SPIRAL STRUCTURE IN A ONE-DIMENSIONAL CHAIN OF SPINS

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A spin chain is considered in a model, which can be solved exactly, involving a Hamiltonian containing the antisymmetric part of the tensor of the interaction constant between the nearest neighbors. The system reduces to an ideal fermion gas with a dispersion law which is not symmetric with respect to change of sign of the quasimomentum. Correlators of the transverse spin components are calculated and it is shown that the system possesses a helical spin structure if the longitudinal magnetic field is smaller than the saturation field.

IN the construction of a microscopic theory of spiral structures in ferro- and antiferromagnetics, one usually starts from a ground state which is determined classically (see, for example, the reviews [1,2].) It remains unclear whether such an approach corresponds to the true state of a quantum system. Therefore, it seems useful to us to investigate the simplest model in which a spiral structure appears and which admits an exact quantum mechanical solution.

As shown by Lieb, Schultz, and Mattis [3], and by Pikin and one of the authors [4], a one-dimensional chain of spins ($s = 1/2$) with nearest-neighbor interaction not containing longitudinal (s^z) spin components, admits of an exact solution. In the present paper it is shown that if the Hamiltonian of the system contains a part antisymmetric in the x and y spin components, this leads to the appearance of a spin structure analogous to the spiral structures in antiferromagnets. In the previous work [3,4], only an interaction symmetric in the spins has been considered.

1. The Hamiltonian for our model is

$$\mathcal{H} = -\sum_n J_{jk} s_n^j s_{n+1}^k - \mu H \sum_n s_n^z, \tag{1}$$

where J_{jk} is a second rank interaction tensor ($j, k = x, y$), H is an external magnetic field directed along the z axis, and μ is the spin magnetic moment at the site.

If one writes the tensor J_{jk} as the sum of its symmetric and antisymmetric parts, and orients the symmetric part along the principal axes, the Hamiltonian (1) takes the form

$$\mathcal{H} = -\sum_n (J_x s_n^x s_{n+1}^x + J_y s_n^y s_{n+1}^y) - J_a \sum_n (s_n^x s_{n+1}^y - s_n^y s_{n+1}^x) - \mu H \sum_n s_n^z. \tag{2}$$

Here J_x and J_y are the principal values of the symmetric part of the tensor J_{jk}, and $J_a = (J_{xy} - J_{yx})/2$ is the antisymmetric part. We shall take the constants J_x, J_y, and J_a to be positive. Going over to the cyclic spin components $s_n^{\pm} = s_n^x \pm s_n^y$ and using the transformation [5]

$$s_n^+ = \prod_{m<n} \sigma_m^z a_n,$$

$$s_n^- = \prod_{m<n} \sigma_m^z a_n^+, \quad 2s_n^z \equiv \sigma_n^z = 1 - 2a_n^+ a_n, \tag{3}$$

where a_n and a_n^+ are Fermi operators, we obtain

$$\mathcal{H} = -\frac{J_x + J_y - 2iJ_a}{4} \sum_n a_n^+ a_{n+1} - \frac{J_x + J_y + 2iJ_a}{4} \sum_n a_{n+1}^+ a_n - \frac{J_x - J_y}{4} \sum_n (a_n^+ a_{n+1}^+ + a_{n+1} a_n) + \mu H \sum_n a_n^+ a_n - \frac{N\mu H}{2}, \tag{4}$$

N is the number of sites in the chain, which later will be made infinite.

To diagonalize the Hamiltonian (4) we transform to the Fourier components $a_\lambda = N^{-1/2} \Sigma_n \exp(-i\lambda n) a_n$, ($-\pi \le \lambda \le \pi$). Then

$$\mathcal{H} = -\frac{N\mu H}{2} + \sum_\lambda \{A_\lambda a_\lambda^+ a_\lambda + B_\lambda a_\lambda a_{-\lambda} + B_\lambda^* a_{-\lambda}^+ a_\lambda^+\}, \tag{5}$$

where

$$A_\lambda = \mu H - {}^1/_2(J_x + J_y)\cos\lambda - J_a \sin\lambda,$$

$$B_\lambda = {}^1/_4(J_x - J_y)e^{-i\lambda}. \qquad (5')$$

Performing a u, v-transformation to new Fermi operators b_λ^+ and b_λ:

$$a_\lambda = u_\lambda b_\lambda + v_{-\lambda}^* b_{-\lambda}^+, \quad a_\lambda^+ = u_\lambda^* b_\lambda^+ + v_{-\lambda} b_{-\lambda}, \qquad (6)$$

we bring the Hamiltonian (5) to a diagonal form:

$$\mathcal{H} = \sum_\lambda \left(b_\lambda^+ b_\lambda - \frac{1}{2}\right)\varepsilon_\lambda. \qquad (7)$$

The dispersion of ϵ_λ is given by

$$\varepsilon_\lambda = {}^1/_2(A_\lambda - A_{-\lambda}) \pm {}^1/_2[(A_\lambda + A_{-\lambda})^2 + 4|B_\lambda - B_{-\lambda}|^2]^{1/2}, \qquad (8)$$

and one can select either sign in front of the radical. As is clear from (8), the two values of ϵ_λ are obtained from each other by the replacement of λ by $-\lambda$ and changing the sign of ϵ_λ ($\epsilon_\lambda^- = -\epsilon_{-\lambda}^+$).[1] One should observe that the dispersion law (8) is not an even function of λ: $\epsilon_{-\lambda} \neq \epsilon_\lambda$.[2] This is connected with the presence of the antisymmetric term ($J_a \neq 0$) in the Hamiltonian. We also give the expressions for the parameters of the u, v-transformation (6):

$$|u_\lambda|^2 = \frac{|B_\lambda - B_{-\lambda}|^2}{(\varepsilon_\lambda - A_\lambda)^2 + |B_\lambda - B_{-\lambda}|^2},$$

$$v_\lambda = \frac{A_\lambda - \varepsilon_\lambda}{B_\lambda^* - B_{-\lambda}^*} u_\lambda. \qquad (9)$$

In the simplest case, in which $J_x = J_y = J$, the Hamiltonian is already diagonal in the operators a_λ, as is evident from formulas (5) and (5′), with

$$\varepsilon_\lambda = A_\lambda = \mu H - J\cos\lambda - J_a \sin\lambda,$$

or (10)

$$\varepsilon_\lambda = \mu H - \sqrt{J^2 + J_a^2}\cos(\lambda - \varphi),$$

where

$$\varphi = \arcsin\frac{J_a}{\sqrt{J^2 + J_a^2}}. \qquad (11)$$

If $\mu H > [J^2 + J_a^2]^{1/2}$, then $\epsilon_\lambda > 0$ for all λ; for $\mu H < [J^2 + J_a^2]^{1/2}$ we have $\epsilon_\lambda < 0$ in the interval $\lambda_1 < \lambda < \lambda_2$, where

$$\lambda_{1,2} = \varphi \mp \arccos\frac{\mu H}{\sqrt{J^2 + J_a^2}} \qquad (12)$$

and $\epsilon_\lambda > 0$ outside this interval. Hence it follows that the energy of the "true" quasiparticle, which represents the excitation above the ground state, equals $-\epsilon_\lambda$ for $\lambda_1 < \lambda < \lambda_2$, and ϵ_λ outside this interval. Within the interval $\lambda_1 < \lambda < \lambda_2$, the transition to elementary excitations thus corresponds to a canonical transformation, for which the creation and annihilation operators change roles. The ground state energy is separated out, being equal to

$$\mathcal{E}_0 = -{}^1/_2\sum_\lambda |\varepsilon_\lambda|.$$

The magnetic moment in the ground state equals

$$M_0(H) = -\frac{\partial \mathcal{E}_0}{\partial H} = \frac{\mu N}{2}\left(1 - \frac{2}{\pi}\arccos\frac{\mu H}{\sqrt{J^2 + J_a^2}}\right).$$

M_0 becomes zero at $H = 0$ and reaches saturation at $\mu H = [J^2 + J_a^2]^{1/2}$.

2. The absence of symmetry relative to the replacement of λ by $-\lambda$ in the dispersion law (10) is connected with the noninvariance of the single-particle excitation Hamiltonian, $\hat{h}$, relative both to the time-reversal operation T and to space inversion I.

In order to verify this, we shall examine the wave function of the state with a single excitation of quasi-momentum λ: $\psi_\lambda = a_\lambda^+ \psi^{(0)}$, where $\psi^{(0)}$ is the wave function of the state without excitation.[3] Inverting relations (3), we obtain $a_n^+ = \prod_{m<n}\sigma_m^z s_n^-$. Hence

$$\psi_\lambda = \frac{1}{\sqrt{N}}\sum_n e^{i\lambda n} s_n^- \psi^{(0)} \qquad (13)$$

(we have used the fact that $\sigma_m^z \psi^{(0)} = \psi^{(0)}$ for all m).

As is evident from (13), the function $g_n = N^{-1/2}\exp(i\lambda n)$ plays the role of the wave function of an elementary excitation in the coordinate (lattice-site) representation, and the Schrödinger equation for it, $\hat{h}g_n = \epsilon g_n$, has the form

$$-{}^1/_2 J(g_{n+1} + g_{n-1}) + {}^1/_2 iJ_a(g_{n+1} - g_{n-1}) + \mu H g_n = \varepsilon g_n. \qquad (14)$$

This is obtained directly from the equation $\mathcal{H}\psi_\lambda = E\psi_\lambda$, where $\mathcal{H}$ is the Hamiltonian (2) with $J_x = J_y = J$, and $E = \epsilon - N\mu H/2$. From Eq. (14) it is

[1] Selecting one of the values of ϵ_λ in (7), we obtain the time-dependent operator $b_\lambda(t) = b_\lambda \exp(-i\epsilon_\lambda t/\hbar)$. Then obviously, the second value will correspond to the time dependence $b_{-\lambda}^+(t)$. Therefore it is clear that the other choice for ϵ_λ in (7) corresponds to the canonical transformation $b_\lambda \to b_{-\lambda}^+$.

[2] An analogous dispersion law with the same symmetry properties, but for boson excitations, is developed in the paper of Bar'yakhtar and Maleev [6], who started from a classical ground state with a spiral structure.

[3] Generally speaking, $\psi^{(0)}$ is not the wave function of the ground state since ϵ_λ for $\mu H < [J^2 + J_a^2]^{1/2}$ is not positive for all λ. For $\mu H > [J^2 + J_a^2]^{1/2}$ we have $\epsilon_\lambda > 0$, and $\psi^{(0)}$ is the ground state.

seen that $\hat{h}^* = T\hat{h}T^{-1} \neq \hat{h}$, that is, the Hamiltonian $\hat{h}$ is not invariant to time-reversal. In exactly the same way it is seen that $I\hat{h}I^{-1} \neq \hat{h}$, where $Ig_n = g_{-n}$, which also leads to an asymmetrical dispersion law. We note that the Hamiltonian (2) of the entire system is not invariant to space inversion,[4] but is invariant under time-reversal T.

3. We turn now to the investigation of the transverse spin structure. In equilibrium, the average value of the transverse spin components at any site is equal to zero, since their operators contain an odd number of Fermi operators. Therefore the spin structure in the xy plane can appear only in the correlation functions. We shall find the probability that if the projection of the spin on the x-axis equals $+1/2$ at site l, then at site $l+m$ the spin projection on the ξ- axis, which makes the angle ϑ with the x-axis in the xy plane, is also $+1/2$. To do this, we consider a two-particle density matrix $f_{l,l+m}$ for the spins at the sites l and $l+m$:

$$f_{l,\,l+m} = \mathrm{Sp}'\, e^{-\beta\mathcal{H}}, \quad \beta = 1/kT,$$

where the prime signifies that the trace is taken on the spin variables at all sites except l and $l+m$. The decomposition of the matrix $f_{l,l+m}$ in a complete set of matrices has the form

$$f_{l,l+m} = {}^1/_4(1 + \mathbf{B}_l\boldsymbol{\sigma}_l + \mathbf{C}_{l+m}\boldsymbol{\sigma}_{l+m} + D^{jk}_{l,l+m}\sigma_l^j\sigma_{l+m}^k),$$

$$j, k = x, y, z,$$

where

$$\mathbf{B}_l = \langle\boldsymbol{\sigma}_l\rangle, \quad \mathbf{C}_{l+m} = \langle\boldsymbol{\sigma}_{l+m}\rangle,$$

$$D^{jk}_{l,l+m} = \langle\sigma_l^j\sigma_{l+m}^k\rangle, \quad \sigma = 2s.$$

The brackets $\langle\ \rangle$ denote an average with an equilibrium Gibbs distribution. The conditional probability $W_{l,l+m}$ of interest is, evidently, represented by the diagonal matrix element of the operator $f_{l,l+m}$ in the state with $\sigma_l^x = 1$ and $\sigma_{l+m}^{\xi} = 1$. Recognizing that in our system $\langle\sigma_l^x\rangle = \langle\sigma_{l+m}^{\xi}\rangle = 0$, and using the equality $\sigma_{l+m}^{\xi} = \sigma_{l+m}^x\cos\vartheta + \sigma_{l+m}^y\sin\vartheta$, we obtain

$$W_{l,\,l+m} = {}^1/_4(1 + \langle\sigma_l^x\sigma_{l+m}^x\rangle\cos\vartheta + \langle\sigma_l^x\sigma_{l+m}^y\rangle\sin\vartheta). \quad (15)$$

The maximum of this probability occurs at the angle given by

$$\tan\vartheta_m = \frac{\langle\sigma_l^x\sigma_{l+m}^y\rangle}{\langle\sigma_l^x\sigma_{l+m}^x\rangle} = \frac{\langle s_l^x s_{l+m}^y\rangle}{\langle s_l^x s_{l+m}^x\rangle}, \quad (16)$$

so that $\cos\vartheta_m$ has the same sign as $\langle\sigma_l^x\sigma_{l+m}^x\rangle$.

The angle ϑ_m characterizes the transverse spin structure of our system. The investigation of the structure thus reduces to the calculation of the correlators $\langle s_l^x s_{l+m}^x\rangle$ and $\langle s_l^x s_{l+m}^y\rangle$.

4. It is convenient to calculate the correlator $\langle s_l^x s_{l+m}^+\rangle = \langle s_l^x s_{l+m}^x\rangle + i\langle s_l^x s_{l+m}^y\rangle$. Using Eq. (3), we obtain

$$2\langle s_l^+ s_{l+m}^+\rangle = \Big\langle (a_l^+ - a_l)\prod_{k=1}^{m-1}(1 - 2a_{l+k}^+ a_{l+k})\,a_{l+m}\Big\rangle.$$

In the following, we shall limit ourselves to the spectrum (10). Since correlation functions with unequal numbers of creation and annihilation operators vanish, all averages containing a_l in the first parentheses are zero, i.e.,

$$2\langle s_l^x s_{l+m}\rangle = \Big\langle a_l^+\prod_{k=1}^{m-1}(1 - 2a_{l+k}^+ a_{l+k})\,a_{l+m}\Big\rangle. \quad (17)$$

Using the identity $1 - 2a^+a = (a^+ + a)(a^+ - a)$, one can also write Eq. (17) as

$$2\langle s_l^x s_{l+m}^+\rangle = \Big\langle a_l^+\prod_{k=1}^{m-1}(a_{l+k}^+ + a_{l+k})(a_{l+k}^+ - a_{l+k})\,a_{l+m}\Big\rangle. \quad (18)$$

The terms in parentheses in Eq. (18) anticommute with each other and with a_l^+ and a_{l+m}. With the aid of Wick's theorem one can write (18) as a sum of products of all possible paired averages, treating the parentheses as single operators. We shall consider the case $H = 0$. The correlation functions $\langle(a_{l+m}^+ + a_{l+k})(a_{l+k}^+ - a_{l+k})\rangle$ then vanish, since

$$\langle(a_{l+k}^+ + a_{l+k})(a_{l+k}^+ - a_{l+k})\rangle = 1 - 2\langle a_{l+k}^+ a_{l+k}\rangle = 2\langle s_{l+k}^z\rangle,$$

and in the absence of a magnetic field the spin average at any site equals zero. Therefore, terms which contain a pairing of operators with identical indices must vanish in (18). Each of the remaining terms in the expansion can be written as a product of m factors, which appear as pair-averages of operators with different indices. On the other hand, from Eq. (17) one sees that after expansion of the brackets, only one term, corresponding to the multiplication of the second terms $2a^+a$ from all parentheses, contains 2m factors, which, after application of Wick's theorem, become a sum of products of m paired-averages. Thus,

$$2\langle s_l^x s_{l+m}^+\rangle = (-2)^{m-1}\Big\langle a_l^+\prod_{k=1}^{m-1}{}'\,a_{l+k}^+ a_{l+k}a_{l+m}\Big\rangle, \quad (19)$$

where the prime means that only pairing of operators with different indices is allowed.

Ordering the operators in (19) such that a_{n+1} follows a_n^+, and noting that the factor $(-1)^{m-1}$ then appears, we obtain

$$2\langle s_l^x s_{l+m}^+\rangle = 2^{m-1}\langle a_l^+ a_{l+1}a_{l+1}^+ a_{l+2}\ldots a_{l+m-1}^+ a_{l+m}\rangle'.$$

[4] We thank I. M. Lifshitz and E. P. Fel'dman for directing our attention to this.

It is easy to see that this average may be written as the determinant[5)]

$$4\langle s_l^x s_{l+m}^+\rangle = 2^m \begin{vmatrix} \langle a_l^+ a_{l+1}\rangle & \langle a_l^+ a_{l+2}\rangle & \dots & \langle a_l^+ a_{l+m}\rangle \\ 0 & \langle a_{l+1}^+ a_{l+2}\rangle & \dots & \langle a_{l+1}^+ a_{l+m}\rangle \\ \dots & \dots & \dots & \dots \\ \langle a_{l+m-1}^+ a_{l+1}\rangle & \dots & 0 & \langle a_{l+m-1}^+ a_{l+m}\rangle \end{vmatrix} \quad (20)$$

Zeroes occur at the positions of the elements $\langle a_n^+ a_n\rangle$. The elements of the determinant depend only upon the differences of the indices (these indicate the number of rows and columns), and are determined by the equation

$$\langle a_n^+ a_{n+r}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\lambda r}}{\exp\{\varepsilon_\lambda/kT\}+1}\, d\lambda. \quad (21)$$

At zero temperature, as is seen from (21) and (10) [for H = 0], one obtains

$$g_{pq}(\varphi) \equiv \langle a_{l+p-1}^+ a_{l+q}\rangle_{T=0}$$

$$= \frac{1}{\pi}\frac{\exp\{i(q-p+1)\varphi\}}{q-p+1}\sin(q-p+1)\frac{\pi}{2}, \quad (21')$$

where φ is given by Eq. (11).

The general term of the determinant (20) has the form, except for sign, $g_{1\alpha}(\varphi)\, g_{2\beta}(\varphi)\cdots g_{m\omega}(\varphi)$; $\alpha, \beta, \cdots, \omega$ denote a certain permutation of the numbers 1, 2, $\cdots$, m. The dependence of this term on φ is given, according to (21), by the exponential

$$\exp\{i\varphi[\alpha + (\beta-1) + \dots + (\omega - m + 1)]\} = e^{im\varphi},$$

since $\alpha + \beta + \cdots + \omega = 1 + 2 + \cdots + m = m(m+1)/2$.

Thus the common factor $\exp(im\varphi)$ can be taken out of the determinant. We find

$$4\langle s_l^x s_{l+m}^+\rangle = (2/\pi)^m e^{im\varphi} D_m, \qquad D_m = |d_{pq}|, \quad (22)$$

so that, by (21),

$$d_{pq} = \begin{cases} 0 & \text{for } p-q- \text{ odd} \\ \dfrac{(-1)^{((p-q)/2)+1}}{p-(q+1)} & \text{for } p-q- \text{ even} \end{cases} \quad (23)$$

To calculate the determinant D_m we proceed as follows. Let q have the same parity as m. We subtract the m-th column, multiplied by $(-1)^{(m-q)/2}$, from the q-th column. The nonzero pq-element is then equal to

$$(-1)^{(p-q)/2}\frac{q-m}{(p-q-1)(p-m-1)}.$$

Doing this for all q, we take the common factor $(p - m - 1)^{-1}$ out of the p-th row, and q − m out of the q-th column, except for the m-th (p and q have the same parity as m). Using an analogous procedure for the m-th row, we can factor the common term m − p from each row except the m-th, and $(m - q - 1)^{-1}$ from each column except the m-th. The remaining determinant equals $-D_{m-1}$. Thus we obtain the recurrence relation

$$D_m = \prod_{n<m}{}' \frac{(m-n)^2}{(m-n)^2-1} D_{m-1}, \qquad m \geqslant 3,$$

where the prime means that n runs through values having the same parity as m. Hence, considering that $D_1 = 1$ and $D_2 = 1$, we obtain

$$D_m = \prod_{1\leqslant n<m/2}\left(\frac{4n^2}{4n^2-1}\right)^{m-2n}. \quad (24)$$

For large m it then follows the asymptotically $D_m \sim (3\sqrt{3/4})^m$, as is easily seen by taking the logarithm of (24) and changing the sum to an integral. For the correlator $\langle s_l{}^x s_{l+m}{}^+\rangle$ we then obtain asymptotically

$$\langle s_l^x s_{l+m}^+\rangle \sim \exp\{-\alpha m + im\varphi\}, \qquad \alpha = \ln\frac{2\pi}{3\sqrt{3}}. \quad (25)$$

Thus we see that the correlation vanishes at infinity,[6)] i.e.,

$$\lim_{m\to\infty}\langle s_l^x s_{l+m}^+\rangle = 0.$$

From Eq. (22) follow the expressions for the transverse correlators of interest to us:

$$\langle s_l^x s_{l+m}^x\rangle = {}^1/_4(2/\pi)^m D_m \cos m\varphi,$$

$$\langle s_l^x s_{l+m}^y\rangle = {}^1/_4(2/\pi)^m D_m \sin m\varphi. \quad (22')$$

Thus, from (16), one sees that the angle ϑ_m of spin "rotation" on going through m sites is $m\varphi$. Thus the angle φ is the "rotation" angle corresponding to a displacement equal to a lattice constant. We note that this same angle occurs in the dispersion law (10), resulting in $\epsilon_{-\lambda} \neq \epsilon_\lambda$.

The dependence of the correlator $\langle s_l{}^x s_{l+m}{}^+\rangle$ on φ in the form $\exp(im\varphi)$ actually is not connected with the condition H = 0 and the representation of the correlator as the determinant (20). When $H < [J^2 + J_a^2]^{1/2}$ such a dependence can be discerned directly from the original formula (17), if one notes that each term of the expansion contains the factor $\exp(im\varphi)$, since the pair correlators have the same dependence upon φ as in the absence of a magnetic field (see (21)). When saturation is reached, at $H = [J^2 + J_a^2]^{1/2}$, both

5)An analogous determinant is encountered in expressions for the correlators in the two-dimensional Ising model (see, e.g., [7]).

6)The absence of correlation at infinity in our case also follows from a limiting formula for the Toeplitz determinants. [8] This result is connected with the one-dimensional system.

transverse correlators become zero. For example, for nearest neighbors

$$\langle s_l^x s_{l+1}^+ \rangle = \frac{1}{2\pi}\left[1 - \frac{(\mu H)^2}{J^2 + J_a^2}\right]^{1/2} e^{i\varphi}.$$

For nonzero temperature, the dependence of the correlators on φ is retained also in the case of the dispersion law (10). To verify this, it is sufficient to replace λ by $\lambda + \varphi$ in Eq. (21).

The longitudinal correlator $\langle s_l^z s_{l+m}^z \rangle$ is expressed by Fermi operators with the help of the simple formula

$$\langle s_l^z s_{l+m} \rangle - \langle s_l^z \rangle \langle s_{l+m}^z \rangle = -|\langle a_l^+ a_{l+m} \rangle|^2.$$

In the absence of a magnetic field at zero temperature,

$$\langle s_l^z s_{l+m}^z \rangle = -\frac{\sin^2(m\pi/2)}{(\pi m)^2}.$$

It is interesting to notice that the longitudinal correlator, unlike the transverse ones, does not decrease exponentially, but is inversely proportional to the square of the separation. We remark that the three-dimensional Fourier component of the transverse correlator has a characteristic maximum at wave vector $k = \varphi$, related to the existence of the spiral structure. The finite width of the peak is connected with the absence of long-range order, and equals α (see Eq. (25)).

5. These results can be interpreted classically. Let us take the spins in the Hamiltonian (2) to be classical vectors with components

$$s_n^x = s \sin\vartheta_n \cos\varphi_n,$$

$$s_n^y = s \sin\vartheta_n \sin\varphi_n,$$

$$s_n^z = s \cos\vartheta_n.$$

Then the Hamiltonian can be expressed in terms of angles:

$$\mathcal{H} = -s^2\sqrt{J^2 + J_a^2}\sum_l \sin\vartheta_l \sin\vartheta_{l+1}\cos\beta_l - \mu H s\sum_l \cos\vartheta_l,$$

where $\beta_l = \varphi_{l+1} - \varphi_l - \varphi$, $\sin\varphi = J_a/[J^2 + J_a^2]^{1/2}$, and it is assumed that $J_x = J_y = J$. By minimizing $\mathcal{H}$ relative to β_l and ϑ_l, it is easily seen that two homogeneous solutions exist:

1) $\sin\vartheta = 0$, β arbitrary;

2) $\sin\beta = 0$, $\cos\vartheta = \mu H / 2s\sqrt{J^2 + J_a^2}$ for $\mu H < 2s\sqrt{J^2 + J_a^2}$.

For $\mu H < 2s[J^2 + J_a^2]^{1/2}$, the minimum corresponds to the second solution with $\beta = 0$, i.e., $\varphi_{l+1} = \varphi_l + \varphi$. This means that the transverse component of a classical spin turns through an angle φ upon translation to a neighboring site; that is, a spiral spin structure exists. For $\mu H = 2s[J^2 + J_a^2]^{1/2}$, saturation occurs, the transverse components become zero, and the structure vanishes. The spiral structure, which in the quantum system appears only in the correlation functions, appears in the corresponding classical system as a rotation of the spin in passing from site to site.

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