LIMITS OF STATISTICAL DESCRIPTION OF A NONLINEAR WAVE FIELD

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A system of nonlinearly interacting oscillations with small nonlinearity in a dispersive medium is considered. A condition is derived under which such a system can be approximately described by statistical means. A criterion is obtained for the randomization of the wave phases as a result of the nonlinear interaction, and the characteristic time of loss of the phase memory of the system is obtained. This makes it possible to derive a kinetic equation for the waves without assuming a priori that the initial phases are random. The spectral limits of applicability of the kinetic equation are found and a connection is established between the phase randomization time and the increment of the decay instability.

THE derivation of a kinetic equation for waves in a nonlinear turbulent medium is based on the well-known assumption that there is no phase correlation between the Fourier amplitudes of the harmonic. An example is the kinetic equation for phonons in a solid and the kinetic equation for waves in a weakly-turbulent plasma (see, e.g.,[1]). The assumption of the separation of the phase correlations is usually called the random phase approximation (RPA). As a result of the development of an instability or some other process, a very large number of waves is excited and interact with one another. The interaction has a decay character, i.e., the resonance condition

$$\sum_j n_j\omega_j = 0$$

(where ω_j are the wave frequencies and n_j are some integers) can be satisfied in an infinite number of ways. The complexity of the interaction and the large number of degrees of freedom lead us to expect a statistical ensemble of waves to be produced and the phase correlation to be lost as a result. In this paper we investigate the conditions under which a system of nonlinearly interacting oscillations can be described, with a certain degree of accuracy, by statistical laws, and we obtain a criterion for the separation of the phase correlation of the waves. A criterion of this kind was derived for interacting harmonics of a nonlinear string (the so-called Fermi-Pasta-Ulam problem[2]) by Izraĭlev and Chirikov.[3] The connection between that problem and the questions considered in the present article will be discussed later. A rigorous derivation of the main kinetic equation for a nonlinear wave field under the assumption that the initial phases are random was obtained by Brout and Prigogine[4]. The system-randomization criterion presented below allows us to dispense with the a priori RPA.

1. DERIVATION OF THE FUNDAMENTAL EQUATIONS

We consider a one-dimensional wave packet, the potential energy of wave interaction being defined by

$$V = \frac{1}{2}\sum_k \omega_k^2 u_k^2 + \beta \sum_{k_1+k_2+k_3=0} V_{k_1k_2k_3}u_{k_1}u_{k_2}u_{k_3} + \beta^2 \sum_{k_1+k_2+k_3+k_4=0} V_{k_1k_2k_3k_4}u_{k_1}u_{k_2}u_{k_3}u_{k_4} + \ldots \qquad (1.1)$$

Here β is a small parameter and the kernels $\hat{V}$ satisfy the usual symmetry properties:

$$V_{k_1k_2k_3} = V_{k_2k_1k_3} = \ldots, \qquad V_{k_1k_2k_3k_4} = V_{k_2k_1k_3k_4} = \ldots,$$

U_k is the amplitude of the k-th harmonic. The spectrum is assumed discrete with a characteristic distance $\sim\Delta k$ between harmonic and with a distance between frequencies

$$\Omega_k = \frac{d\omega_k}{dk}\Delta k. \qquad (1.2)$$

In addition we shall assume that the spectrum is of the decay type in first order. This means that simultaneously with satisfaction of the phonon momentum conservation law $k_3 = k_1 + k_2$ there can be satisfied the following energy-conservation law (resonance condition):

$$\omega_3 = \omega_1 + \omega_2 \qquad (\omega_i \equiv \omega_{k_i}). \qquad (1.3)$$

We shall consider later the more general case of the spectrum $\omega_k = \omega(k)$.

We confine ourselves in (1.1) to the terms written out, and take account of the fact that the term proportional to β^2 gives rise to a nonlinear correction to the frequency:

$$\Delta\omega_k \sim \frac{1}{\omega_k}\beta^2 \sum_{k+k_1=0} V_{kk_1}u_{k_1}^2, \quad V_{kk_1} = V_{kk_1k_2k_3}\delta_{kk_2}\delta_{k_1k_3}. \quad (1.4)$$

The expression (1.4) for $\Delta\omega_k$ is accurate apart from a numerical coefficient, the exact value of which is immaterial. We change to action and phase variables (I_k, φ_k) and rewrite the total Hamiltonian of the system of interacting waves in the form [5]

$$H = \sum_k (\omega_k I_k + \Delta\omega_k(I)I_k) + \beta \sum_{k_1+k_2+k_3=0} \left(\frac{I_{k_1}I_{k_2}I_{k_3}}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}\right)^{1/2}$$
$$\times \Big\{ V_{k_1k_2k_3}\exp\{i(\varphi_{k_1}+\varphi_{k_2}+\varphi_{k_3})\}$$
$$+ 3V_{k_1,k_2,-k_3}\exp\{i(\varphi_{k_1}+\varphi_{k_2}-\varphi_{k_3})\} + \text{c.c.}\Big\}$$
$$\doteq H_0 + V_{int}; \quad H_0 = \sum_k \omega_k I_k. \quad (1.5)$$

Here $\Delta\omega_k(I) = \Delta\omega_k(I_1, I_2, \ldots)$ and does not depend on the phases; the action (which has the meaning of the number of "quasiparticles") and the phase are defined by the relations

$$u_k = \sqrt{\frac{I_k}{\omega}}\, e^{i\varphi_k} + \sqrt{\frac{I_{-k}}{\omega_{-k}}}\, e^{-i\varphi_{-k}},$$
$$\dot{u}_k = i\sqrt{\omega_k I_k}e^{i\varphi_k} - i\sqrt{\omega_{-k}I_{-k}}e^{-i\varphi_{-k}}$$

$$\omega_{-k} = \omega_k, \quad V_{k_1k_2k_3} = \overset{*}{V}_{-k_1,-k_2,-k_3}, \quad (1.6)$$

the letters c.c. stand for terms that are complex-conjugates of those preceding them.

We introduce the density function $f(I, \varphi, t)$ in phase space, satisfying the Louiville equation:

$$\frac{\partial f}{\partial t} + \sum_k \omega_k \frac{\partial f}{\partial \varphi_k} = \sum_k \left(\frac{\partial V_{int}}{\partial \varphi_k}\frac{\partial f}{\partial I_k} - \frac{\partial V_{int}}{\partial I_k}\frac{\partial f}{\partial \varphi_k}\right) \quad (1.7)$$

and containing no additional information other than the solutions of the equations of motion

$$\dot{\varphi}_k = \omega_k + O(\beta^2), \quad \dot{I}_k = O(\beta).$$

If we go over to the interaction representation, then the second term in the left side of (1.7) vanishes, and it is necessary to replace φ_k in the right side by $\varphi_k - \omega_k t$ throughout. Inasmuch as f is a periodic function of the phases φ_k, we can write

$$f(I, \varphi, t) = \sum_n \{f^{(n)}(I,t)e^{i(n,\varphi)} + \text{c.c.}\},$$
$$(n, \varphi) \equiv \sum_k n_k\varphi_k, \quad f^{(n)} = (f^{(-n)})^*, \quad (1.8)$$

or, in the interaction representation,

$$f(I, \varphi, t) = \sum_n \{f^{(n)}(I,t)e^{i(n,\varphi-\omega t)} + \text{c.c.}\}. \quad (1.9)$$

We neglect temporarily the nonlinear correction to the frequency. Then substitution of (1.5) and (1.8) into (1.7) and a changeover to the interaction representation yields

$$\frac{\partial f^{(n)}}{\partial t} = -i\beta\{Q_{n,n+1}f^{(n+1)}e^{-i[\omega]t} + Q_{n,n-1}f^{(n-1)}e^{i[\omega]t}\}, \quad (1.10)$$

where

$$Q_{n,n\pm1} = 3\sum_{k_1k_2k_3}\Big\{V_{k_1,k_2,-k_3}\left(\frac{n_{k_1}}{2I_{k_1}} + \frac{n_{k_2}}{2I_{k_2}} - \frac{n_{k_3}}{2I_{k_3}} \pm \frac{\partial}{\partial I_{k_1}}\right.$$
$$\left.\pm\frac{\partial}{\partial I_{k_2}} \mp \frac{\partial}{\partial I_{k_3}}\right)\left(\frac{I_{k_1}I_{k_2}I_{k_3}}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}\right)^{1/2}\delta_{[k],0} + \text{c.c.}\Big\};$$
$$[\omega] \equiv \omega_{k_1} + \omega_{k_2} - \omega_{k_3}, \quad [k] \equiv k_1 + k_2 - k_3. \quad (1.11)$$

In addition, only the resonant terms, for which the decay conditions (1.3) are satisfied and which make the main contribution to $\partial f/\partial t$, have been picked out in the expression for Q.

So far, the derivation of (1.10) has been the same as that of the main kinetic equation for waves.[5] The main difference in what follows will be connected with two factors: 1) elimination of the random-phase hypothesis when choosing the initial conditions, and 2) allowance for the nonlinear frequency correction $\Delta\omega_k$.

For $t = 0$ we put

$$f(I, \varphi, 0) = \sum_n \{f^{(n)}(I, 0)e^{i(n,\varphi)} + \text{c.c.}\}. \quad (1.12)$$

The RPA is usually equivalent to the initial condition $f(I, \varphi, 0) = f(I)$, i.e., all $f^{(n)} = 0$ with the exception of $n = 0$.

Taking the Laplace transform of (1.10), we get

$$pf_p^{(n)} - f^{(n)}(I, 0) = -i\beta\{Q_{n,n-1}f^{(n-1)}_{p-i[\omega]} + Q_{n,n+1}f^{(n+1)}_{p+i[\omega]}\}. \quad (1.13)$$

From (1.13) we get an equation for $f_p^{(0)}$:

$$pf_p^{(0)} - f^{(0)}(I, 0) = -i\beta\{Q_{0,-1}f^{(-1)}_{p-i[\omega]} + Q_{0,1}f^{(1)}_{p+i[\omega]}\}. \quad (1.14)$$

We iterate (1.14) up to terms in β^2 inclusive:

$$pf_p^{(0)} - f^{(0)}(I, 0) = -i\beta\Big\{Q_{0,-1}\frac{f^{(-1)}(I,0)}{p-i[\omega]} + Q_{0,1}\frac{f^{(1)}(I,0)}{p+i[\omega]}\Big\}$$
$$-\beta^2 Q_{0,-1}Q_{0,1}\frac{f^{(0)}(I,0)}{p^2+[\omega]^2}. \quad (1.15)$$

Going asymptotically to $t \to \infty$, i.e., to $p \to 0$, and transforming back to the t-representation, we get ultimately:

$$\frac{\partial f^{(0)}}{\partial t} = -i\beta\{Q_{0,-1}e^{-i[\omega]t}f^{(-1)}(I,0) + Q_{0,1}e^{i[\omega]t}f^{(1)}(I,0)\}$$
$$+ 6\pi\beta^2\sum_{k_1k_2k_3}\frac{|V_{k_1,k_2,-k_3}|^2}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}\delta([\omega])\delta_{[k],0}\left[\frac{\partial}{\partial I}\right]I_{k_1}I_{k_2}I_{k_3}\left[\frac{\partial}{\partial I}\right]f^{(0)},$$

$$\left[\frac{\partial}{\partial I}\right] \equiv \frac{\partial}{\partial I_{k_1}} + \frac{\partial}{\partial I_{k_2}} - \frac{\partial}{\partial I_{k_3}} \qquad (1.16)$$

The difference between (1.16) and the fundamental Prigogine-Brout kinetic equation, which has the Fokker-Planck form, lies in the presence of terms in β, which conserve the phase memory of the system with respect to the initial conditions. We note that even if $f^{(\pm 1)}(I, 0) = 0$, the terms containing the phase memory appear in a higher order in β. Namely, they will be of the order of β^{n_0}, where n_0 corresponds to the smallest number of the nonzero harmonic in the expansion (1.12).

We introduce for future use the distribution function $\Phi(I, t)$ obtained from $f^{(0)}(I, t)$ by averaging over the initial phases $\varphi_{k,(0)}$:

$$\Phi(I, t) = (2\pi)^{-N} \int d\varphi_{k_1,(0)} \ldots d\varphi_{k_N,(0)} f^{(0)}(I, t), \qquad (1.17)$$

where N is the number of degrees of freedom, i.e., the number of oscillations excited in the plasma. When $t = 0$ we get

$$\Phi(I, 0) = f^{(0)}(I, 0).$$

The equation for Φ can be obtained from (1.16) by integrating the latter over the initial phases. A most important fact is that under the assumptions made in the derivation of (1.16) the equation for $\Phi(I, t)$ has exactly the same form. In other words, averaging over the initial phases of Eq. (1.16) does not change this equation, and the phase memory of the system is conserved. This is connected with the fact that (1.16) does not contain as yet terms that depend on $\varphi_{k,(0)}$. The situation changes if account is taken of the nonlinear correction to the frequency. It will be shown below that this makes the term of first order in β dependent on $\varphi_{k,(0)}$, and we shall obtain a condition under which averaging over $\varphi_{k,(0)}$ leads to a kinetic equation of the Fokker-Planck type.

2. ANALYSIS OF EQUATIONS OF MOTION

When the resonance conditions (1.3) are satisfied there develops for a certain set of three waves a coherent, instability, called decay instability,[6] which leads to growth of the amplitudes of the oscillations having frequencies ω_2 and ω_3. The presence of a nonlinear frequency correction can lead to violation of the resonance conditions and to a cessation of the instability. The following situation, however, is also possible: violation of the resonance condition of a certain fixed oscillation ω_1 with a pair of waves ω_2 and ω_3, owing to the nonlinearity of the frequencies, makes resonance possible between ω_1 and another pair of waves ω_2' and ω_3'. In the case when

$$\frac{d\omega_k}{dI_{k'}} \Delta I_{k'} \gg \Omega_k, \qquad (2.1)$$

the harmonic with frequency ω_k rapidly goes out of resonance with any pair of waves, owing to the strong nonlinearity, but on the other hand it always enters into resonance with some other pair of waves. The left side of (2.1) represents the change in frequency due to the passage through resonance, while the right side (according to (1.2)) represents the characteristic distance between harmonics; ΔI_k is the change in number of quasiparticles (action) on going through resonance. A condition similar to (2.1) was considered in [3, 7–9] as a condition for the phase randomization of a nonlinear oscillation in an external periodic field and for the transition from a dynamic description of the system to a statistical one. We shall study in detail this question for Eq. (1.16).

We turn for the time being to the initial equation (1.1), which leads (when (1.4) is taken into account) to the equations of motion

$$\ddot{u}_k + (\omega_k + \Delta\omega_k)^2 u_k \approx F(t),$$
$$F(t) = -3\beta \sum_{k_1, k_2 \neq k} V_{kk_1k_2} u_{k_1} u_{k_2} \delta_{k_1+k_2, k}. \qquad (2.2)$$

The right side of (2.2) can be regarded as the external force acting on the k-th mode. If we use for u_k the zeroth approximation:

$$u_k = u_k^{(0)} \cos(\omega_k t + \varphi_{k,(0)}),$$

then

$$F(t) = -3\beta \sum_{k_1, k_2 \neq k} V_{kk_1k_2} u_{k_1}^{(0)} u_{k_2}^{(0)} \cos(\omega_{k_1} t + \varphi_{k_1,(0)}) \cos(\omega_{k_2} t + \varphi_{k_2,(0)}) \delta_{k_1+k_2, k}. \qquad (2.3)$$

If we now assume that the characteristic distance between the harmonics of the spectrum Ω_k changes little over the interval of the excited wave packet, then it is easy to see that (2.3) is a Fourier expansion of a certain periodic function with period $2\pi/\Omega_k$.

We now make the very important assumption that the frequency interval of the excited oscillations is sufficiently broad:

$$N\Omega_k \gg \omega_k, \qquad (2.4)$$

where N is the number of excited oscillations, Ω_k and ω_k pertain to the considered interval of frequencies, which are far enough from the right edge of the packet. Then the force $F(t)$ for such frequencies ω_k constitutes a sequence of very narrow time pulses (of width $\sim 1/N\Omega_k$) that follow each other at a frequency $\sim \Omega_k$. Each pulse is accompanied by a change in the adiabatic invariant of the oscillation ("scattering") by an amount δI_k. To estimate δI_k, we note that owing to the very narrow width of the pulse $F(t)$ (compared with the period of the oscillation $2\pi/\omega_k$) it can be approximately replaced by a δ-function. We then get in lieu of (2.3)

$$\ddot{u}_k + (\omega_k + \Delta\omega_k)^2 u_k \approx -F_0 \sum_n \delta(t - 2\pi n/\Omega_k),$$

$$F_0 \sim V_k I_k/\omega_k\Omega_k, \qquad V_k = V_{k,k,2k}. \tag{2.5}$$

From (2.5) we easily get

$$\frac{\delta I_{k,(n)}}{I_{k,(n)}} \approx \frac{F_0}{\omega_k u^{(0)}_{k,(n)}} \sin 2\varphi_{k,(n)} = \beta \frac{V_k I_{k,(n)}}{\omega_k{}^2 \Omega_k u^{(0)}_{k,(n)}} \sin 2\varphi_{k,(n)}, \tag{2.6}$$

where the index n in the superscript pertains to scattering by the n-th pulse. In general, Eq. (2.5) can be replaced by the following finite-difference system:

$$I_{k,(n+1)} = I_{k,(n)} + \delta I_{k,(n)},$$

$$\varphi_{k,(n+1)} = \varphi_{k,(n)} + 2\pi\frac{\omega_k}{\Omega_k} + \frac{2\pi}{\Omega_k}\sum_{k'} \frac{\partial\Delta\omega_k}{\partial I_{k',(n)}} \delta I_{k',(n)} \sin 2\varphi_{k',(n)}$$

$$= \varphi_{k,(n)} + 2\pi\frac{\omega_k}{\Omega_k} + \sum_{k'} K_{kk',(n)} \sin 2\varphi_{k',(n)}, \tag{2.7}$$

$$K_{kk',(n)} = \frac{2\pi}{\Omega_k}\frac{\partial\Delta\omega_k}{\partial I_{k',(n)}} \delta I_{k',(n)}. \tag{2.8}$$

When $K \gg 1$ the oscillation phase changes rapidly as a result of scattering, and we can expect (this will be demonstrated below) transition to take place in this case from the dynamic description of the system to a statistical one.

Formulas (2.6) and (2.8) can be rewritten in a more compact form by using the expression for the increment ν_k of the coherent three-wave decay instability (see, e.g.,[1]):

$$\frac{\delta I_{k,(n)}}{I_{k,(n)}} = \frac{\nu_k}{\omega_k}\sin 2\varphi_{k,(n)} \ll 1, \qquad \Delta\omega_k \approx \sum_{k'} \frac{\nu_{k'}{}^2}{\omega_{k'}},$$

$$K_{kk',(n)} \approx \frac{1}{\Omega_k}\frac{\nu_k \nu_k{}^2}{\omega_k\omega_{k'}}, \qquad \nu_k = \beta\frac{V_k I_{k,(n)}}{\omega_k\Omega_k u^{(0)}_{k,(n)}}. \tag{2.9}$$

If we consider a narrow packet, such that $N\Omega_k \ll \omega_k$, then the characteristic width of the pulse $F(t)$ becomes much larger than the period of the wave. Such a force is adiabatic. The change δI as a result of scattering is exponentially small:

$$\frac{\delta I_k}{I_k} \sim \exp\left\{-1\Big/\frac{N\Omega_k}{\omega_k}\right\},$$

and consequently K is also exponentially small.

The results (2.9) can be obtained also by another more general method. The solutions of (2.2) in the interval between two successive pulses can be represented in the form of the solutions that arise when the WKB method is used [8]; the latter is applicable, in view of the smallness of the nonlinearity. Let A_0 and B_0 be the complex amplitudes of the solutions before the scattering and let A and B be the same quantities after scattering. The general transformation relating (A, B) with (A_0, B_0) is [8]

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1, \tag{2.10}$$

where a and b are certain parameters characterizing the scattering. If we put $B = A^*$ and $B_0 = A_0^*$, then

$$|B|^2 = |A|^2 = I, \qquad |B_0|^2 = |A_0|^2 = I_0, \tag{2.11}$$

where I_0 and I are respectively the actions before and after the scattering. From (2.10) and (2.11) we get

$$\frac{I}{I_0} = \frac{1 + 2\varepsilon\cos(2\varphi_0 + \psi) + \varepsilon^2}{1 - \varepsilon^2}, \tag{2.12}$$

where

$$A_0 = |A_0| e^{i\varphi_0}, \quad \left|\frac{b}{a}\right| = \varepsilon, \quad \frac{b}{a} = \varepsilon e^{-i\psi}.$$

When $\epsilon \ll 1$, a small change in the action takes place, $\delta I = I - I_0$. In this case it follows from (2.12) that

$$\delta I/I \approx 2\varepsilon\cos(2\varphi_0 + \psi).$$

This expression is similar to (2.6). In particular, for (2.5) we have $\epsilon \sim \nu/\omega$ and $\psi \approx -\pi/2$, so that we arrive at the formula (2.9).

In concluding this section, we note the quantity ΔI_k introduced in (2.1) characterizes the change of the action as a result of passage through a single resonance, and does not coincide with δI_k. Let us estimate ΔI_k. At resonance there is added to the main oscillation a forced oscillation with amplitude $\Delta u_{k'} \sim \beta V_k u_{k'}{}^2/\Delta$, where Δ is the deviation of the frequency from the resonant value. In this case the frequency deviation is connected with its nonlinear dependence on u_k, i.e.,

$$\Delta \sim \frac{\partial\omega_k}{\partial I_{k'}}\Delta I_{k'}.$$

Hence

$$\frac{\Delta I_{k'}}{I_{k'}} = \frac{\Delta u_{k'}}{u_{k'}} \sim \beta V_k u_{k'}\Big/\left(\frac{\partial\omega_k}{\partial I_{k'}}\Delta I_{k'}\right)$$

or

$$\Delta I_{k'} = \left(\beta V_k u_{k'} I_{k'}\Big/\frac{\partial\omega_k}{\partial I_{k'}}\right)^{1/2}. \tag{2.13}$$

Substitution of (2.13) in (2.1) yields

$$\frac{\partial \omega_k}{\partial I_{k'}} \frac{\beta V_k u_k I_{k'}}{\Omega_k^2} = \frac{\nu_{k'}^2 \nu_k}{\omega_{k'}\omega_k \Omega_k} = K_{kk'} \gg 1. \qquad (2.14)$$

Thus, condition (2.1) denotes, according to (2.7), a very rapid change in the phase of the oscillation as a result of the scattering.

The foregoing results enable us to proceed to the solution of the main problem—the derivation of the condition under which a transition from the dynamic to the statistical description is possible for the system (1.5).

3. DERIVATION OF THE CONDITIONS FOR THE WAVE PHASE RANDOMIZATION

We return to Eq. (1.16) and take now into account the nonlinear correction to the frequency, $\Delta\omega$. As shown in the preceding section, the interaction between this oscillation and all others reduces to an effective scattering of a phonon periodically in time and to a change of the action by an amount δI per scattering act. This means that the quantity $\Delta\omega(I)$ is a function of the time. Let us take this circumstance into account in the factors $\exp\{\pm i[\omega]t\}$ which enter in the terms proportional to β in (1.16). To this end we note that, according to (1.6),

$$(\omega + \Delta\omega)t = \varphi(t) - \varphi_{(0)} = \varphi_{(m)} - \varphi_{(0)}, \qquad (3.1)$$

where m is the number of scattering acts at the instant of time t, namely $m \approx t\Omega_k \gg 1$. With the aid of (2.7) we get

$$(\omega_k + \Delta\omega_k)t = \varphi_{k,(m-1)} + 2\pi\frac{\omega_k}{\Omega_k} + \sum_{k'} K_{kk'} \sin 2\varphi_{k',(m-1)}$$
$$- \varphi_{k,(0)} = \varphi_{k,(m-2)} + 4\pi\frac{\omega_k}{\Omega_k} + \sum_{k'} K_{kk'} \sin 2\varphi_{k',(m-2)}$$
$$+ \sum_{k'} K_{kk'} \sin 2\Big\{\varphi_{k',(m-2)}$$
$$+ \sum_{k''} K_{k'k''} \sin 2\varphi_{k'',(m-2)}\Big\} - \varphi_{k,(0)} = \dots \qquad (3.2)$$

After the iteration process in (3.2) is complete, the quantity $(\omega_k + \Delta\omega_k)t$ becomes a function of t (of the number m) and of $\varphi_{k,(0)}$.

For the function Φ, as already noted in Sec. 1, Eq. (1.16) takes the form

$$\frac{\partial\Phi}{\partial t} = -i\beta\,\{Q_{0,-1}\langle\!\langle e^{-i[\varphi(t)-\varphi_{(0)}]}\rangle\!\rangle f^{(-1)}(I, 0)$$
$$+ Q_{0,1}\langle\!\langle e^{i[\varphi(t)-\varphi_{(0)}]}\rangle\!\rangle f^{(1)}(I, 0)\}$$
$$+ 6\pi\beta^2 \sum_{k_1k_2k_3} \frac{|V_{k_1,k_2,-k_3}|^2}{\omega_{k_1}\omega_{k_2}\omega_{k_3}} \delta[\omega]\,\delta_{[k],0}\left[\frac{\partial}{\partial I}\right] I_{k_1}I_{k_2}I_{k_3}\left[\frac{\partial}{\partial I}\right]\Phi,$$

$$[\varphi(t) - \varphi_{(0)}] \equiv \varphi_{k_1}(t) - \varphi_{k_1,(0)} + \varphi_{k_2}(t)$$
$$- \varphi_{k_2,(0)} - \varphi_{k_3}(t) + \varphi_{k_3,(0)}, \qquad (3.3)$$

where only the linear frequencies can be retained under the δ-function sign; this will be justified later. In addition, we have used the notation

$$\langle\!\langle\dots\rangle\!\rangle \equiv (2\pi)^{-N}\int \dots d\varphi_{1,(0)}\dots d\varphi_{N,(0)}.$$

We now consider two limiting cases. Let $K \ll 1$ in almost the entire frequency interval under consideration. Then

$$(\omega + \Delta\omega)t = \omega t + O(K),$$
$$\langle\!\langle e^{\pm i[\omega+\Delta\omega]t}\rangle\!\rangle = e^{\pm i[\omega]t}(1 + O(K)). \qquad (3.4)$$

It follows from (3.4) that the equation for Φ has precisely the same form as Eq. (1.16) for $f^{(0)}$, and the phase memory of the system is conserved.

Let now $K_{kk'} \gg 1$ for almost all k and k'. In this case, retaining in (3.2) the terms that vary most rapidly, we have

$$\varphi_k(t) - \varphi_{k,(0)} = (\omega_k + \Delta\omega_k)t \approx \omega_k t - \varphi_{k,(0)}$$
$$+ \sum_{k_1} K_{kk_1} \sin\Big\{\sum_{k_2} K_{k_1k_2} \sin\Big[\sum_{k_3} K_{k_2k_3}$$
$$\dots\sin\Big(\sum_{k_m} K_{k_{m-1},k_m} \sin 2\varphi_{k_m,(0)}\Big)\dots\Big]\Big\}. \qquad (3.5)$$

Recognizing that $K_{kk'} \gg 1$, we can estimate the required integral by the stationary-phase method. This yields

$$\langle\!\langle e^{\pm i[\omega+\Delta\omega]t}\rangle\!\rangle \sim e^{-t/\tau}, \qquad \tau^{-1} = {}^1\!/_2\Omega N \ln K, \qquad (3.6)$$

where Ω and K are certain values of Ω_k and $K_{kk'}$ averaged over the packet.

This result solves our problem. The term of (3.3) of first order in β vanish after a time on the order of τ, and the equation for Φ takes the Fokker-Planck form.

$$\frac{\partial\Phi}{\partial t} = 6\pi\beta^2 \sum_{k_1k_2k_3} \frac{|V_{k_1,k_2,-k_3}|^2}{\omega_{k_1}\omega_{k_2}\omega_{k_3}} \delta[\omega]\,\delta_{[k],0}\left[\frac{\partial}{\partial I}\right]$$
$$\times I_{k_1}I_{k_2}I_{k_3}\left[\frac{\partial}{\partial I}\right]\Phi. \qquad (3.7)$$

The time τ can thus be regarded as the time of vanishing of the phase correlation in the system (1.1). The equilibrium solution of (3.7) is that Φ for which [5]

$$\langle I_k/\omega_k\rangle = \text{const}, \qquad (3.8)$$

where the angle brackets $\langle\dots\rangle$ denote averaging over Φ.

The characteristic time τ_0 for the establishment of the stationary state is determined from (3.7), viz., $\tau_0 \sim \nu^2/\omega$. This time should be much longer than the time connected with the smearing of $\delta([\omega])$ in (3.7) as a result of the nonlinear frequency correction. According to (2.9), this yields

$$\tau_0\omega^2/\nu^3 \sim \omega/\nu \gg 1.$$

The last inequality justifies the neglect of the nonlinear corrections to the frequency in the argument of the δ-function in (3.3).

From (3.7) follows immediately a kinetic equation for the waves [1]. Indeed, multiplying (3.7) by I_k and integrating over the entire phase space of the function Φ, we get

$$\frac{\partial\langle I_k\rangle}{\partial t} = 18\pi\beta^2 \sum_{k_1k_2k_3} \frac{|V_{k_1,k_2,-k_3}|^2}{\omega_{k_1}\omega_{k_2}\omega_{k_3}} \delta(\omega_{k_1} + \omega_{k_2} - \omega_{k_3})(\delta_{k_1k} + \delta_{k_2k} - \delta_{k_3k})\delta_{k_1+k_2,k_3}(\langle I_{k_2}I_{k_3}\rangle + \langle I_{k_1}I_{k_3}\rangle - \langle I_{k_1}I_{k_2}\rangle). \quad (3.9)$$

Taking into account the symmetry properties of $V_{k_1,k_2,-k_3}$ and of (1.6) and assuming, as usual, separation of the moments,

$$\langle I_{k_1}I_{k_2}\rangle = \langle I_{k_1}\rangle\langle I_{k_2}\rangle,$$

we get

$$\frac{\partial\langle I_k\rangle}{\partial t} = 18\pi\beta^2 \sum_{k_1k_2} \frac{|V_{k,k_1,-k_2}|^2}{\omega_{k_1}\omega_{k_2}\omega_k} \{2[\langle I_{k_1}\rangle\langle I_{k_2}\rangle + \langle I_k\rangle\langle I_{k_2}\rangle - \langle I_k\rangle\langle I_{k_1}\rangle]\delta(\omega_k + \omega_{k_1} - \omega_{k_2})\delta_{k+k_1,k_2} - [\langle I_k\rangle\langle I_{k_1}\rangle + \langle I_k\rangle\langle I_{k_2}\rangle - \langle I_{k_1}\rangle\langle I_{k_2}\rangle] \times \delta(\omega_k - \omega_{k_1} - \omega_{k_2})\delta_{k,k_1+k_2}\}. \quad (3.10)$$

4. DISCUSSION OF RESULTS

1. As shown in the preceding section, satisfaction of the condition (2.14) for almost all k and k′ leads to randomization of the wave phases within a time of the order of τ. This result has an intuitive interpretation. It is well known that the trajectories of motion are very strongly unstable in a statistical system against small perturbations of the initial conditions. This means that two phase points that start moving under nearly equal initial conditions can move arbitrarily far apart after a certain time. Let us consider for simplicity the change in the phase of a single oscillation. Let $d\varphi_{k,(n)}$ be the distance between two values of the phase on the unit circle at the instant of time characterized by the number n. Then, according to (2.7),

$$\frac{d\varphi_{k,(n+1)}}{d\varphi_{k',(n)}} = K_{kk'}\cos 2\varphi_{k',(n)} \equiv T_n(k, k'). \quad (4.1)$$

When $K_{kk'} \gg 1$ the transformation (4.1) is a stretching transformation, with the exception of a small region $\varphi_{k'}$ of dimension $\sim K^{-1} \ll 1$. In view of the large number of degrees of freedom, the statistical weight of regions of this kind is very small during the time corresponding to $m \gg 1$ steps. Therefore the transformation T_n, when applied a sufficiently large number of times for $K \gg 1$,[1)] signifies the presence of the instability discussed above. The condition $K \sim 1$ can be regarded as the limit of the stochasticity.

2. Let us discuss the consequences of the phase randomization condition (2.14). We rewrite it in the form

$$K_{kk'} = \frac{\partial\Delta\omega_k}{\partial I_{k'}} I_{k'} \frac{\nu_k}{\omega_k}\frac{1}{\Omega_k} \gg 1, \quad (4.2)$$

or, as is the case for the majority of real systems in a plasma, $\Delta\omega$ depends in power-law fashion on I, so that

$$K_{kk'} = \frac{\Delta\omega_{kk'}}{\Omega_k}\frac{\gamma_k}{\omega_k} \gg 1, \qquad \Delta\omega_k = \Big|\sum_{k'}\Delta\omega_{kk'}. \quad (4.3)$$

In addition, usually $K_{kk'}$ is a positive power of k and k′. It follows therefore that oscillations with sufficiently large wavelengths do not become randomized. Below a certain limit k_0 Eq. (3.10) becomes meaningless. The value of k_0 can be estimated from

$$K_{k_0k_0} \sim 1. \quad (4.4)$$

The action of the lower limit k_0 is similar to the presence of a reflecting wall for the quasiparticles.[3] It follows also from (4.2) that in the regions of anomalous dispersion, where

$$\Omega_k = \frac{\partial\omega_k}{\partial k}\Delta k \to \infty$$

the stoichasticity may cease. In addition, the nonlinearity $\Delta\omega$ should be sufficiently large if (4.2) is to be satisfied.

3. In the model considered thus far, the resonance condition

$$\sum_j n_j\omega_j = 0 \quad (4.5)$$

took the form

$$\omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \qquad k_1 = k_2 + k_3. \quad (4.6)$$

Satisfaction of (4.6) is ensured by the corresponding form of the dispersion law $\omega = \omega(k)$. It may turn out that processes of the type (4.6) are forbidden for the spectrum $\omega(k)$ under considera-

1)On going over to the continuous spectrum $\Omega_k \to 0$ the condition $K \gg 1$ would seemingly be always satisfied. Actually, however, such a transition calls for a special investigation.

tion, and the resonance conditions are satisfied only when the number of waves exceeds three. Thus, for example, assuming random phases, the kinetic equation for four-plasmon interactions was obtained in [10]. In the arbitrary case, the criterion (4.3) can be retained by making suitable substitutions for $\Delta\omega_k$ and ν_k. The problem considered in [3] pertains to a spectrum $\omega(k)$ for which the process (4.6) is forbidden and decays are possible only in the next higher order.

Satisfaction of the conditions (4.5) ensures the possible appearance of a "bare" (decay) instability. The latter leads to a radical change of the adiabatic invariant of the oscillation.

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