THE KAPITZA-DIRAC EFFECT IN A STRONG RADIATION FIELD

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Scattering of electrons under the action of a strong standing electromagnetic wave is considered. It is shown that scattering occurs in the directions defined by the Laue conditions. Equations are derived for the electron scattering probability amplitudes. Formulas for determining saturation of the scattering probability in the case of prolonged interaction between the electrons and field are also deduced. Asymptotic expressions are obtained under some assumptions for the scattering amplitudes in a strong radiation field. Numerical estimates are presented and compared with the experiments.

1. The scattering of electrons in the field of a strong standing wave was first considered by Kapitza and Dirac [1] (see also [2]). Recently this effect has again become the object of numerous investigations.[3-8] This is due to the fact that the scattering probability of electrons becomes according to Kapitza and Dirac appreciable only in sufficiently strong fields. Only the construction of powerful sources of coherent monochromatic radiation (lasers) has made it possible to realize the experimental observation of the Kapitza-Dirac effect.

From the quantum point of view this effect constitutes stimulated Compton scattering, the frequencies of the radiated and absorbed photons being equal and the momenta being equal and antiparallel. Such a process is possible only in the case when a definite condition relating the energy of the electron, the frequency of the photons and their direction of relative propagation is fulfilled. The specific nature of the standing wave consists in the fact that in this case the Compton scattering of the electrons can be interpreted as diffraction by a periodic "lattice" with a period $\lambda/2$ (λ is the wavelength of the radiation). A process with absorption of n photons propagating in one direction and the emission of n photons propagating in the opposite direction corresponds to a diffraction peak of the order n. The condition determining the direction of motion of the electron for a given energy and frequency of the photons coincides in this case with the Bragg condition, the wavelength of the scattered particles being the de Broglie wavelength of the electron $\lambda_e = \hbar/mv$.

Restricting ourselves to the first diffraction maximum, we shall first of all consider the problem of how strict the Bragg condition is when weak nonmonochromaticity of the electromagnetic field is taken into account. Let the electron absorb a photon with a frequency ω and momentum $(-\mathbf{k})$ and emit a photon $(\omega', \mathbf{k}')$. The expression for the electron scattering probability can be readily obtained from well-known formulas for the Compton scattering cross section [9] if the presence of the external field is taken into account and when one goes over from a spontaneous to a stimulated process. By virtue of the conservation laws for weak nonmonochromaticity of the field the fraction of deflected electrons is proportional to

$$\int d\omega d\omega' I_\omega I_{\omega'} \delta(\varepsilon' - \varepsilon - \hbar\omega + \hbar\omega'), \tag{1}$$

where

$$\varepsilon' = \{\varepsilon^2 + c^2[2p(k+k') + (k+k')^2]\}^{1/2},$$

$I_\omega = c\hbar\omega N_\omega / V$ is the spectral intensity of the external field, $(N_\omega/V)\,d\omega$ is the number of photons per unit volume in the frequency interval $[\omega, \omega + d\omega]$ propagating in a given direction, ϵ is the energy of the incident electron, and p is the projection of the electron momentum on the propagation direction of the photons k. (For simplicity we neglect the spread of the photon momentum with respect to the direction.) As a result we find that the fraction of reflected electrons is given by the formula

$$w = \frac{8\pi^3 e^4 t}{\hbar^2 m^2 \omega^4 c^2} \int I_\omega I_{\omega'(\omega)} d\omega, \quad \omega' = \omega \frac{\varepsilon + cp}{\varepsilon - cp + \hbar\omega}, \tag{2}$$

where t is the time during which the electron traverses the laser beam.

For optical frequencies $\hbar\omega \ll \epsilon$, so that we have

$$\omega' \cong \omega\left[1 - 2\frac{\hbar\omega}{\varepsilon}\left(1 - \frac{\sin\theta}{\sin\theta_0}\right)\right]. \quad (3)$$

$\pi/2 - \theta$ is the angle between the direction of motion of the electron and the vector $\mathbf{k}$, θ_0 is the angle determined by the Bragg condition

$$\sin\theta_0 = \lambda_e/\lambda = \hbar\omega/c|\mathbf{p}|, \quad (4)$$

and $\mathbf{p}$ is the total momentum of the electron. When the condition $\theta = \theta_0$ is rigorously fulfilled, the probability of electron deflection is maximum. We denote by $\Delta\theta$ that deviation from the Bragg angle for which the effect decreases considerably. Obviously this occurs when the frequency ω' differs from ω by a quantity of the order of the width $\Delta\omega$ of the spectral line of the laser radiation. Assuming that $\theta_0 \ll 1$ and $|\Delta\theta| \ll 1$, we find from (4)

$$\frac{\Delta\theta}{\theta_0} = \frac{\varepsilon}{\hbar\omega}\frac{\Delta\omega}{\omega} \cong \frac{mc^2}{\hbar\omega}\frac{\Delta\omega}{\omega}. \quad (5)$$

Thus the nonmonochromaticity of the field leads to a washing out of the Bragg maxima. The critical value of the relative width of the radiation spectrum for which a value $\Delta\theta/\theta_0 \sim 1$ is reached is given by (5): $\Delta\omega/\omega \sim \hbar\omega/mc^2$. In the case of optical frequencies $\omega = 3\times 10^{15}$ sec^{-1} the nonmonochromaticity of the field becomes appreciable in this sense when $\Delta\omega/\omega \sim 3\times 10^{-6}$.

Estimates in accordance with the Kapitza-Dirac formula show that when $\omega \sim 3\times 10^{15}$ sec^{-1} and $\Delta\omega/\omega \sim 10^5$, at an electron energy $\epsilon \sim 10$ keV the reflection probability of electrons reaches a value ~ 1 for fields corresponding to an energy flux density $I \sim 1$ MW/cm^2. Such fields are fully attainable with the aid of laser sources and do not constitute the limit of present-day experimental technique. Thus at present fields are attainable for which the Kapitza-Dirac formula is not valid by virtue of the inapplicability of perturbation theory.

The Kapitza-Dirac result also loses its validity on going over to a highly monochromatic field. Here one must allow for the finite time of interaction of the electron with the field. Obviously perturbation theory yields in the first order the following expression for the reflection probability of electrons:

$$w = \frac{4\pi^2e^4c^2}{\hbar^4\omega^8}I^2\left(1 - \frac{\sin\theta}{\sin\theta_0}\right)^{-2}\sin^2\left[\frac{\hbar\omega^2t}{mc^2}\left(1 - \frac{\sin\theta}{\sin\theta_0}\right)\right]. \quad (6)$$

This formula shows that the reflection probability of electrons in the case of a monochromatic field increases sharply on approaching a Bragg direction. The relative width of the maximum is determined by the interaction time $\Delta\theta/\theta_0 \sim mc^2/\hbar\omega^2t$. The value of the probability at the maximum is $w \cong 4\pi^2e^4I^2t^2/\hbar^2\omega^4c^2$. Equation (6) is not valid for very strong fields and for sufficiently long interaction of the electrons with the field. Estimates indicate that for sensible values of the interaction time t Eq. (6) leads to values $w > 1$ for fields attainable at present.

2. The results of perturbation theory are thus inapplicable for sufficiently long interaction and for sufficiently large field intensities, the critical values of the fields being definitely attainable with present-day laser-type light sources. It is therefore of interest to extend the theory to the case of strong fields and long interactions of electrons with the field. Below we shall restrict ourselves to consideration of a strictly monochromatic field. The treatment will be based on a semiclassical approach in which the motion of the electron is described quantum mechanically whereas the external field is described classically. With such an approach a rigorous consideration of the stimulated Compton scattering requires a knowledge of the exact wave functions of the electron in the field of two plane waves neither of which can be considered weak. The problem of finding such wave functions is generally speaking very complicated. However, in the case when the field is a standing monochromatic wave and the velocity of the translational motion of the electron and the velocity of its oscillations in the external field is much smaller than the speed of light one can make a certain simplifying assumption.

Under these conditions the time during which the electron traverses a distance of the order of a wavelength is large compared with the period of the field oscillations. One can therefore assume that the time oscillations of the field are less appreciable than the spatial oscillations and consider the motion of the electron in some average field. In practice this reduces to time averaging of the nonrelativistic Hamiltonian of the electron in the external field. Such a procedure fully corresponds to going over to gauge potentials.[10] The possibility of such an approach has also been indicated in[5]. When perturbation theory is applicable one can verify directly that the discarded parts of the Hamiltonian make a small contribution $\sim v^2/c^2$ or $\sim \hbar\omega/mc^2$. As a result we find that the stationary Schrödinger equation reduces to the Mathieu equation[11]

$$y'' + (a - 2q\cos 2z)y = 0,$$
$$a = 2m\varepsilon/k^2, \quad q = \frac{4\pi e^2c}{\hbar^2\omega^4}I, \quad z = \frac{k\tilde{z}}{\hbar}, \quad (7)$$

where $\tilde{z}$ is the spatial coordinate along the vector k.

The standard statement of the quantum mechanical problem consists in the fact that at the initial moment the electron has a momentum p and it is required to determine the probability that at the instant t it will go over into a state with momentum p′. The wave function of the electron at the instant t can be formally written in the form

$$\hat{S}|p\rangle = \exp\left\{i\tau\left(\frac{\partial^2}{\partial z^2} - 2q\cos 2z\right)\right\}|p\rangle. \tag{8}$$

Here $|p\rangle$ is the wave function of the electron at the initial instant $\tau = 0$. It is assumed that this is a plain wave; $\tau = \hbar\omega^2 t/2mc^2$ is the dimensionless time. Here and everywhere below we understand p to be the projection of the electron momentum onto the direction of the vector k expressed in dimensionless units, i.e., referred to the quantity $k = \hbar\omega/c$.

The amplitude of the scattering probability of the electron with a transition to the state with momentum p′ is obviously given by

$$A_{pp'}(\tau) = \langle p'|\hat{S}(\tau)|p\rangle. \tag{9}$$

The simplest method of analysis of expressions (8) and (9) consists in expanding the plane wave $|p\rangle$ in eigenfunctions of the Hamiltonian $\partial^2/\partial z^2 - 2q\cos 2z$, i.e., in Mathieu functions. In this case we obtain for the amplitude of the scattering probability $A_{pp'}$ the expression

$$A_{pp'}(t) = \frac{1}{2\pi\hbar}\int d\nu \exp\left\{-i\frac{k^2 t}{2m\hbar}a_\nu\right\} \times\left[\frac{y_{1\nu}^*(p)\,y_{1\nu}(p')}{N_{1\nu}} + \frac{y_{2\nu}^*(p)\,y_{2\nu}(p')}{N_{2\nu}}\right]; \tag{10}$$

$y_{1\nu}(z)$ and $y_{2\nu}(z)$ are two linearly independent solutions of Eq. (7) corresponding to definite values of the parameters (q, a_ν), where ν is some index which determines the solutions; $y_{i\nu}(p)$ are Fourier transforms of the solutions

$$y_{i\nu}(p) = \int dz\, y_{i\nu}(z)\exp[-ipz], \tag{11}$$

$N_{i\nu}$ are normalizing factors

$$\int y_{i\nu}(z)\,y_{j\nu'}^*(z)\,dz = N_{i\nu}\delta_{ij}\delta(\nu-\nu'). \tag{12}$$

Integration over ν extends over all stability regions of the Mathieu equation (7).

The probability amplitude is normalized in such a way that the fraction of electrons undergoing scattering at the instant t is given by the expression

$$\frac{2\pi\hbar}{L}\int dp'\,|A_{pp'}(t)|^2, \tag{13}$$

where L is the normalization length.

Unfortunately, only series expansions in powers of q, valid for sufficiently small q which corresponds to sufficiently weak fields (at a frequency $\omega = 3\times 10^{15}\ \mathrm{sec}^{-1}$ a value $I \cong 10\ \mathrm{MW/cm^2}$ corresponds to $q = 1$), are known in practice for the Mathieu functions. Therefore the exact solution (10) hardly permits an investigation of the asymptotic properties at $q \gg 1$. However, Eq. (10) makes it possible to obtain a result which differs under certain conditions from the usual formulas of perturbation theory even when $q < 1$.

We choose one of the solutions of the Mathieu equation in the form [11]

$$y_{1\sigma} = \exp(\mu(\sigma)z)\varphi(z,\sigma). \tag{14}$$

One can convince oneself that when $q \ll 1$ the most appreciable contribution to expansion (10) is due to solutions near the first instability region of the Mathieu equation (this is only true for the first diffraction maximum). For $\varphi(z,\sigma)$, $\mu(\sigma)$, and $a_\sigma(q)$ we have the following expansions:

$$\begin{aligned}\varphi(z,\sigma) &\cong \sin(2z-\sigma)+\dots,\\ \mu(\sigma) &\cong -\tfrac{1}{2}q\sin 2\sigma+\dots,\\ a_\sigma(q) &\cong 1 - q\cos 2\sigma + \tfrac{1}{8}q^2\cos 4\sigma+\dots\end{aligned} \tag{15}$$

The parameter σ takes on values $i\nu$ or $(-\pi/2 + i\nu)$, $\nu \geq 0$ in the stability regions. The second solution is obtained from (14) by changing the sign of σ.

Calculating the norm, carrying out the integration in (10), and taking into account expansions (15), we obtain the following expression for the fraction of electrons deflected in the direction of the first diffraction maximum:

$$w = q^2\frac{\sin^2\tau\sqrt{q^2+\xi^2}}{q^2+\xi^2}, \tag{16}$$

where

$$\xi = \frac{mc^2\Delta E}{(\hbar\omega)^2} = 2\left(1-\frac{\sin\theta}{\sin\theta_0}\right) \cong 2\left(1-\frac{\theta}{\theta_0}\right)$$

[formula (16) is valid for $\xi \ll 1$].

The result indicates that the usually employed formal expansion in powers of q^2 is correct only in the region $\xi \gg q$, and for $\xi \to 0$ it contains divergences in the higher orders of q^2 (beginning with q^6).

The region of applicability of perturbation theory is thus limited by the condition $q\tau \ll 1$. In practice this condition can turn out to be considerably stricter than the condition $q \ll 1$. If one makes the time of interaction t of the electron with the radiation field go to infinity, then the equations of perturbation theory become in general

inapplicable to an analysis of the stimulated process in a strictly monochromatic field.

In the case when $q\tau \gg 1$, the width of the Bragg maximum is determined by the magnitude of the field $\Delta\theta/\theta_0 \sim q$. Inside the maximum the probability of reflection of electrons oscillates rapidly on changing the angle of incidence. Under the actual conditions of the experiment these oscillations are averaged out on account of the spread of the velocities of the electrons due to the nonmonochromaticity of the field. The average value of the probability increases sharply on approaching the Bragg angle θ_0 and reaches a value $\sim 1/2$.

3. Thus the use of the plane-wave expansion in Mathieu functions makes it possible to obtain a generalization of perturbation theory to the case of long interaction $q\tau \gg 1$ assuming the field to be weak $q \ll 1$, and only for the first Bragg maximum. An extension of the theory to the case of higher-order maxima and an analysis of the asymptotics of a very strong field by the above method are very difficult to carry out.

It turns out, however, that considerably more information can be obtained by determining certain general properties of the scattering amplitude. From the periodicity of the operator $\hat{S}(\tau)$ (8) in the spatial coordinate it follows that it can be represented in the form

$$\hat{S}(\tau) = \sum_{n=-\infty}^{+\infty} (-i)^n e^{2inz} F_n\left(\tau, -i\frac{\partial}{\partial z}\right) \exp\left\{i\tau\frac{\partial^2}{\partial z^2}\right\}, \quad (17)$$

where n takes on integer values. Bearing in mind that henceforth the operator $\hat{S}(\tau)$ will always act only on the wave function of the electron in the initial state $|p\rangle$, we can consider the operator quantities F_n to be ordinary functions $F_n = F_n(\tau, p)$.

Thus the amplitude of the electron scattering probability differs from zero only for changes of the electron momentum $\Delta p = p' - p$ by $-2n$, where $n = 0, \pm 1, \pm 2, \ldots$, and is given by the functions $F_n(\tau, p)$. Consequently $F_n(\tau, p)$ is the amplitude of the electron scattering probability in the direction of the n-th diffraction maximum. The condition $\Delta p = -2n$ can be written in the form

$$1/2\lambda(\sin\theta - \cos\theta \operatorname{tg}\theta') = n\lambda_e, \quad (18)$$

where $\lambda_e = \hbar/mv$, v is the electron velocity, θ and θ' are the glancing angles of the incident and scattered electron beams, i.e., the angles between the direction of motion and the equal-phase planes in the standing wave.

For small angles θ and not very large n condition (18) coincides with the Laue condition

$$1/2\lambda(\sin\theta - \sin\theta') = n\lambda_e.$$

The difference between condition (18) and the Laue condition is due to neglect of the finite width of the light beam. Under practically realizable conditions [2,3] $\theta \sim 10^{-5}$ and consequently condition (18) differs from the Laue condition only for $n \gtrsim 10^5$. The intensity of diffraction maxima of such high order is practically always very small. One can therefore practically always assume that the directions of propagation of the scattered electrons are determined by the Laue conditions.

Differentiating (8) and (17) with respect to time, we obtain for the amplitudes the following equations:

$$\partial F_n/\partial\tau = i\gamma_n F_n + q(F_{n+1} - F_{n-1}), \quad (19)$$

where $\gamma_n = -4n(n + p)$ with the initial conditions $F_n(0, p) = \delta_{n,0}$.

The problem of determining all the amplitudes $F_n(\tau, p)$ from Eq. (19) can be reduced to the following equivalent problem. We introduce the function

$$S(z) = \sum_{n=-\infty}^{+\infty} e^{2inz} F_n(\tau, p).$$

The system of equations (19) is equivalent to the equation for $S(z)$:

$$i\frac{\partial S}{\partial\tau} = -\frac{\partial^2 S}{\partial z^2} + 2ip\frac{\partial S}{\partial z} + 2qS\sin 2z, \qquad S(0, z) = 1, \quad S(\tau, z) = S(\tau, z + \pi). \quad (20)$$

The amplitudes F_n are determined with the aid of a Fourier transform of the function $S(z)$:

$$F_n(\tau, p) = \frac{1}{\pi}\int_0^{\pi} S(\tau, z)e^{-2inz}\,dz. \quad (21)$$

Equation (20) is an analog of the equation for Bloch functions, except that in solid state theory one usually seeks stationary solutions of the Schrödinger equation. One can of course seek $S(\tau, z)$ in the form of an expansion in terms of stationary solutions, but with this we again go back to the expansion in Mathieu functions.

4. The system of equations (19) is readily solved by successive approximations in the case when $q \ll 1$. These are the usual results of perturbation theory. With accuracy up to quantities of higher order of smallness in q the amplitudes F_n are given by the expressions

$$F_0 = 1 + q^2[\gamma_1^{-2}(e^{i\gamma_1\tau} - 1) + \gamma_{-1}^{-2}(e^{i\gamma_{-1}\tau} - 1) - i\tau(\gamma_1\gamma_{-1})^{-1}(\gamma_1^{-1} + \gamma_{-1}^{-1})] + O(q^4), \quad (22)$$

$$F_1 = iq\gamma_1^{-1}(e^{i\gamma_1\tau} - 1) + O(q^3), \quad (23)$$

$$F_2 = q^2\gamma_1^{-1}[\gamma_2^{-1}(1 - e^{i\gamma_2\tau}) - (\gamma_1 - \gamma_2)(e^{i\gamma_1\tau} - e^{i\gamma_2\tau})] + O(q^4), \quad (24)$$

$$F_n = O(q^n). \tag{25}$$

It is seen from these formulas that in addition to an oscillatory time dependence of the type $\exp(i\gamma n\tau)$ the amplitudes F_n also contain terms which depend on τ according to the power law $\tau^m \exp(i\gamma_n\tau)$. This is explained by the following reasons.

The exact solution of the system (19) is a superposition of oscillating functions, the eigenfrequencies being the exact solutions of the corresponding characteristic equation. Therefore the expansions (22)—(25), being successive expansions of the amplitudes in powers of q, contain also expansions of oscillating exponents for deviations of the eigenfrequencies from the quantities γ_n.

These are the so-called secular terms. They lead to a power dependence on τ. The other reason for the appearance of the power dependence on the time is specific for the system (19). The eigenfrequencies of the zeroth approximation depend on the parameter p. For $p = -m$, $m = 0, \pm 1, \pm 2, \ldots$ all the eigenfrequencies (or all except one) turn out to be pairwise degenerate $\gamma_n = \gamma_{m-n}$. This is related to the sharp increase of certain of the amplitudes F_n when the momentum p approaches an integer value. The condition $p = -m$ coincides with the Bragg condition of order $|m|$. When the finiteness of the parameter is taken into account this degeneracy is lifted. For this reason the exact solution of the system (19) contains only an oscillatory dependence on τ.

An analysis of perturbation theory formulas shows that the expansion of the amplitudes F_n in powers of q begins with q^n where in the first nonvanishing order in q the amplitude F_n contains oscillatory terms of the type $\exp(i\gamma_k\tau)$, $0 \le k \le n$. The power dependence on τ appears only for integer values of the momentum on account of the degeneracy. For $\tau \to \infty$ these terms lead to δ functions in the reflection probabilities.

On approaching integer values of the momentum $p = -m$ (for simplicity we assume $m \ge 0$) the degeneracy begins to play a role in amplitudes F_n with $n \ge n_0$ where $n_0 = 1 + m/2$ for even m and $n_0 = (m+1)/2$ for odd m (in the first nonvanishing approximation in q).

The form of perturbation theory under consideration (when the finite width of the light ray is not taken into account) predicts thus that on approaching the Bragg angle of order m ($p \cong -m$) the scattering occurs primarily with a momentum change Δp of $-2n_0$. Thus, according to perturbation theory, the angle of incidence is equal to the angle of reflection only in the case of Bragg angles of both the first and second order $m = 1$ and $m = 2$.

5. The presence of a power dependence in expansions (22)—(25) leads to the circumstance that the conditions of applicability of the perturbation theory formulas impose restrictions not only on the magnitude of the field q but also on the length of the interaction τ. We shall attempt to obtain more accurate formulas assuming, as before, the field to be weak $q \ll 1$. This problem consists of two parts: it is necessary to find more accurate expressions for the eigenfrequencies and then to determine the contribution of the various oscillating functions to the amplitudes F_n. We shall restrict ourselves to a consideration of angles close to Bragg angles of first, second, and third order. We shall seek more accurate expressions only for these eigenfrequencies which lead according to perturbation theory to a power dependence of the amplitudes F_n already in the first nonvanishing orders in q.

In the case when the direction of motion of the electrons is close to a direction determined by the Bragg condition of the first order we have $|1 + p| \ll 1$. The eigenfrequencies of interest are those solutions of the characteristic equation corresponding to the system (19) which for $q \to 0$ go over to γ_0 and γ_1. The characteristic equation for determining these roots can be written approximately in the form

$$\gamma(\gamma + 4(1+p)) - q^2 = 0. \tag{26}$$

Obviously this equation has the following solutions:

$$\tilde{\gamma}_{0,1}(q) \cong -2(1+p) \pm \operatorname{sgn}(1+p)\sqrt{4(1+p)^2 + q^2}. \tag{27}$$

On approaching the Bragg angle of the second order $|p + 2| \ll 1$. One must find eigenfrequencies going over for $q \to 0$ to γ_0 and γ_2. The approximate characteristic equation is of the form

$$144\gamma^2 + 48\gamma(q^2 + 24(2+p)) - q^2(5q^2 - 192(2+p)) = 0. \tag{28}$$

Hence we obtain expressions for the eigenfrequencies

$$\gamma_{0,2} = \frac{1}{12}\{-2(q^2 + 24(2+p)) \pm 3\operatorname{sgn}(2+p) \times \sqrt{[16(2+p)]^2 + q^4}\}. \tag{29}$$

Equations (26) and (29) are obtained when consistent account is taken in the characteristic equation of terms $\sim q^2$ and $\sim q^4$ respectively.

If the glancing angle of the electrons is close to the Bragg angle of the third order then $|p + 3| \ll 1$ and the eigenfrequencies $\tilde{\gamma}_0(q)$ and $\tilde{\gamma}_3(q)$

are important. In this case it is impossible to write a finite characteristic equation by making use of the smallness of the parameter q only. This is connected with the fact that terms $\sim q^4$ determine the shift of the eigenfrequencies γ_0 and $\tilde{\gamma}_3$ but do not lift their degeneracy. The degeneracy is lifted when account is taken in the characteristic equation of terms $\sim q^6$. The number of such terms in the characteristic equation is infinite. One can write only a characteristic equation whose coefficients contain infinite sums. But the terms in these sums decrease with increasing n like n^{-4}. This makes it possible to restrict oneself to a few of the first terms. The characteristic equation is of the form

$$(1+\alpha q^2)\gamma^2+\gamma(12(3+p)+\beta q^2+\delta q^4)+\varkappa q^4-\rho q^6=0 \tag{30}$$

Solutions of this equation determine the eigenfrequencies $\tilde{\gamma}_{0,3}(q)$:

$$\tilde{\gamma}_{0,3}\cong -6(3+p)-q^2/16\pm \operatorname{sgn}(3+p)\sqrt{36(3+p)^2+\lambda q^6}, \tag{31}$$

where $\lambda=\beta\delta/2-\alpha\varkappa+\rho$.

The coefficients α, β, δ, $\varkappa$, and ρ can be expressed in terms of the quantities $\gamma_n(p)$. The corresponding formulas are not cited because of their unwieldiness. In calculating the coefficient λ in $\gamma_n(p)$ one should substitute the value of the momentum $p=-3$. As a result we find $\lambda\cong 8^{-4}$.

When the initial momentum is close to Bragg directions of higher order $|p+m|\ll 1$ the degeneracy of the eigenfrequencies γ_0 and γ_m is lifted apparently only when terms $\sim q^{2m}$ are taken into account in the characteristic equation. The degree of the equation increases at the same time, and this complicates considerably the problem of finding accurate eigenfrequencies for $m>3$.

A general solution of the system (19) can obviously be written as

$$F_k=\sum C_k^n \exp(i\tilde{\gamma}_n\tau), \tag{32}$$

where $\tilde{\gamma}_n$ are the exact solutions of the characteristic equation.

After substituting the solution (32) in the system of equations (19) we can express approximately all coefficients C_k^n in terms of C_0^n. Introducing the notation $x_k^n=-C_k^n/C_0^n$, we find

$$x_{-1}^n=iq/(\tilde{\gamma}_n-\gamma_{-1}), \tag{33}$$

$$x_1^n=iq/(\tilde{\gamma}_n-\gamma_{-1})-i\tilde{\gamma}_n/q, \tag{34}$$

$$x_2^n=-1-(\tilde{\gamma}_n-\gamma_1)/(\tilde{\gamma}_n-\gamma_{-1})+\tilde{\gamma}_n(\tilde{\gamma}_n-1)/q^2, \tag{35}$$

$$x_3^n=iq/(\tilde{\gamma}_n-\gamma_{-1})-(i/q)(2\tilde{\gamma}_n-\gamma_2)-(i/q)(\tilde{\gamma}_n-\gamma_1)(\tilde{\gamma}_n-\gamma_2)(\tilde{\gamma}_n-\gamma_{-1})^{-1}+(i/q^3)\gamma_n(\gamma_n-\gamma_1)(\gamma_n-\tilde{\gamma}_2)\ \text{etc.} \tag{36}$$

The coefficients C_0^n must be determined from the initial conditions which yield

$$\sum_n C_0^n=1,\quad \sum_n x_k^n C_0^n=0. \tag{37}$$

Let us consider, like before, the case of angles close to the Bragg angles $|p+m|\ll 1$, $m=1, 2, 3$.

1) $m=1$. It is readily seen that in this case only the coefficients C_0^0 and C_0^1 are not small. Solving the system (37) for these two quantities with account of formulas (27) and (34), we obtain

$$C_0^0=-q^2/(\tilde{\gamma}_1-\tilde{\gamma}_0),\quad C_0^1=1-C_0^0. \tag{38}$$

The amplitude of the probability of electron scattering $F_1(\tau, p)$ is of the form

$$F_1\cong -qe^{2i(1+p)\tau}\frac{\sin[\tau\sqrt{4(1+p)^2+q^2}]}{[4(1+p)^2+q^2]^{1/2}}. \tag{39}$$

This formula leads to a reflection probability given by Eq. (16) and describes the effect of probability saturation on increasing the interaction time τ. On the other hand it goes over into the usual expression obtained in accordance with perturbation theory for $q\tau\ll 1$.

2) $m=2$. An analysis of Eqs. (33)—(36) shows that only the coefficients C_0^0 and C_0^2 make an appreciable contribution to Eq. (37) (the second one for $k=2$). We obtain a system of equations with two unknowns from which we find

$$C_0^0=[\gamma_1\tilde{\gamma}_2+q^2(1+\gamma_1/\gamma_{-1})]\gamma_1^{-1}(\tilde{\gamma}_2-\tilde{\gamma}_0)^{-1}, \tag{40}$$

$$C_0^2=-[\gamma_1\tilde{\gamma}_0+q^2(1+\gamma_1/\gamma_{-1})]\gamma_2^{-1}(\tilde{\gamma}_2-\tilde{\gamma}_0)^{-1}. \tag{41}$$

This makes it possible to find the only appreciable (other than F_0) scattering amplitude $F_2(\tau, p)$:

$$F_2\cong -iq^2\exp\left\{i\frac{\tilde{\gamma}_2+\tilde{\gamma}_0}{2}\tau\right\}\frac{\sin\left[\frac{1}{4}\tau\sqrt{[16(2+p)]^2+q^4}\right]}{([16(2+p)]^2+q^4)^{1/2}}, \tag{42}$$

where $\tilde{\gamma}_0$ and $\tilde{\gamma}_2$ are given by Eq. (29). This expression coincides with the results of perturbation theory in the case $q^2\tau\ll 1$.

3) $m=3$. Generally speaking, in this case the coefficients C_0^0 and C_0^3 as well as C_1^1 and C_1^2 can be appreciable. This follows from a comparison with the results of perturbation theory. However, in Eqs. (37) (for $k=3$) the contribution due to these coefficients contains a small parameter. This makes it possible to solve, as before, a system of two equations with two unknowns. In full analogy with the preceding cases we obtain

$$C_0^0=[\gamma_1\tilde{\gamma}_3+q^2(1+\gamma_1\gamma_{-1}^{-1})]\gamma_1^{-1}(\tilde{\gamma}_0-\tilde{\gamma}_3)^{-1}, \tag{43}$$

$$C_0^3=-[\gamma_1\tilde{\gamma}_0+q^2(1+\gamma_1\gamma_{-1}^{-1})]\gamma_1^{-1}(\tilde{\gamma}_0-\tilde{\gamma}_3)^{-1}, \tag{44}$$

$$F_3 \cong -\frac{q^3}{8}\exp\left(i\tau\frac{\tilde{\gamma}_0+\tilde{\gamma}_3}{2}\right)\frac{\sin[\tau\sqrt{36(3+p)^2+8^{-4}q^6}]}{[36(3+p)^2+8^{-4}q^6]^{1/2}}, \quad (45)$$

where $\tilde{\gamma}_0$ and $\tilde{\gamma}_3$ are given by expression (31).

The condition for the applicability of perturbation theory in this case is of the form $q^3\tau \ll 1$.

The coefficients C_1^1 and C_1^2 should be determined with the aid of two of Eqs. (37) (for $k = 1$ and $k = 2$), in which one should substitute expressions (43) and (44) for C_0^0 and C_0^3. Calculation shows that in this case there is compensation of the terms not containing the small parameter q. Consequently, under conditions of saturation, on approaching the Bragg angle of the third order the scattering takes place basically in such a way that the angle of incidence is equal to the angle of reflection (unlike the conclusions from perturbation theory). One can assume that this result is also valid in the case $m > 3$.

6. Let us consider the question of the asymptotic behavior in a strong field. It is seen directly from the system (19) that in the case $q \gg 1$ the first term in the right-hand side of the equations becomes less important. Neglecting this term, we find that the system of equations (19) goes over into the system of recurrence relations for the Bessel functions. Consequently the system (19) has the following solution that satisfies the initial conditions

$$F_n(\tau, p) \cong J_n(-2q\tau). \quad (46)$$

For not very small interaction times $q\tau \gg 1$ one can utilize the asymptotic values of the Bessel functions for large values of the argument, which yield

$$F_n \cong (-1)^n(\pi q\tau)^{-1/2}\cos\left(2q\tau - \frac{n\pi}{2} - \frac{\pi}{4}\right). \quad (47)$$

It is readily seen that on substituting the solution (47) in the system (19) the discarded part of the equations tends to zero like $q^{-1/2}$ as $q \to \infty$, whereas the parts taken into account increase like $\sqrt{q}$. This speaks for the fact that expression (47) is indeed the asymptotic solution of the system (19) as $q \to \infty$. For a more rigorous determination of the limits of applicability of solutions (46) and (47), we turn to Eq. (20). Introducing the notation $S(\tau, z) = \exp(-i\Psi)$, we arrive at an equation for the function $\Psi(\tau, z)$:

$$\frac{\partial\Psi}{\partial\tau} = 2p\frac{\partial\Psi}{\partial z} + \left(\frac{\partial\Psi}{\partial z}\right)^2 + i\frac{\partial^2\Psi}{\partial z^2} + 2q\sin 2z, \qquad \Psi(0, z) = 0. \quad (48)$$

Neglecting the derivatives of Ψ with respect to the spatial coordinate, we obtain the solution of the zeroth approximation

$$\Psi_0 = 2q\tau\sin 2z, \quad (49)$$

which corresponds [according to (21)] to the solution (46) for the amplitudes F_n. Account of the first power of the derivative $\partial\Psi/\partial z$ and of the second derivative $\partial^2\Psi/\partial z^2$ leads to a correction $\Psi_1' = 4q\tau^2(p\cos 2z - i\sin 2z)$. Allowance for the square of the derivative $(\partial\Psi/\partial z)^2$ yields $\Psi_1'' = (16/3)\,q^2\tau^3\cos^2 2z$.

Thus expression (49) is indeed the solution of Eq. (40) with definite restrictions on the duration of the interaction, $\tau : \tau \ll 1$ and $\tau \ll q^{-1/2}$. Since $q \gg 1$, the second condition is stricter, but neither contradicts the condition $q\tau \gg 1$ for which the asymptotic representation (47) is valid. The restriction $\tau \ll q^{-1/2}$ can be removed if one neglects in (48) the second derivative $\partial^2\Psi/\partial z^2$ and then seeks an exact solution of the equation. The resulting nonlinear equation in partial derivatives can be solved.[12] However, the solution is very unwieldy. It is expressed in terms of a function defined by a complex implicit algebraig equation and admits apparently only numerical investigation. Thus in the case of a strong field $q \gg 1$ the initial electron beam splits into a series of beams whose directions are determined by conditions (18). The intensity of the diffracted electron beams depends weakly on the initial direction of motion. The time after which there occurs an appreciable decrease of the intensity of the initial beam, $\tau_0 \sim q^{-1}$. Under the condition $\tau \ll q^{-1/2}$ the propagation of electrons, the splitting of the initial beam, and the intensities of the beams of scattered electrons are given by solutions (46) and (47). In the case of $q\tau \gg 1$ the directions along which intensive scattering occurs are located symmetrically with respect to the direction of initial electron motion. The number of scattered beams is $\sim q\tau$. The fraction of the electrons scattered in the direction of each of these beams is $\sim(q\tau)^{-1}$.

7. Thus, our treatment shows that the magnitude of the field in the Kapitza-Dirac effect is determined by the parameter q. In the case of a weak field $q \ll 1$ the probability of electron reflection can reach values ~ 1 only for long duration of the interaction $q\tau \gg 1$. In the case of a strong field $q \gg 1$ the Bragg maxima are strongly washed out, only the Laue condition is fulfilled, and the initial beam splits into a fan. The nonmonochromaticity of the field in the case $q \ll 1$ leads to a washing out of the Bragg maxima and to a weakening of the effect. In the case of a strong field the role of the nonmonochromaticity is apparently not very large.

The first experimental observation of the Kapitza-Dirac effect was carried out in [5]. Ac-

cording to the estimate of the authors the intensity of the field was 1—10 MW/cm^2 and the relative width of the radiation spectrum $\Delta\omega/\omega \lesssim 2\times 10^{-5}$. The values of the parameters q and τ were: $q \cong 0.98$ and $\tau \cong 2.24$. An experimental observation of stimulated scattering of electrons in a standing wave was also carried out in a paper by Schwartz et al,[6] who apparently used in their estimates somewhat too low a value of the field and too high values of the spectral width. Comparing the conditions of the experiment of[6] with those in[5], we can estimate the parameters q and τ in the case of[6]. As a result we obtain: $q \cong 6$, $\tau \cong 0.3$, and $\Delta\omega/\omega \sim 3\times 10^{-5}$. Possibly in[6] the field and its spectral width were in fact somewhat smaller. In both cases the values of the parameters correspond thus to the intermediate situation between the cases of a strong and of a weak field. One can expect then the appearance of rather strong diffraction maxima, including maxima of higher orders. Their widths should be rather large.

For a more detailed comparison of theory with experiment it is apparently necessary to carry out detailed quantitative measurements. It is essential to observe experimentally the dependence of the effect on the magnitude of the field and the duration of the interaction, to measure the intensities of the diffraction maxima of various orders for various values of the field etc.

The observation of the saturation effect of the probability of the case of a weak field for long interaction of the electrons with a standing wave is possible, for example, when the following experimental conditions are fulfilled. At a frequency $\omega = 3\times 10^{15}$ sec^{-1} and a field power in the standing wave of $I \cong 2$ MW/cm^2 (which corresponds to a value $q \cong 0.2$) one must insure a spectral width $\Delta\omega/\omega \lesssim 2\times 10^{-7}$. The energy of the electron beam should not exceed ~0.14 keV, which corresponds to an electron velocity $v \sim 7\times 10^8$ cm/sec and a Bragg angle $\theta_0 \sim 1.7\times 10^{-4}$. According to the theory presented, a splitting of the initial electron beam into a whole series of beams should occur in strong fields.

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