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AN INVARIANT FORM OF THE INTEGRAL LAW OF CONSERVATION OF ENERGY

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Invariant expressions are introduced for the energy density and the covariant Umov-Poynting vector, by means of which an invariant formulation is given for the integral law of conservation of energy. In the case of the special theory of relativity, in inertial reference systems the invariant formulation is the same as the usual one. The invariant expression for the integral flux of energy is given a concrete form for the case of the general theory of relativity. An invariant Hamiltonian is constructed for a system of point masses in a gravitational field.

1. The usual integral form of the conservation laws for a system described by an energy-momentum tensor $T^{\mu\nu}$ is not invariant relative to arbitrary coordinate transformations. This gives rise to known difficulties in the passage to curvilinear coordinates and the pseudo-riemannian space of the general theory of relativity. We shall here give an invariant formulation of the integral law of conservation of energy, which in the case of the Galilean coordinates of special relativity theory (STR) agrees with the generally accepted expression.

Let us consider a family of spacelike hypersurfaces

$$\tau(x^0, x^1, x^2, x^3, C) = 0, \tag{1}$$

with

$$\frac{\partial\tau}{\partial x^0} > 0, \quad g^{\alpha\beta}\frac{\partial\tau}{\partial x^\alpha}\frac{\partial\tau}{\partial x^\beta} > 0. \tag{2}$$

We denote by A the norm of the vector $\partial\tau/\partial x^\alpha$ orthogonal to the hypersurface (1):

$$g^{\alpha\beta}\frac{\partial\tau}{\partial x^\alpha}\frac{\partial\tau}{\partial x^\beta} = A^2. \tag{3}$$

We further introduce a timelike unit vector

$$n_\alpha = A^{-1}\partial\tau/\partial x^\alpha. \tag{4}$$

Using the energy-momentum tensor $T^{\mu\nu}$ of the system and the vector n_α, we construct the following vector and scalar:

$$S^\mu = T^{\mu\nu}n_\nu, \quad S = T^{\mu\nu}n_\mu n_\nu. \tag{5}$$

We introduce into the discussion the time interval

$$d\eta = n_\alpha dx^\alpha, \tag{6}$$

which is smaller than the integral interval $d\tau$ by a factor A.

From (6) there follows the obvious relation

$$n_\alpha dx^\alpha / d\eta = 1. \tag{7}$$

We define a new vector Π^μ:

$$\Pi^\mu = S^\mu - S\, dx^\mu/d\eta. \tag{8}$$

It is easy to see that the vector Π^μ has only three independent components, since it satisfies one identity:

$$\Pi^\mu n_\mu \equiv 0. \tag{9}$$

2. In a physical space with the metric $g_{\mu\nu}$ let us consider a symmetric tensor field $a_{\mu\nu}$. In a particular case $a_{\mu\nu}$ may be identical with $g_{\mu\nu}$. Furthermore, let us introduce a covariant derivative $\underset{a}{\nabla}_\mu$ relative to the tensor $a_{\mu\nu}$ and the following invariant elements of four-dimensional volume, three-dimensional volume, and area:

$$\underset{a}{d\Omega} = \sqrt{\frac{a}{g}}\,d\Omega, \quad \underset{a}{d\sigma_\alpha} = \sqrt{\frac{a}{g}}\,d\sigma_\alpha, \quad \underset{a}{d\Sigma_{\mu\nu}} = \sqrt{\frac{a}{g}}\,d\Sigma_{\mu\nu}, \tag{10}$$

where a and g are the determinants of the tensors $a_{\mu\nu}$ and $g_{\mu\nu}$ and $d\Omega$, $d\sigma_\alpha$, and $d\Sigma_{\mu\nu}$ are given by

$$d\Omega = \frac{1}{4!}\sqrt{-g}\,\varepsilon_{\alpha\beta\gamma\delta}\,dx^\alpha\,dx^\beta\,dx^\gamma\,dx^\delta,$$

$$d\sigma_\alpha = \frac{1}{3!}\sqrt{-g}\,\varepsilon_{\alpha\beta\gamma\delta}\,dx^\beta\,dx^\gamma\,dx^\delta,$$

$$d\Sigma_{\mu\nu} = \frac{1}{2!}\sqrt{-g}\,\varepsilon_{\alpha\beta\gamma\delta}\,dx^\gamma\,dx^\delta. \tag{11}$$

Let the vector S^μ satisfy the differential conservation law

$$\underset{a}{\nabla}_\mu S^\mu = 0. \tag{12}$$

For this it is necessary that $T^{\mu\nu}$ be symmetric and satisfy the identical relation

$$\underset{a}{\nabla}_\nu T^{\mu\nu} \equiv 0, \tag{13}$$

and that n_α satisfy the Killing identities

$$\underset{a}{\nabla}_\beta n_\alpha + \underset{a}{\nabla}_\alpha n_\beta \equiv 0. \tag{14}$$

Integrating (12) over the invariant three-volume, we get (see Appendix) the following invariant integral conservation law:

$$\frac{d}{d\tau}\int S\,\underset{a}{dL} = \oint \Pi^\mu \frac{dx^\nu}{d\tau}\,\underset{a}{d\Sigma_{\mu\nu}} \quad \left(\underset{a}{dL} = \underset{a}{d\sigma_\alpha}\frac{dx^\alpha}{d\eta}\right). \tag{15}$$

It follows from (15) that S is an invariant energy density and Π^μ a covariant Umov-Poynting vector.

In the case of the STR the spacelike hypersurface can be chosen in the form $\tau = t^0$. Then

$$n_0 = 1, \quad n_i = 0,$$
$$S = T^{00}, \quad \Pi^0 = 0, \quad \Pi^i = T^{i0},$$
$$dL = dV, \quad dx^0/d\tau = 1, \quad dx^i/d\tau = 0$$

and consequently (15) can be rewritten in the form

$$\frac{d}{dt}\int_V T^{00}\,dV = \oint T^{i0}\,df_i. \tag{15a}$$

3. There is an arbitrariness in the choice of the vector n_α, and consequently in S and Π^α, owing to the arbitrariness in the choice of the hypersurface (1). This arbitrariness is connected with the possibility of choosing various reference systems in the sense of [1,2].

In the STR there exist distinguished inertial systems of reference. The choice of the hypersurface (1) in the form

$$t^0(x^0, x^1, x^2, x^3) = C, \tag{16}$$

where t^0 is a temporal Galilean coordinate and x^α are curvilinear coordinates, corresponds to the use of an inertial system of reference (in an arbitrary curvilinear mesh). A different choice of the hypersurface (1) corresponds to various accelerated systems.

Accordingly, Eq. (15) allows us to write the integral conservation law in any coordinate mesh from the point of view of any arbitrarily chosen reference system.

In the general theory of relativity there are no inertial coordinates,[2] and therefore one or another choice of the hypersurface (1) is dictated by the concrete physical statement of the problem.

We note that the integral conservation laws in invariant form in the STR have also been considered by Fock,[3] but Fock used a special form of the vector n_α, corresponding to the choice of a noninertial system of reference.

4. Let us now consider the integral conservation laws when a gravitational field is taken into account. In [4-6] it is shown that one can introduce an energy-momentum tensor $S^{\mu\nu}$ of the matter and the gravitational field, satisfying the conservation law

$$\underset{0}{\nabla}_\nu S^{\mu\nu} = 0, \tag{17}$$

where $\underset{0}{\nabla}_\mu$ is the covariant derivative relative to the metric tensor of the auxiliary flat space. When we introduce in addition to the vector n_α a vector $\underset{0}{n}_\alpha$ which satisfies the identities

$$\underset{0}{\nabla}_\alpha \underset{0}{n}_\beta + \underset{0}{\nabla}_\beta \underset{0}{n}_\alpha = 0 \tag{18}$$

and define an invariant energy density and a covariant Umov-Poynting vector in the form

$$\underset{0}{S} = S^{\mu\nu} \underset{0}{n}_\mu n_\nu, \quad \underset{0}{\Pi}^\mu = S^{\mu\nu} \underset{0}{n}_\nu - \underset{0}{S}\, dx^\mu/d\eta, \tag{19}$$

we get the conservation law in the form

$$\frac{d}{d\tau}\int \underset{0}{S}\, \underset{0}{dL} = \oint \underset{0}{\Pi}^\mu \frac{dx^\nu}{d\tau} \underset{0}{d\Sigma}_{\mu\nu}. \tag{20}$$

For spaces of Petrov's type I,[7] in coordinates which are Galilean at infinity, the conservation law (20) reduces to the usual Einstein form. If, however, in the Einstein form of the conservation law the total energy can become infinite when we go over to new (for example spherical) coordinates, Eq. (20) is an invariant continuation of the physically justified results of the Einstein conservation law in asymptotically Galilean coordinates on an arbitrary coordinate net.

It must be pointed out that if $T^{\mu\nu}$ is the source of a gravitational field with the metric tensor $g_{\mu\nu}$, possessing a certain group of motions, then the Killing equations in this space will be soluble, and by the general theorem expounded in Sec. 2 a conservation law will hold for the energy of the matter without the gravitational field (besides the conservation of the total energy of the matter and the gravitational field).

The conservation law (20) differs somewhat from the invariant integral form of the laws of conservation of gravitational energy which the writer derived previously,[8] but the total value found for the energy in spaces of Petrov's type I is the same in both versions. On the other hand the conservation laws for momentum and angular momentum cannot be formulated in invariant form without covariant integration (this last has already been done previously[8]).

The invariant expression for the flux of energy gives us an invariant criterion for solving the problem of the existence of gravitational waves. The whole question is complicated, however, by the fact that the expression which is invariant relative to arbitrary coordinates depends on the choice of the hypersurfaces, and consequently of the reference system. In the case of electrodynamics this difficulty does not arise, because in STR there are preferred inertial systems. Nevertheless, in the case of spaces of Petrov's type I, for physical reasons we must give preference to the class of reference systems in which the metric tensor approaches the Galilean form at infinity. It is not hard to show that the flux of gravitational energy is the same in all such reference systems, i.e., our criterion becomes invariant not only relative to arbitrary coordinate transformations, but also relative to the choice between different reference systems which leave the asymptotic behavior of the metric tensor $g_{\mu\nu}$ at infinity unchanged.

5. Let us define the invariant Hamiltonian of the system in the form

$$H = \int_L S\, dL. \tag{21}$$

Substituting in (21) the expression for the energy-momentum tensor for a system of point masses, we get the invariant Hamiltonian of a system of material points in a gravitational field. In particular, for one particle in a Schwarzschild field (with the hypersurfaces chosen to be $\tau = x^0$), we get an expression which agrees with that given by Landau and Lifshitz[9]

$$H = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1 - v^2/c^2}}, \tag{22}$$

where

$$v = c\frac{dl}{d\eta} = \frac{c}{\sqrt{g_{00}}}\frac{dl}{dx^0}.$$

Accordingly, the formalism given here allows us to construct quantities which have the same physical meaning as the noncovariant quantities, but which have good transformation properties.

APPENDIX

Let us consider the integral

$$\int \underset{a}{\nabla}_\mu S^\mu \cdot \frac{1}{A} \underset{a}{dL}. \tag{I}$$

In transforming the integral (I) we use the following relations:

$$d\Omega = d\sigma_\alpha\, dx^\alpha = \left(d\sigma_\alpha \frac{dx^\alpha}{d\eta}\right)\frac{1}{A}\, d\tau = \frac{1}{A}\, dL\, d\tau,$$
$$d\sigma_\alpha = d\Sigma_{\alpha\beta}\, dx^\beta.$$

Then

$$\int_L \underset{a}{\nabla_\mu S^\mu}\cdot\frac{1}{A}\,\underset{a}{dL} = \int_\Omega \underset{a}{\nabla_\mu S^\mu}\delta(\tau-\tau_0)\,\underset{a}{d\Omega} = \int_\Omega \frac{\partial}{\partial x^\mu}(\sqrt{-a}\,S^\mu)\,\delta(\tau-\tau_0)\, d\Omega^*$$
$$= \int_\Omega \frac{\partial}{\partial x^\mu}(\sqrt{-a}\,S^\mu\delta(\tau-\tau_0))\, d\Omega^* - \int_\Omega S^\mu \frac{\partial\delta(\tau-\tau_0)}{\partial x^\mu}\,\underset{a}{d\Omega},$$

where

$$d\Omega^* = \frac{1}{4!}\,\varepsilon_{\alpha\beta\gamma\delta}\, dx^\alpha\, dx^\beta\, dx^\gamma\, dx^\delta.$$

We transform the first integral:

$$\int_\Omega \frac{\partial}{\partial x^\mu}(\sqrt{-a}\,S^\mu\delta(\tau-\tau_0))\, d\Omega^* = \int_\sigma S^\mu\delta(\tau-\tau_0)\,\underset{a}{d\sigma_\mu}$$
$$= \int_\sigma S^\mu\delta(\tau-\tau_0)\,\underset{a}{d\Sigma_{\mu\nu}}\frac{dx^\nu}{d\tau}\, d\tau = \oint S^\mu \frac{dx^\nu}{d\tau}\,\underset{a}{d\Sigma_{\mu\nu}} = \oint \Pi^\mu \frac{dx^\nu}{d\tau}\,\underset{a}{d\Sigma_{\mu\nu}}.$$

We now transform the second integral:

$$\int_\Omega S^\mu \frac{\partial\delta(\tau-\tau_0)}{\partial x^\mu}\,\underset{a}{d\Omega} = \int_\Omega S^\mu \frac{\partial\tau}{\partial x^\mu}\,\delta'(\tau-\tau_0)\,\underset{a}{d\Omega}$$
$$= \int_\Omega S^\mu n_\mu A\,\delta'(\tau-\tau_0)\,\underset{a}{d\Omega} = \int_\Omega S^\mu n_\mu \delta'(\tau-\tau_0)\,\underset{a}{dL}\, d\tau$$
$$= \int_\Omega S\delta'(\tau-\tau_0)\,\underset{a}{dL}\, d\tau = \frac{d}{d\tau}\int_L S\,\underset{a}{dL}.$$

From this we get the expression given in the text.

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