field is also obvious. It reduces to changing the scalar function $I(\mathbf{x}, \mathbf{s})$ and the coefficients κ and $f(\mathbf{s}, \mathbf{s}')$ into a vector and matrix respectively.[1,9]

The conditions of applicability of the single-group approximation in the case of a normal distribution of the effective potential cited in[14] coincide with the conditions formulated by Andreev.[16] They differ from the condition $d \gg \lambda_0$ where $d = \kappa^{-1}$ is the extinction length cited by most authors in that they impose a limitation not only on the imaginary but also on the real part of the Fourier transform of the mass operator.

In a number of papers one encounters also other conditions for the applicability of the transport equation. In these use is at times made of graphic concepts of the theory of diffraction of waves by an isolated scatterer. Thus, for example, Borovoĭ[7] postulates the condition $n^{-1/3} \gg a^2/\lambda_0$, where a is the radius of the scatterer, starting from the usual concepts about the Fraunhofer region of diffraction. Conditions of this type appear, in our opinion, unfounded, since one cannot transfer automatically the concepts of diffraction theory for an isolated scatterer to an ensemble of scatterers.

We would still like to dwell on the physical interpretation of the obtained results. In Sec. 2 we introduced with the aid of group expansions the concept of inhomogeneities. In the approximation made their mutual effect breaks up essentially into two parts—a coherent and an incoherent part. The coherent effects are taken into account by the operators M_1 and K_1, or by the coeffi-

cients κ and f in the transport equation; these depend on the correlations of the random potential or of random scatterers. The incoherent part of the effect of inhomogeneities manifests itself in the multiple scattering of waves in which the inhomogeneities appear as independent inhomogeneities. The described situation is in agreement with the point of view of Rozenberg.[9]

[1] S. Chandrasekhar, Radiative Transfer, Dover, 1960.
[2] G. V. Rozenberg, Candidate's dissertation, Institute of Theoretical Geophysics, 1946.
[3] V. V. Sobolev, Izv. AN SSSR, ser. geogr. i geofiz. 8, 173 (1944); Uch. zap. LGU, No. 116, 3 (1944).
[4] H. Hönl, A. Maue, and K. Westphal, Diffraction Theory, (Russ. Transl.), Mir, 1964.
[5] L. S. Dolin, Izv. Vyssh. uch. zav., Radiofizika, 7, 559 (1964).
[6] Yu. N. Gnedin and A. Z. Dolginov, Zh. Eksp. Teor. Fiz. 45, 1136 (1963) [Sov. Phys.-JETP 18, 784 (1964)].
[7] A. G. Borovoĭ, Izv. Vyssh. uch. zav., Fizika, No. 6, 50 (1966).
[8] Yu. N. Barabanenkov, Dokl. Akad. Nauk SSSR 174, 53 (1967) [Sov. Phys.-Dokl., 12, 431 (1967)].
[9] G. V. Rozenberg, Usp. Fiz. Nauk 69, 57 (1959) [Sov. Phys.-Usp. 2, 666 (1960)].
[10] L. L. Foldy, Phys. Rev. 67, 107 (1945).
[11] R. Bourret, Nuovo cimento 26, 1 (1962).
[12] V. Frisch, Wave Propagation in Random Media. A Theory of Multiple Scattering. Institut d'Astrophysique, Paris, 1965.
[13] V. Frisch, Wave Propagation in Random Media. II. Multiple Scattering by N Bodies (Provisional Version), Institut d'Astrophysique, Paris, 1965.
[14] V. M. Finkel'berg, Zh. Eksp. Teor. Fiz. 53, 401 (1967) [Sov. Phys.-JETP, 26, 268 (1968)].
[15] Yu. N. Barabanenkov and V. M. Finkel'berg, The Optical Theorem in the Theory of the Multiple Scattering of Waves, ITÉF Preprint No. 486, 1967.
[16] I. V. Andreev, Zh. Eksp. Teor. Fiz. 48, 1437 (1965) [Sov. Phys.-JETP 21, 961 (1965)].

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