NON-EINSTEINIAN GRAVITATIONAL EQUATIONS

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A general form for a Lagrangian is derived in agreement with present empirical data on the gravitational field. The general character of the corresponding gravitational equations is investigated. It is shown that the latter can formally differ arbitrarily much from the Einsteinian equations. However, in some cases they agree quantitatively with the Einstein equations as a result of special boundary conditions, the form of which is uniquely determined by the required asymptotic behavior of the field. Owing to the incompleteness of the empirical data, it is at present impossible to make a final selection of the equations.

1. SELECTION OF THE LAGRANGIAN

We determine the general form of the gravitational equations corresponding to the principles of the general theory of relativity and to the present empirical data. The equations are derived from a variational principle $\delta S = 0$, where

$$S = \frac{1}{c}\int \sqrt{-g}(\Lambda + M)\, d^4x, \tag{1}$$

Here c is the velocity of light, $d^4x = dx^0dx^1dx^2dx^3$, g is the determinant of the metric tensor, and Λ and M are invariant Lagrangians[1)] corresponding respectively to the gravitational field and to the gravitating matter.

The Lagrangians are subjected to the conditions imposed by the general theory of relativity. According to the equivalence principle the covariant structure of M is preserved under transformations to an inertial system; this condition determines M uniquely[2)], as well as the law of interaction of matter with the gravitational field. If the Lagrangian M is selected in this manner, Λ does not contain the dynamical variables of matter and is a metric invariant[3)] which vanishes in a flat space-time (cf. infra).

The equations of gravitation have the form

$$D_{ik} = T_{ik}, \tag{2}$$

where

$$\sqrt{-g}\, D_{ik} = 2(\sqrt{-g}\,\Lambda)_{ik}, \quad \sqrt{-g}\, T_{ik} = -2(\sqrt{-g}\, M)_{ik}, \tag{3}$$

where we have used the notation $(\ )_{ik} = \delta/\delta g_{ik}$,[4)] and T_{ik} is the energy-momentum tensor of the gravitating matter ([1], Sec. 94). The restriction

$$\Lambda \equiv 0 \text{ for } R_{iklm} \equiv 0, \tag{4}$$

where R_{iklm} is the Riemann tensor ([1], Sec. 92), is necessary to make in Eq. (2) compatible with a flat space-time in the absence of matter[5)]. The Lagrangian Λ and the tensor D_{ik} (which will be called the dynamical tensor) are determined from the empirical data.

Quantitative data are available at present only for weak fields $|\varphi| \ll c^2$ (φ denotes the gravitational potential). The data can be reduced to two empirical laws. The first is Poisson's equation

$$\Delta\varphi = 4\pi G\rho, \tag{5}$$

where ρ denotes the mass density and G the Newtonian gravitational constant; (5) is valid if one neglects all relativistic effects and all powers of the expansion parameter φ/c^2. The external gravitational field is better known. It follows from astronomical data that in free space outside a source

$$R_{ik} = 0, \tag{6}$$

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1)The invariance is not mandatory, owing to the possibility of a gauge transformation of the Lagrangian (the addition of an arbitrary divergence) without affecting the field equations. In the sequel all Lagrangians are compared to within such a gauge transformation.

2)The result is unique if M does not contain second order (and higher order) derivatives of nonscalar fields; this is true for all known forms of matter.

3)Λ depends only on the metric tensor g_{ik} and on its derivatives, taken at the same 4-dimensional point as Λ; nonlocality is excluded by the causalty principle.

4)$(f)_{ik} = f_{ik} - \partial_l f_{ik}{}^l + \partial_{lm} f_{ik}{}^{lm} \ldots$, where $\partial_l = \partial/\partial x^l$, $f_{ik} = \partial f/\partial g^{ik}$, $f_{ik}{}^l = \partial f/\partial g_l^{ik}, \ldots$, $g_l^{ik} = \partial_l g^{ik}, \ldots$; the f_{ik} are symmetrized in their upper and lower indices.

5)The condition (4) excludes the cosmological constant, independent of the concrete form of Λ.

where $R_{ik} = R_{ikab}g^{ab}$; Eq. (6) is verified not only in the nonrelativistic approximation (5), but up to the first order relativistic corrections to the spherically symmetric static harmonic of the gravitational field of the Sun.[6] At present there is no quantitative information in addition to Eqs. (5) and (6),[7] therefore the present-day data allow only to establish the general character of the relativistic equations of gravitation.

We consider Poisson's equation. In the approximation which is linear in φ the left-hand side of this equation is proportional to the component R_{00} ([1], Sec. 96), hence in a vanishingly weak gravitational field ($R_{iklm} \to 0$) the dynamical tensor and the Lagrangian Λ depend linearly on the components of the Riemann tensor[8]. The only invariant of this kind is the scalar curvature R, and accordingly (5) and (4) necessarily lead to the limiting condition [3] $\Lambda \to \text{const} \cdot R$ for $R_{iklm} \to 0$. This condition is satisfied by any Lagrangian of the following form

$$\Lambda = \frac{1}{2\varkappa_1}(R+X), \tag{7}$$

where κ_1 is a universal coupling constant (with dimension cm/erg) and X is an arbitrary metric invariant satisfying the condition $X/R \to 0$ as $R_{iklm} \to 0$. The latter does not exclude the possibility $X \equiv 0$, when the structure of the Lagrangian does not depend on the intensity of the gravitational field. If $X \neq 0$, then according to (7) there must be an additional universal constant in nature [3]. Indeed, X/R is a dimensionless metric invariant of the type indicated in footnote[3], and therefore it must depend on invariant contractions of components of the Riemann tensor and its covariant derivatives; the contractions have the dimension cm^{-k}, $k \geq 2$, and consequently they can occur in X/R only together with a factor l^k, where l is a universal constant having the dimension of a length, and such that $X \equiv 0$ if $l = 0$. It will be seen in Sec. 3 that at present it is impossible to estimate l uniquely.

According to (2) and (3) the Lagrangian (7) implies the gravitational equations

$$R_{ik} - {}^1/_2 g_{ik}R + X_{ik} = \varkappa_1 T_{ik}, \tag{8}$$

where

$$\sqrt{-g}\, X_{ik} = (\sqrt{-g}\, X)_{ik}. \tag{9}$$

The equations (8) must obviously not contradict the law (6). For this it is sufficient to assume that R-symmetry holds. This means that the tensor X_{ik} must vanish in those regions of space-time where $R = 0$. It is easy to see that in this case the Eqs. (8) admit in free space ($T_{ik} = 0$) the solution $R_{ik} = 0$, independent of the concrete form of the invariant X and of the value of the constant l. We arrive at the conclusion that an infinite set of R-symmetric gravitational equations[9] is quantitatively compatible with the empirical laws (5) and (6). A final selection among these equations is impossible at the present time because of lack of additional data.

2. BOUNDARY CONDITIONS

For the Einstein field ($X \equiv 0$) the gravitational equations establish a direct relation between the curvature of space-time and the distribution of matter. For a non-Einsteinian field the tensor X_{ik} contains the covariant derivatives of the Riemann tensor up to order $2(n-1)$ inclusive, with n the order of the Lagrangian[10], and the equations (8) are differential equations (of order at least two) in the curvature tensor. In this case the connection between curvature and matter is determined not only by Eq. (8), but also by additional conditions (boundary conditions and initial conditions). As a consequence of R-symmetry, such additional conditions may affect the character of the gravitational equations.

In order to investigate this important circumstance we consider the simplest R-symmetric Lagrangian that depends only on the scalar curvature. According to (7) such a Lagrangian has the form

$$\Lambda = \frac{1}{2\varkappa_1} Rf(\xi), \quad \xi = l^2 R, \quad f(0) = 1, \tag{10}$$

where $f(\xi)$ is an empirical function. The corresponding gravitational equations are

$$\eta(\xi)R_{ik} - {}^1/_2 Rf(\xi) g_{ik} + g_{ik}\Box\zeta(\xi) - \zeta_{ik}(\xi) = \varkappa_1 T_{ik}, \tag{11}$$

where $\zeta_{ik} = \zeta_{;i;k}$ (; denotes covariant differentiation),

[6]The other relativistic effects corresponding to (6) are at present smaller than the measurement errors.

[7]About strong fields, $|\varphi| \gtrsim c^2$ one can at present assert only that inside the source $R_{ik} \neq 0$; this result corresponds to $q \neq 0$ for the cosmological slowing-down parameter [2].

[8]Linear combinations of covariant derivatives of R_{iklm} are excluded by a gauge transformation of the Langrangian.

[9]One of the special cases of R-symmetric equations are the Einstein equations ($l = 0$, $X \equiv 0$).

[10]The maximal order of derivatives of g_{ik} which cannot be lowered by gauge transformations. For the Einsteinian field $n = 1$ ([4]; [1], § 93); for all non-Einsteinian fields $n \geq 2$.

$$\eta(\xi) = 1 + \zeta(\xi) = \frac{d}{d\xi}\{\xi f(\xi)\},$$

$$\overline{\square}\,\zeta = \frac{1}{\sqrt{-g}}\partial_n(\sqrt{-g}\,g^{nm}\,\partial_m\zeta), \tag{12}$$

here $\overline{\square}\zeta = g^{ik}\zeta_{ik}$. The contraction (11) leads to the equation

$$\overline{\square}\zeta - F = {}^1/_3\varkappa_1 T, \tag{13}$$

where

$$F = {}^1/_3 R\{2f(\xi) - \eta(\xi)\}, \quad T = g^{ik}T_{ik}. \tag{14}$$

Let the gravitational field be created by matter with $T \neq 0$ only inside a closed space-time hypersurface Σ. We consider the external field. In the Einsteinian case $\zeta \equiv 0$, and (13) implies $R \equiv 0$ outside Σ. In the non-Einsteinian case $F = \psi(\zeta)$ (the form of the function ψ is determined from (12) and (14)) and (13) implies

$$\overline{\square}\zeta - \psi(\zeta) = 0 \text{ outside } \Sigma. \tag{15}$$

The solution of Eq. (15) must vanish at infinity and thus ζ must satisfy the condition of being Galilean[11)]:

$$\zeta \to 0, \quad d\zeta/dr \to 0 \text{ as } r \to \infty, \tag{16}$$

where r is some invariant distance from the source. Consequently only boundary conditions

$$\zeta = \zeta(\Sigma), \quad \zeta_n = \zeta_n(\Sigma) \text{ on } \Sigma, \tag{17}$$

which do not contradict (16) are admissible for ζ and its normal derivative ζ_n.

The simplest method of making (16) and (17) compatible is to select vanishing values $\zeta(\Sigma) = \zeta_n(\Sigma) \equiv 0$ (we call these pseudo-Einsteinian). A solution of Eqs. (15) corresponding to pseudo-Einsteinian conditions is of the form $\zeta \equiv 0$, $R \equiv 0$ outside Σ, and thus the pseudo-Einsteinian field will, according to (11) satisfy the Einstein equations

$$R_{ik} = \varkappa_1 T_{ik} \text{ outside } \Sigma \tag{18}$$

in the external region, regardless of the concrete form of the Lagrangian[12)]. If there is no matter outside Σ (or if T_{ik} is negligible outside Σ), (18) can be replaced by the equation

$$R_{ik} = 0 \text{ outside } \Sigma, \tag{19}$$

which coincides exactly with (6). Consequently, if the law (6) is indeed valid starting from the boundary surface of the source, then this circumstance may be either a consequence of the fact that $l = 0$ and $X \equiv 0$, or a consequence of the vanishing (for $l = 0$) of the R-symmetric tensor X_{ik} in Eqs. (8) outside the surface Σ, as a result of the pseudo-Einsteinian boundary conditions. We show that in certain cases these latter conditions are unique consequences of the Galilei condition.

We consider a static gravitational field generated by a fixed[13)] spherically symmetric mass distribution of finite radius a. We select the time t and the spherical coordinates r, ϑ, φ according to the metric

$$ds^2 = c^2p^2(r)\,dt^2 - dr^2 - q^2(r)\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2) \tag{20}$$

and write Eq. (15) for the external field in the form:

$$\zeta'' = -\frac{dU(\zeta)}{d\zeta} - \gamma\zeta' \quad (r > a), \tag{21}$$

where the prime denotes differentiation with respect to r,

$$U(\zeta) = -\int_0^\zeta \psi(\zeta)\,d\zeta, \quad \gamma = \frac{d}{dr}\ln(pq^2); \tag{22}$$

the Galilei conditions imply

$$\zeta \to 0, \quad \zeta' \to 0, \quad \gamma \to 2/r \text{ for } r \to \infty. \tag{23}$$

In order to investigate the function $\zeta(r)$ we note that Eq. (21) coincides with a nonrelativistic equation of motion of a point of unit mass moving in a potential field $U(\zeta)$ with a friction force $-\gamma\zeta'$, where ζ is the "coordinate" and r plays the role of "time" (for briefness we write the terms relating to the mechanical analogue with quotation marks). The "motion" corresponds to arbitrary boundary conditions (at r = a), with the exception of pseudo-Einsteinian conditions; in this latter case $\zeta \equiv 0$ in the whole external space and the "point does not move." We write the law of "motion" in the following form

$$r - r_1 = \pm\int_\zeta^{\zeta_1}\frac{dz}{\sqrt{2\{W(z) - U(z)\}}} \quad (r \geqslant r_1), \tag{24}$$

where $\zeta_1 = \zeta(r_1)$, the sign is chosen depending on the direction of "motion," and the "total energy"

$$W(\zeta) = {}^1/_2\zeta'^2 + U(\zeta) \tag{25}$$

depends on the "coordinate" as a result of the "friction." The Galilei condition (23) requires

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11)The vanishing of the derivative is necessary in order to ensure a finite flux of gravitational energy.

12)For pseudo-Einsteinian boundary conditions the results (18) and (19) are valid not only in the case (10), but also for an arbitrary R-symmetric Lagrangian.

13)In a non-Einsteinian field the Birkhoff theorem ([5], § 61) is not valid.

that the integral (24) diverge at the lower limit $\zeta \to 0$.

We consider the limiting procedure. According to (22), (23), and (25) we have $U(\zeta \to 0) \to 0$ and $W(\zeta \to 0) \to 0$, and two cases are possible: 1) $U(\zeta)$ goes to zero not faster than $W(\zeta)$, and 2) $W(\zeta)$ goes to zero slower than $U(\zeta)$. In the first case the Galilei condition takes the form

$$\int_0 \frac{d\zeta}{\sqrt{|U(\zeta)|}} = \infty. \tag{26}$$

In the second case $W(\zeta) \gg U(\zeta)$ for $\zeta \to 0$ and $r \to \infty$. Consequently one can neglect the quantity $dU/d\zeta$ in (21), and according to (23) one can set $\gamma = 2/r$, which reduces (21) to a Laplace equation with the unique Galilean solution $\zeta = \text{const} \cdot r^{-1}$ and $\zeta' = \text{const} \cdot \zeta^2$. For $\zeta \to 0$ the condition $U(\zeta) \ll W(\zeta) \sim \zeta^4$ leads to a divergent integral (26), and we see that (26) is a general expression of the Galilei condition for the "motion." Consequently if

$$\int_0 \frac{d\zeta}{\sqrt{|U(\zeta)|}} < \infty, \tag{27}$$

the gravitational field must be pseudo-Einsteinian. We see that under definite properties of the invariant X, which are compatible with the R-symmetry, the pseudo-Einsteinian field is the only physical solution of (8).

We apply these results in order to estimate the constant l from the observable properties of the external gravitational field for a spherical static mass. We assume that the function $f(\xi)$ in (10) can be expanded in the power series

$$f(\xi) = 1 + \alpha\xi^s + \dots \quad (\xi \to 0), \tag{28}$$

where $s > 0$, and the terms that have not been written out vanish for $\xi \to 0$ faster than ξ^s. The expansion (28) corresponds to $|U(\zeta)| = \text{const.} |\zeta|^{1+1/s}$, at $\zeta \to 0$, and it follows from (27) that $s > 1$. In this case the external field is pseudo-einsteinian and is rigorously subject to the Schwarzschild metric, irrespective of the values of s and l, which makes it impossible to estimate the magnitude of l from the behavior of the external field. If one assumes s to be an integer (in order to avoid branch points of $f(\xi)$ at $\xi \to 0$), then other boundary conditions than the pseudo-Einsteinian ones become admissible for $s = 1$ and the external field may depend on l. For $s = 1$ the series (28) becomes

$$f = 1 + \alpha l^2 R + \dots \tag{29}$$

and the numerical value of $|\alpha|$ can be (partially or totally) absorbed in the definition of l.

Setting $\alpha = \epsilon/6$ and $\epsilon = \pm 1$, it is easy to show that at sufficiently large distances from the source Eq. (15) has the form

$$\Box R - \frac{\varepsilon}{l^2} R = 0, \tag{30}$$

where $\Box$ is the usual D'Alembertian; (30) implies that $\epsilon = +1$ (for $\epsilon = -1$ the group velocity of the R-waves would exceed the velocity of light). The Galilean asymptotic behavior of the spherically symmetric static solution of (30),

$$R \sim \frac{b}{r} e^{-r/l}, \tag{31}$$

shows that at the surface of the source there appears a gravitational "skin-effect" of thickness l, beyond which $R = 0$, so that the external field is subject to the law (6) with exponential accuracy. Up to the present time the gravitational skin-effect has not been observed[14)]. Consequently, either l is imperceptibly small (including the possibility $l = 0$), or the real field is pseudo-Einsteinian. In the latter situation the magnitude of l could be estimated from the behavior of the internal field.

3. THE NONRELATIVISTIC FIELD

We set $x^0 = ct$ (t is the time) and choose orthogonal cartesian coordinates for the spatial coordinates $x^\alpha = x^{1,2,3}$. In this system the nonrelativistic metric has the form

$$g_{ik} = \gamma_{ik} - 2h_{ik}, \tag{32}$$

where $\gamma_{00} = -1$, $\gamma_{0\alpha} = 0$, $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$, h_{ik} are small quantities, of the order of $1/c^2$, and $h_{00} \neq \varphi/c^2$, where φ is the gravitational potential. The transition to the nonrelativistic limit is performed in Eqs. (11) and (13) by means of linearization with respect to h_{ik}. The operator $\partial_0 = c^{-1}\partial_t$ is not taken into account, being relativistically small, and the operator $\bar{\Box}$ is replaced by the Laplacian Δ. The quantity $\Delta\zeta$ is retained, its nonlinear dependence on h_{ik} notwithstanding, in order to take into account the effect of higher order derivatives. Assuming $T_{00} = \rho c^2 = -T$ (ρ is the mass density) we write the nonrelativistic limit of Eq. (11) for $i = k = 0$:

$$\Delta\varphi - c^2\Delta\zeta + {}^1/{}_2 c^2 R = \varkappa_1 \rho c^4, \tag{33}$$

and the nonrelativistic limit of Eq. (13) as:

$$\Delta\zeta - {}^1/{}_3 R = -{}^1/{}_3 \varkappa_1 \rho c^2. \tag{34}$$

The spatial and mixed components of (11) may be disregarded.

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14)The experimental proof of the existence of gravitational skin-effect would be a clear-cut rejection of the pseudo-Einsteinian law.

We note first that according to (33) and (34) the quantity $\Delta\varphi$ is not proportional to ρ in the non-relativistic limit, if $\zeta \neq 0$. In other words for the nonrelativistic non-Einsteinian field Poisson's equation does not hold. However, for an adequate choice of the parameters, Poisson's equation may be satisfied to an arbitrary degree of approximation, which allows one to obtain some estimates for the constants l and κ_1.

We consider the internal field. Let us first assume that gravitational skin-effect exists. According to Sec. 2, the skin-depth l must be very small compared to dimensions of ordinary bodies. Therefore practically throughout the interior $\Delta\zeta \ll R$, and $\Delta\zeta$ can be neglected in Eqs. (33) and (34). In this case $\Delta\varphi = \kappa_1 \rho c^4/2$, which coincides with Poisson's equation (5) if

$$\varkappa_1 = \varkappa = 8\pi c^{-4} G; \tag{35}$$

(35) determines the value of κ_1 corresponding to the possible case of small l (including also $l = 0$). It is essential that Poisson's equation can also be valid for large $\Delta\zeta \gg R$. In this case (33) and (34) imply

$$\Delta\varphi = {}^2/_3 \varkappa_1 \rho c^4, \tag{36}$$

$$\Delta\zeta = -{}^1/_3 \varkappa_1 \rho c^2. \tag{37}$$

Eq. (36) leads to the value

$$\varkappa_1 = {}^3/_4 \varkappa, \tag{38}$$

which differs from (35); consequently $\Delta\zeta \gg R$ corresponds not to small values of l, which are possible only in the pseudo-Einsteinian field. Therefore the solution of Eq. (37) must be subjected to the boundary conditions

$$\zeta = 0, \quad \zeta_n = 0 \text{ on } \Sigma, \tag{39}$$

where ξ_n denotes the normal derivative and Σ is the boundary surface of the source.

The behavior of the function ζ inside the source is determined by the source geometry and by the character of the mass distribution inside the source. Inside a homogeneous sphere of radius a

$$\zeta = -r_0/2r \quad (r \leqslant a), \tag{40}$$

where r is the distance from the center, r_0 is the gravitational radius of the sphere, (mG/c^2). Eq. (40) shows that in a central region, of radius $r \sim r_0$, the gravitational field becomes relativistic ($\zeta \sim 1$) even for $r_0 \ll a$. The latter fact is caused by the spherical symmetry of the source and may not be true for nonspherical bodies. For example for an oblate spheroid

$$|\zeta|_{max} \sim r_0/\Delta \tag{41}$$

(r_0 is the gravitational radius, Δ the interfocal distance), and for $r_0 \ll \Delta$ the gravitational field remains nonrelativistic throughout the internal region. If the conditions of the Earth correspond to $\Delta\zeta \gg R$, one can obtain an estimate for l. Setting, $\zeta \sim (l^2 R)^s$ according to (28), and taking (41) into account, we obtain

$$R \lesssim l^{-2} (r_0/\Delta)^{1/s}, \tag{42}$$

and according to (37) $\Delta\zeta \sim \kappa\rho c^2 = \delta^{-2}$. Therefore it follows from $R/\Delta\zeta \ll 1$ that

$$l \gg \delta (r_0/\Delta)^{1/2s}, \tag{43}$$

which determines those values of l to which the magnitude (38) corresponds for the constant κ_1. For the parameters of the Earth, and for the smallest pseudo-Einsteinian integer $s = 2$ the inequality (43) leads to the estimate $l \gg 10^{11}$ cm. In conjunction with the preceding result, this shows that at present a unique estimate of l is impossible.

CONCLUSION

It has been shown that a restriction of the order of the gravitational equations, as introduced by Einstein,[15)] is not a necessary requirement for the quantitative interpretation of the empirical data on weak gravitational fields. The latter imposes only a general requirement of R-invariance of the Lagrangian (7).

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15)In the "Project" [6] equations of arbitrary order are not excluded. Judging by the note at the end of Einstein's last paper [7], he did not consider consideration of such equations as experimentally necessary.

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167