THE QUANTUM MECHANICAL HARMONIC OSCILLATOR IN THE PHASE REPRESENTATION AND THE UNCERTAINTY RELATION BETWEEN THE NUMBER OF QUANTA AND THE PHASE

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The number operator and the creation and annihilation operators for the quanta are written down in the phase representation. A complete set of eigenfunctions of the Hamilton operator for the harmonic oscillator is found in this representation. The uncertainty relation between the phase and the number of quanta is derived. The relation has the form (16′).

1. RECENTLY, a number of papers[1-4] have appeared which are devoted to the description of the quantum mechanical harmonic oscillator in the variables n and φ, where n is the number of quanta and φ is the phase of the oscillator. This renewed interest is connected with the fact that the traditional description of the quantum oscillator in the variables n and φ (cf., for example, [5-8]) is not completely correct. Usually, the transition to the description in the variables n and φ is made by writing the creation and annihilation operators in the form

$$a = e^{i\varphi}\sqrt{n}, \quad a^+ = \sqrt{n}\, e^{-i\varphi}. \tag{1}$$

From this and from the commutation relations for a and a^+ ($[a, a^+] = 1$) it follows that

$$[e^{i\varphi}, n] = e^{i\varphi}. \tag{2}$$

This relation is considered equivalent to

$$[\varphi, n] = -i. \tag{3}$$

The criticism of this description of the harmonic oscillator essentially concerns two points: All operators under discussion are defined on basis functions which are periodic in φ. The multiplication operator φ itself transforms a function from this class to a class of functions which are not periodic in φ, and cannot therefore be Hermitian. Therefore the relation (3) is incorrect. [This is usually shown by taking matrix elements between eigenstates $|n\rangle$ and $|m\rangle$ of both sides of (3):

$$\langle m[\varphi, n]n\rangle = (n-m)\langle m|\varphi|n\rangle = -i\delta_{mn}.$$

The last relation is clearly meaningless.]

The other objection relates to the following circumstance: even if one stays within the class of functions periodic in φ and uses the relation (2), one also arrives at a contradiction. Acting with the relation (2) on the ground state function $|0\rangle$ we obtain

$$ne^{i\varphi}|0\rangle = -e^{i\varphi}|0\rangle,$$

i.e., n has the negative eigenvalue -1, which is obviously meaningless. Susskind and Glogower[2] try to avoid this difficulty by introducing the operators $\hat{e}^{i\varphi}$ and $\hat{e}^{-i\varphi}$ which are not multiplication (by $e^{\pm i\varphi}$) operators in the φ representation and do not commute with one another.

In the present paper we propose another method by defining a complete system of eigenfunctions of the operator n in the φ representation so that in this representation the operator $e^{i\varphi}$ is the usual operator of multiplication by $e^{i\varphi}$.

2. First of all we introduce the operator I (as will be seen in the following, it has the meaning of an operator of the sense of rotation, where φ is the angular coordinate). An arbitrary periodic function $\Psi(\varphi)$ can be expanded in a series in $e^{im\varphi}$, where m is an integer or a half-integer. The bilinear combination $\Psi^+(\varphi)\Psi(\varphi)$ will then be periodic with the period 2π. The effect of the operator I is to multiply $e^{im\varphi}$ by 1 if $m > 0$ and by -1 if $m < 0$. The operator of the number of quanta is defined by

$$n = -iI\frac{\partial}{\partial\varphi} - \frac{1}{2}. \tag{4}$$

As is easily seen, the normalized eigenfunctions of this operator and of the operator I have the form

$$\Psi_{n,1} = \frac{1}{\sqrt{2\pi}}e^{i(n+1/2)\varphi}, \quad \Psi_{n,-1} = \frac{1}{\sqrt{2\pi}}e^{-i(n+1/2)\varphi}; \quad n \geqslant 0, \tag{5}$$

where $\Psi_{n,1}$ are the eigenfunctions with I = 1, and $\Psi_{n,-1}$ are those with I = −1.

We further introduce the operators a and a^+:

$$a=\left(e^{-i\varphi}\frac{1+I}{2}+e^{i\varphi}\frac{1-I}{2}\right)\sqrt{n},$$
$$a^+=\sqrt{n}\left(\frac{1+I}{2}e^{i\varphi}+\frac{1+I}{2}e^{-i\varphi}\right). \qquad (6)$$

It is easy to show that these operators have the properties of creation and annihilation operators:

$$a^+\Psi_{n;\pm1}=\sqrt{n+1}\Psi_{n+1;\pm1},\qquad a\Psi_{n;\pm1}=\sqrt{n}\Psi_{n-1;\pm1}.$$

It follows from this that the operators a and a^+ satisfy the commutation relation $[a, a^+] = 1$ when acting on an arbitrary function expanded in the complete set of functions (5) [i.e., on a function periodic with period 4π and satisfying the cyclic condition $\Psi(\varphi + 2\pi) = -\Psi(\varphi)$]. From (4) we also obtain the commutation relations

$$[n,f(\varphi)]=-iI\frac{\partial f(\varphi)}{\partial\varphi}+(If(\varphi)I-f(\varphi))\left(n+\frac{1}{2}\right), \qquad (7)$$

where $f(\varphi)$ is an arbitrary periodic function with period 2π. In particular,

$$[n,e^{i\varphi}]=Ie^{i\varphi}+(Ie^{i\varphi}I-e^{i\varphi})(n+{}^1/_2). \qquad (8)$$

By defining now the Hamiltonian of the system in the form

$$\mathscr{H}=\left(n+\frac{1}{2}\right)\hbar\omega=\omega I\left(-i\hbar\frac{\partial}{\partial\varphi}\right), \qquad (9)$$

we have completed the description of the quantum mechanical harmonic oscillator in the φ representation. It is easy to verify that our description is free of the difficulties mentioned above.

3. Let us now turn to the interpretation of the relations just obtained. We first discuss how the harmonic oscillator is described in terms of the variables action—angular variable in the classical case (cf., for example, [9]). The action variable is defined as

$$J=\frac{1}{2\pi}\oint p\,dq. \qquad (10)$$

Let us write q in the form

$$q=\frac{\sqrt{2\mathscr{H}}}{\omega}\cos\varphi. \qquad (11)$$

From the expression for the energy in terms of p and q,

$${}^1/_2\,p^2+{}^1/_2\,\omega^2q^2=\mathscr{H}$$

we find

$$p=\pm\sqrt{2\mathscr{H}}\sin\varphi. \qquad (12)$$

From this and from (10) and (11) we find

$$\mathscr{H}=\mp\omega J. \qquad (13)$$

The variable J has the dimension of an angular momentum and its sign is determined by the sign of the velocity of the representative point in the p, q plane, i.e., by the sign of $d\varphi/dt$. The minus sign corresponds to a negative J, the plus sign to a positive J. The remaining possible solutions do not satisfy the condition that $\mathscr{H}$ must be positive. It is evident that (6) and (9) are the quantum generalizations of the relations (11), (12), and (13). The operator $-i\hbar\partial/\partial\varphi$ in (9) represents the quantity J (it is the operator of angular momentum of the representative point in the q, p plane), and the operator I is determined by the sense of rotation with respect to φ and replaces the ± signs in (13). In the quantum as well as in the classical case there is a degeneracy with respect to the sense of rotation in the variables J and φ (a state is determined by the sense of rotation as well as by the energy). However, when we go over to a description in terms of the variables q and p, this degeneracy does not show up at all. A state with arbitrary (but definite) value of I as well as a superposition of such states are identical as far as the probability distribution in q and p is concerned.

In concluding this section we emphasize that in both the classical and the quantum cases the quantity canonically conjugate to φ is the action variable, which coincides with $\mathscr{H}/\omega$ [or $\hbar(n + {}^1/_2)$] only up to a sign or up to the factor I in the quantum case. This is the origin of the contradiction in (2), which must hold for a genuine canonical variable, which n is not.

It should also be noted that already in 1926 Dirac[12] has shown that the action takes only half-integer values for spinless systems [cf. the well-known quantization rule for the harmonic oscillator $(2\pi)^{-1}\oint p\,dq = (n + 1/2)\hbar$]. Connected with this is the circumstance that the wave functions of the harmonic oscillator satisfy the condition

$$\Psi(\varphi+2\pi)=-\Psi(\varphi).$$

4. Let us now show by a physical example how I can be fixed at a definite value.

Consider a system consisting of a magnetic moment located in a constant magnetic field H_0; the magnetic moment is proportional to the angular momentum. The Hamiltonian for this system is

$$\mathscr{H}=-\hbar\omega L_z,\qquad \omega=\gamma H_0>0,$$

where γ is the gyromagnetic ratio, and $\hbar L_z$ is the projection of the angular momentum on the direction of the magnetic field. Let the absolute value of the angular momentum be $L \gg 1$; then the levels of the system with

$$L - M \ll L, \quad M > 0,$$

where M is the eigenvalue of L_z, are well approximated by the Hamiltonian

$$\mathcal{H} = -\hbar\omega L + \hbar\omega a^+ a;$$
$$a^+ = L^-/\sqrt{2L}, \quad a = L^+/\sqrt{2L},$$
$$[a, a^+] = L_z/L \approx 1.$$

The second term in $\mathcal{H}$ describes the harmonic oscillator

$$\mathcal{H}_1 = \hbar\omega a^+ a = \hbar\omega n, \quad n = L - L_z;$$

the eigenfunctions have the form

$$\Psi_M = (2\pi)^{-1/2} e^{-in\varphi + iL\varphi}.$$

If we disregard the inessential phase factor (which is the same for all states), then this is a state with $I = -1$. The operators a and a^+ have the form

$$a = e^{i\varphi}\sqrt{n}, \quad a^+ = \sqrt{n}\, e^{-i\varphi}.$$

Changing the sign of the magnetic field H_0 (thus changing the sense of rotation), we obtain the other states (close to the ground state) with $I = -1$.

5. We now consider the uncertainty relation. Clearly (as noted in the papers quoted at the beginning) the traditional relation $\Delta n \Delta\varphi \geq 1/2$ has no meaning, if for no other reason than that for $\Delta n \to 0$ we would have $\Delta\varphi \to \infty$, whereas the indeterminacy in φ must not be larger than 2π. We note further that an uncertainty relation between ΔL_z and $\Delta\varphi$ can easily be obtained, where the angular momentum operator is $L_z = -i\partial/\partial\varphi$.

Indeed, the commutation relation for L_z and $f(\varphi)$ [where $f(\varphi)$ is an arbitrary periodic function with period 2π] has the form

$$[L_z, f(\varphi)] = -i\partial f(\varphi)/\partial\varphi;$$

from this we find in the usual way

$$\Delta L_z \Delta f \geqslant 1/2 |\langle \partial f/\partial\varphi \rangle|. \tag{14}$$

For $\Delta\varphi \ll 1$ the mean square fluctuation $\Delta\varphi$ is naturally defined as

$$\langle (\Delta f)^2 \rangle = |\langle \partial f/\partial\varphi \rangle|^2 (\Delta\varphi)^2;$$

then we find

$$\Delta L_z \Delta\varphi \geqslant 1/2, \quad \Delta\varphi \ll 1. \tag{15}$$

If we do not restrict ourselves to small values of $\Delta\varphi$ and define the quantity $\Delta\varphi$ correspondingly,[10, 11, 4] we obtain

$$\Delta L_z \Delta\varphi \geqslant 1/2 [1 - 3(\Delta\varphi)^2/\pi^2]. \tag{16}$$

In order to find the uncertainty relation between the number of quanta and the phase φ [or $f(\varphi)$], we note that the following relation is satisfied identically:[1)]

$$\langle (\Delta L_z)^2 \rangle = \langle (\Delta n)^2 \rangle + (\langle n \rangle + 1/2)^2 - \langle L_z \rangle^2. \tag{17}$$

Replacing $\langle (\Delta L_z)^2 \rangle$ in (14), (15), and (16) by $\langle (\Delta n)^2 \rangle + (\langle n \rangle + 1/2)^2$, we only strengthen the inequalities and find

$$[\langle (\Delta n)^2 \rangle + (\langle n \rangle + 1/2)^2]^{1/2} \Delta f \geqslant 1/2 |\langle \partial f/\partial\varphi \rangle|, \tag{14$'$}$$

$$[\langle (\Delta n)^2 \rangle + (\langle n \rangle + 1/2)^2]^{1/2} \Delta\varphi \geqslant 1/2, \quad \Delta\varphi \ll 1, \tag{15$'$}$$

$$[\langle (\Delta n)^2 \rangle + (\langle n \rangle + 1/2)^2]^{1/2} \Delta\varphi \geqslant 1/2 [1 - 3(\Delta\varphi)^2/\pi^2]. \tag{16$'$}$$

These inequalities solve completely the problem of finding the uncertainty relation between the number of quanta and the phase.

In the quasi-classical approximation the inequalities (14′) to (16′) can be replaced by more stringent ones:

$$\Delta n \Delta f \geqslant 1/2 |\langle \partial f/\partial\varphi \rangle|. \tag{14$''$}$$

Indeed, in this case Eq. (13) holds for $n \gg 1$, and hence

$$n = \pm L_z.$$

From this and from (17) we find that in the quasi-classical limit

$$\Delta n = \Delta L_z$$

and (14″) follows from (14).

On the other hand, in the case where the quasi-classical approximation is inapplicable, it is meaningful to give an example illustrating the possibility of reaching an equality of both sides of (14′) for $\Delta n = 0$ (when according to the traditional uncertainty relation $\Delta f \to \infty$, if $\langle \partial f/\partial\varphi \rangle \neq 0$). Let

$$f(\varphi) = \sin(2n+1)\varphi, \quad \Psi(\varphi) = \frac{1}{\sqrt{\pi}} \cos\left(n + \frac{1}{2}\right)\varphi,$$

then

$$\langle (\Delta f)^2 \rangle = \langle f^2(\varphi) \rangle = 1/4, \quad \langle \partial f/\partial\varphi \rangle = n + 1/2,$$

and we obtain $(n + 1/2)\Delta f = \frac{1}{2} \partial f/\partial\varphi$.

[1)]Indeed, $n = IL_z - 1/2$; $\langle (\Delta n)^2 \rangle = \langle L_z^2 \rangle - (\langle n \rangle + 1/2)^2$ (since $I^2 = 1$); $\langle (\Delta L_z)^2 \rangle = \langle L^2{}_z \rangle - \langle L_z \rangle^2$. Comparison of these two equations yields (17).

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