TRANSFORMATION OF THE ENERGY ACCUMULATED IN AN ACTIVE MEDIUM INTO THE ENERGY OF A LONGITUDINAL WAVE IN A PLASMA

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The possibility of employing an excited active medium (three-level system) for amplifying plasma waves is investigated. It is shown that the energy confined to the medium can be transformed into the energy of a plasma wave with a prescribed phase velocity. The conditions for such a transformation are determined.

THE interaction between a powerful laser light beam and a plasma is considered in a number of recent theoretical papers. In particular, several papers consider the transformation of a transverse wave into a longitudinal (plasma) wave as a result of nonlinear interaction between the electromagnetic field of the wave and the plasma, an interaction which becomes appreciable at large field intensities.[1-3] The production of longitudinal large-amplitude waves in a plasma is of interest in connection with plasma heating, and also for acceleration in a plasma.

We consider in this paper the direct transformation of the internal energy of an active medium used as the working medium in a laser into the energy of a longitudinal wave in a plasma, bypassing the stage of transverse-wave generation.

We shall simulate a nonlinear active medium by a system of N three-level molecules with energy levels $E_1 < E_2 < E_3$, and we shall assume that the polarization vector $\mathbf{P}$ of the medium is determined by the sum of the dipole moments of the individual molecules: $\mathbf{P} = N\mathbf{p}$. In order to obtain the dipole moment $\mathbf{p}$ of one molecule as a function of the amplitude of the electric field, let us consider the behavior of the molecule in an external electric field $\mathbf{E}$. We use here Schrödinger's equation, assuming that the operator describing the interaction between the molecule and the field is of the form $\hat{H}_1 = -\hat{\mathbf{d}}\cdot\mathbf{E}$:

$$i\hbar\,\partial\psi/\partial t = (\hat{H}_0 - \hat{\mathbf{d}}\mathbf{E})\psi, \tag{1}$$

Where $\hat{H}_0$ describes the stationary state of the molecule (in the absence of the field), and $\hat{\mathbf{d}}$ is the operator of the dipole moment of the molecule.

We seek a solution of (1) in the form

$$\psi(\mathbf{r},t) = A(t)\psi_1(\mathbf{r}) + B(t)\psi_2(\mathbf{r}) + C(t)\psi_3(\mathbf{r}), \tag{2}$$

where $\psi_1(\mathbf{r})$, $\psi_2(\mathbf{r})$, and $\psi_3(\mathbf{r})$ are the wave functions corresponding to the energy levels $E_1 < E_2 < E_3$, and $A(t)$, $B(t)$, and $C(t)$ are complex functions of the time.

Substituting (2) in (1) and then multiplying equation (1) by $\psi_i(\mathbf{r})$ we get, after integrating with respect to $\mathbf{r}$, a system of equations for the coefficients $A(t)$, $B(t)$, and $C(t)$:

$$\begin{aligned} i\hbar\, dA/dt &= E_1A - (\mathbf{d}_{12}B + \mathbf{d}_{13}C)\mathbf{E},\\ i\hbar\, dB/dt &= E_2B - (\mathbf{d}_{12}A + \mathbf{d}_{23}C)\mathbf{E},\\ i\hbar\, dC/dt &= E_3C - (\mathbf{d}_{13}A + \mathbf{d}_{23}B)\mathbf{E}, \end{aligned} \tag{3}$$

where

$$E_i = \langle\psi_i^*|\hat{H}_0|\psi\rangle, \quad \mathbf{d}_{ik} = \langle\psi_i^*|\hat{\mathbf{d}}|\psi_k\rangle$$

with $i \neq k$ and $\mathbf{d}_{ii} = 0$, and the brackets denote averaging over $\mathbf{r}$. The dipole moment of the molecule then takes the form

$$\mathbf{p} = \mathbf{d}_{12}(AB^* + A^*B) + \mathbf{d}_{13}(AC^* + A^*C) + \mathbf{d}_{23}(BC^* + B^*C). \tag{4}$$

The system (3), together with Maxwell's equations for the fields, in which it is necessary to substitute the polarization current $\mathbf{j}_p = \partial^2\mathbf{P}/\partial t^2$, is a self-consistent system of equations describing the interaction of the electromagnetic waves with the system of three-level molecules.

We shall assume henceforth that the active medium is chosen in such a way that the transitions between the levels E_2 and E_1 are completely forbidden ($\mathbf{d}_{12} = 0$). Then, if we assume that at the initial instant of time the system is in a state such that all the particles are at the level E_2, then such a state of the medium is stable, since the transition to the level E_3 is not favored from the energy point of view. However, if such a medium is placed in a high frequency field of a frequency close to the frequency of the transition between the levels E_2 and E_3, namely $\Omega_{32} = (E_3 - E_2)/\hbar$, then the particles of the medium can go over to the level E_3 after absorbing energy from the field.[5] This opens up the possibility of the spontaneous transition[1)] $E_3 \to E_1$ with emission of a transverse wave of frequency $\Omega_{31} = (E_3 - E_1)/\hbar$. This is precisely the frequency-conversion mechanism which is used in three-level quantum amplifiers.[6] [2)] We are interested, however, in the case when the system considered above can radiate a longitudinal wave, that is, a wave in which the projection of the electric field on the direction of wave propagation differs from zero. One of the main conditions for obtaining a longitudinal wave is in this case the creation of a waveguide in which such a wave could propagate.

[1)] We define as spontaneous transitions any collective transition which occur in the absence of an external field.

[2)] The interaction between a three-level scheme and electromagnetic fields of resonant frequencies was investigated by Fain and Khasin [7] (stationary regime), and the generation of optical harmonics in a nonresonant linear medium was investigated by Akhmanov, Khokhlov [8], and Bloembergen [9].

The simplest waveguide of this type is a plasma waveguide, which can be obtained by producing a plasma in the volume occupied by the active medium, for example, by passing an electron beam through a gas or through a channel in a solid.

We shall neglect below processes of energy dissipation in the active medium, and disregard likewise the thermal motion of the molecules of the medium. In this case we can assume that the dipole moments of the molecules of the medium are parallel to the electric field, and the vector quantities are replaced by scalar quantities.

The initial system of equations of the problem in question consists of the equations of the system (3), in which we must put $d_{12} = \varepsilon_{12} = 0$, and the Maxwell equation that describes the variation of the electromagnetic field with frequency Ω_{31}. In addition, we shall assume that a plane transverse wave of frequency $\omega = \Omega_{31}$ and of phase velocity v_{ph} is incident on the system:

$$E_{32} = \varepsilon_0 \sin[\Omega_{32}(t - z/v_{ph})]. \qquad (5)$$

The Maxwell equation describing the variation of the longitudinal field E_{31} can be obtained by considering a system of equations consisting of linearized equations of motion and the continuity equation, and also the Poisson equation

$$\frac{\partial}{\partial z}(E_{31} + 4\pi P_{31}) = 4\pi e(n - n_0),$$

where $P_{31} = Nd_{13}(AC^* + A^*C)$, and n_0 is the density of the beam. It can be shown that in this case the dependence of E_{31} on P_{31} is described by the following equation:

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)^2 (E_{31} + 4\pi P_{31}) + \omega_0^2 E_{31} = 0 \qquad (6)$$

(v_0 is the directional velocity of the beam, $\omega_0^2 = 4\pi e^2 n_0/m$ is the plasma frequency of the beam.) In the derivation of (6) we have assumed that the velocity of the beam particles can be represented in the form $v = v_0 + \tilde{v}$, where $\tilde{v}$ is a small addition to v_0 ($|\tilde{v}| \ll v_0$), and neglect the terms of the order $\tilde{v}\partial\tilde{v}/\partial z$. The condition for this approximation will be presented below.

Equations (3) and (6) make up a self-consistent system of nonlinear partial differential equations. One of the presently available methods of solving such equations is that developed by Bogolyubov and Mitropol'skiĭ,[10] which we shall use below.

We seek the solution of (3) and (6) in the form of traveling waves with amplitudes that vary slowly in time[3)] (the dimensionless parameters are in this case $q = (2Nd_2/\hbar\Omega)^{1/2}$ and $\alpha = d\varepsilon_0/2\hbar\Omega$):

$$A = a(t)\exp\left[-\frac{i}{\hbar}E_1\left(t - \frac{z}{v_{ph}}\right)\right], \quad B = b(t)\exp\left[-\frac{i}{\hbar}E_2\left(t - \frac{z}{v_{ph}}\right)\right],$$

$$C = c(t)\exp\left[-\frac{i}{\hbar}E_3\left(t - \frac{z}{v_{ph}}\right)\right], \quad E_{31} = \varepsilon_{31}(t)\sin\left[\Omega_{31}\left(t - \frac{z}{v_{ph}}\right)\right]. \qquad (7)$$

An important feature of the choice of solutions in this form is that we regard the phase velocity v_{ph} of the wave as constant and investigate the dependence of the amplitudes on the time, that is, actually the change produced in the wave frequency by the nonlinear effect. Another formulation of the problem, such as used in [8, 9], is to fix the frequency and determine the dependence of the phase velocity on the coordinate (amplification mode).

Substituting (7) and (5) in (3) and retaining only the resonant terms,[4)] we obtain a nonlinear system of equations for the functions a(t), b(t), and c(t):

$$\dot{a} = \frac{d_{13}}{2\hbar}\varepsilon_{31}c, \quad \dot{b} = \frac{d_{23}}{2\hbar}\varepsilon_0 c, \quad \dot{c} = -\frac{1}{2\hbar}(d_{13}\varepsilon_{31}a + d_{23}\varepsilon_0 b). \qquad (8)$$

In order to obtain an equation for $\varepsilon_{31}(t)$, we substitute $P_{31} = 2Nd_{13}ac\cos[\Omega_{31}(t - z/v_{ph})]$ and $E_{31} = \varepsilon_{31}(t)\sin[\Omega_{31}(t - z/v_{ph})]$ in equation (6). In the same approximation as above, and also assuming that the relation

$$\Omega_{31}(1 - v_0/v_\Phi) = \omega_0 \qquad (9)$$

is satisfied, we get

$$\varepsilon_{31} = 4\pi\omega_0 N d_{13}\, ac. \qquad (8a)$$

Relation (9), which determines the dependence of v_{ph} on Ω_{31} and ω_0, is a condition for the Doppler effect: the frequency of the wave emitted by the stationary atoms of the medium, recalculated in the reference frame connected with the beam, is equal to the frequency ω_0 of the natural oscillations of the beam. It is just a wave with a phase velocity satisfying (9) which will be resonantly absorbed by the beam particles.[5)]

Before we proceed to solve the system (8), let us introduce the following notation:

$$\varepsilon = \frac{\varepsilon_{31}}{Nd_{13}}, \quad q^2 = \frac{2Nd_{13}^2\omega_0}{\hbar}, \quad \alpha = \frac{d_{23}\varepsilon_0}{2\hbar}, \quad \tau = \alpha t \qquad (10)$$

(Here q^{-1} is the time of spontaneous transmission of the system from the level E_3 to E_1 with emission of a longitudinal quantum,[12] and α is the frequency of the induced transitions between the levels E_2 and E_3.[5]) In terms of the variables (10), we can rewrite the system (8) in the form

$$\ddot{b} + b = -\frac{1}{4}\frac{q^2}{\alpha}\sqrt{1 - b^2 - \dot{b}^2}\,\varepsilon, \qquad \dot{\varepsilon} = \frac{4\pi}{\alpha}\sqrt{1 - b^2 - \dot{b}^2}\,\dot{b}, \quad (11)$$

The dot denotes differentiation with respect to the dimensionless time τ.) In the derivation of (11) we have used the relation $a^2 + b^2 + c^2 = 1$, which represents the law of conservation of the probabilities for a three-level molecule; this law can be obtained from the equations of the system (8).

Differentiating the first equation of (11) and substituting $\dot{\varepsilon}$ from the second, we obtain a third-order non-

3) In the derivation of (3) we assumed that A, B, and C do not depend on the coordinate, since only one molecule was considered. However, in the medium consisting of active molecules, a traveling wave may propagate and therefore it is necessary to assume in the analysis of electromagnetic processes in the medium that A, B, and C are functions of the time and of the coordinate.

4) It is possible to disregard the spontaneous transition between the levels E_3 and E_2 only if the condition $\varepsilon_0^2 \gg 8\pi N\hbar\Omega_{32}$ is satisfied.

5) It should be noted that when a beam of charged particles passes through a gas of oscillators, two-stream instabilities can set in, and can in principle change the dynamics of the emission in the longitudinal wave. However, as shown by Neufeld [11], instability can arise in such a case only if the conditions $v_{ph} = v_0$ (the Cerenkov effect) and $\Omega(1 - v_0/v_{ph}) = -\omega_0$ (anomalous Doppler effect) are satisfied.

linear equation for the function b(t):

$$\frac{d}{d\tau}\frac{\ddot{b}+b}{\sqrt{1-b^2-\dot{b}^2}}=-\frac{\pi q^2}{\alpha^2}\dot{b}\sqrt{1-b^2-\dot{b}^2}. \quad (12)$$

Changing in (12) to the variables $a=\sqrt{1-b^2-\dot{b}^2}$ and $db=\dot{b}d\tau$, we obtain a second-order linear equation for the function a(b):

$$\frac{d^2a}{db^2}-\frac{\pi q^2}{\alpha^2}a=0, \quad (13)$$

the solution of which, satisfying the initial conditions $a(0)=a_0$ and $b(0)=b_0$, is

$$a(b)=a_0\,\mathrm{ch}\left[\sqrt{\pi}\frac{q}{\alpha}(b-b_0)\right]. \quad (14)$$

Substituting a in (14) as a function of b and $\dot{b}$, and then solving the resultant equation with respect to $\dot{b}$, we obtain

$$\left(\frac{db}{dt}\right)^2=\alpha^2\left\{1-b^2-a_0^2\,\mathrm{ch}^2\left[\sqrt{\pi}\frac{q}{\alpha}(b-b_0)\right]\right\}. \quad (15)$$

Solving (15), we can find the dependence of b on the time t. However, since the solution of (15) cannot be expressed in terms of elementary functions, it is of interest to investigate it qualitatively.

We assume that at the initial instant of time all the particles are at the level E_2: $b_0^2\approx 1$ and $a_0^2\ll 1$, and we consider the character of the solutions of (15) as functions of the relation between the parameters q and α.

We assume first that the time of the spontaneous transition $E_3\rightarrow E_1$ is large compared with the time of the stimulated transition $E_3\rightarrow E_3$: $1/q\gg 1/\alpha$. In this case the parameter entering in the right side of (15) is small, $q/\alpha\ll 1$, so that we can assume that $\cosh[\sqrt{\pi}q(b-b_0)/\alpha]\approx 1$, and obtain a solution of (15) in explicit form: $b=b_0\sin\alpha t$. Thus, the molecules of the medium execute stimulated transitions between the levels E_2 and E_3, with frequency α, and the population of the level E_1 remains unchanged ($a\sim qb_0/\alpha\ll 1$).

In the opposite limiting case, when the parameter q/α is not small ($q/\alpha>1$), the character of the solution of (15) can change, for in this case the last term of (15) is comparable to unity and the right side of (15) can vanish at small values of b(t): $b=b_{min}\ll 1$. The minimum value of the function b_{min} is obtained by equating $\dot{b}(b_{min})$ to zero:

$$1-b_{min}^2-a_0^2\,\mathrm{ch}^2\left[\sqrt{\pi}\frac{q}{\alpha}(b_{min}-b_0)\right]=0. \quad (16)$$

Since $b_{min}^2\ll 1$ and $c(b_{min})=\dot{b}(b_{min})=0$, we get $a(b_{min})\equiv a_{max}\approx 1$, that is, practically all the particles go over to the level E_1, emitting a longitudinal wave of frequency Ω_{31}. According to (15) the transition time is of the order of $1/\alpha$.

In order to determine the maximum energy of the generated longitudinal wave, we can use the energy conservation law which follows from (11):

$$b^2+\frac{1}{\alpha^2}\dot{b}^2+\frac{\varepsilon_{31}^2}{8\pi}\frac{1}{N\hbar\Omega_{31}}=1. \quad (17)$$

(We chose the constant on the right side of (17) from the condition $\dot{b}_0=\varepsilon_{31}(0)=0$ and $b_0=1$.) If we assume that all the particles have gone over to the level E_1, namely $b_{min}=\dot{b}(b_{min})=0$, then, according to (17), we have

$$\varepsilon_{31\,max}^2/8\pi=N\hbar\omega_0. \quad (18)$$

Thus, the amplitude of the longitudinal wave can be increased by increasing the plasma density (however, the resonance condition (9) must not be violated). The maximum energy of the longitudinal wave cannot exceed in this case the value $N\hbar\Omega_{31}$, that is, the maximum energy stored in the medium.

We now obtain the conditions under which it is possible to use the linear equations of motion of the beam. Using relations (9) and (18) we can show that the approximation considered above is valid if

$$n_0mv_0^2/2\gg\varepsilon_m^2/8\pi=N\hbar\omega_0, \quad (19)$$

that is, if the kinetic energy of the beam particles is much higher than the wave energy. In the case when an inequality inverse to (19) is satisfied, the decisive nonlinear effects are the effects of nonlinear interaction of the beam particles with the wave field.[13-15] In such a case it is necessary to use the nonlinear equations of motion of the beam, and the equations of motion of the medium can be regarded as linear.

In conclusion let us investigate the stability of the obtained nonlinear solutions. Differentiating both sides of (15) with respect to time and substituting in it $b=b(t)+x(t)$, where b(t) is the solution obtained above and x(t) is a small perturbation, $|b(t)|\gg|x(t)|$, we obtain a second-order linear equation with variable coefficients for the function x(t):

$$\ddot{x}+\left\{\alpha^2+a_0^2\pi q\,\mathrm{ch}\left[2\sqrt{\pi}\frac{q}{\alpha}(b(t)-b_0)\right]\right\}x=0. \quad (20)$$

In the case considered above we were unable to find the dependence of the function b(t) on the time in explicit form, and we therefore confine ourselves below to a qualitative investigation of (20).

Since the processes connected with energy dissipation were not taken into account, the nonlinear solution b(t) obtained above has a periodic character[6)] (with the period $\sim 1/\alpha$); consequently Eq. (20) can be reduced to a Hill equation.[16] As is well known from the general theory of this equation, under certain relations between the parameters which enter into the equation there exist instability regions in which the equation has exponentially growing solutions: $x(t)\sim x_0e^{\gamma t}$. The growth coefficient γ is proportional to the coefficient of the alternating part of the frequency, that is, in our case to a_0q. Therefore, letting a_0 go to zero (in practice $a_0\ll\alpha/q$) and assuming that $b(t)\sim b_0$ at the start of the process, we can always make the time of instability development $\sim 1/\gamma$ larger than the period of the nonlinear oscillations $\sim 1/\alpha$: $1/\gamma\gg 1/\alpha$.

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6)That is, after all the particles go over to the level E_1, and the amplitude of the longitudinal wave reaches a maximum, an inverse process is possible – absorption of energy by the medium.

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66