THE ABSORPTION OF ULTRASOUND IN SUPERCONDUCTING ALLOYS IN THE MIXED STATE

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The absorption coefficient of long-wave ultrasound (longitudinal and transverse) in an "impure" superconducting alloy in the mixed state is calculated in the local approximation. The magnetic field is assumed to be not very large, $H_{c1} < H_0 \ll H_{c2}$. The total absorption consists of two parts—the absorption due to the viscosity of the electron gas and the absorption due to the viscosity of the vortex lines. The local value of the sound absorption coefficient due to the first absorption mechanism is determined, independently of the polarization, by a certain isotropic function of the temperature and the velocity of the superconducting condensate; this function is not identical with the local value of the BCS function $2(e^{\Delta/T} + 1)^{-1}$. After averaging over the vortex lattice in a weak field, the absorption coefficient is computed from Eqs. (19) and (20). The second absorption mechanism is connected with oscillations of the vortex lattice by the sound wave and with the losses due to the viscous motion of the vortices. This part of the absorption is strongly anisotropic in its dependence on the direction of propagation of the sound wave and the polarization, and is characterized by a complicated dependence on the magnetic field.

ULTRASOUND absorption in metals represents an effective means of study of the features of the energy spectrum of the conduction electrons. In particular, the threshold dependence of the ultrasound absorption in a superconducting metal on the temperature has been shown to be one of the direct experimental confirmations of the existence of the energy gap in the spectrum of a superconductor.[1,2] In the further development of the microscopic theory of superconductivity, it has become clear that the presence of the energy gap is not a necessary condition for the superconducting state. In superconductors with a sufficiently large number of paramagnetic impurities, the gap in the spectrum disappears for a superconducting ordering of the electrons that differs from zero.[3] The corresponding calculation of the ultrasonic absorption in such superconductors[4] shows that this vanishing of the gap can be found experimentally by measurements of the sound absorption coefficient.

Another important case of gapless superconductivity is realized in type II superconductors (in particular, in "impure" alloys) in the mixed state.[5] In such superconductors, in a magnetic field larger than H_{c1}, the Meissner effect is absent: the magnetic flux penetrates partially into the interior of the bulk superconductor in the form of vortex current lines, forming a regular lattice in the equilibrium state. Here the parameter of superconducting ordering is a function of the coordinates, and vanishes at the center of the vortex line. Calculations show[6] that in the center of a vortex with radius a of the order of δ_0/κ (δ_0 is the penetration depth of a weak magnetic field and κ is the parameter of the Ginzburg-Landau theory[7]) the gap in the spectrum of the superconductor is populated by densely distributed local levels, which are close to the electron levels in the normal metal. The presence of these gapless regions in type II superconductors in the mixed state was confirmed by direct observation of the dissipative current states in a magnetic field.[8] Evidently the existence of these regions should also be indicated by the ultrasonic absorption in the superconductor.

In the present paper, the coefficient of electron absorption of long-wave sound is computed ($ql \ll q\delta_0 \ll 1$, where q is the wave number, l the length of the free path of the electrons)[1)] in the mixed state for "impure" superconducting alloy ($\tau T_c \ll 1$, where $\tau = l/v_0$ is the relaxation time, T_c the temperature for the superconducting transition, and v_0 the velocity on the Fermi surface).

[1)]In what follows, the much more rigid condition $qd \ll 1$ will be assumed, if $d > \delta_0$, where d is the distance between the vortex lines.

The external magnetic field is assumed to be not too large: $H_{C1} < H_0 \ll H_{C2}$. This allows us to regard the current structure of the superconductor in a mixed state as a lattice of isolated vortex lines parallel to the external field.[5]

The enumerated conditions make it possible to use as the basic calculation the so-called local approximation in the electrodynamics of type II superconductors.[9] The essence of this approximation is that locally, at distances less than δ_0, the superconductors in a magnetic field are considered as being in a homogeneous current state (analogous to the state of a thin film) and all quantities, for example, the superconducting flow and the ordering parameters, can be regarded as local functions of the velocity of the superconducting condensate at the given point, $\mathbf{v}_S(\mathbf{r})$, determined by the expression $\mathbf{v}_S = (\nabla\chi - 2e\mathbf{A})/2m$, where $\hbar = c = 1$, e and m are the charge and mass electron, χ the phase of the superconducting ordering parameter, $\mathbf{A}$ the vector potential of the magnetic field, and $\mathbf{H} = \operatorname{curl} \mathbf{A}$. The shape of the function $\mathbf{v}_S(\mathbf{r})$ for the mixed state is then determined by the solution of the Maxwell equations, just as had been done by Abrikosov.[5] Application of the local approximation materially simplifies the calculations while at the same time guaranteeing the necessary accuracy, as will be seen from what follows.

For the calculation of the electronic absorption of the ultrasound, it is necessary to determine the work done by the sound wave on the electron gas per unit time. For long-wave sound, in the hydrodynamic approximation, this work can be represented in the form[10, 16]

$$\overline{\langle\dot{\mathscr{H}}\rangle}^t = \int dV[\overline{\langle\mathbf{j}\rangle\mathbf{E}}^t - \overline{\langle\tau_{ik}\rangle\dot{\varepsilon}_{ik}}^t]. \quad (1)$$

Here $\mathbf{j} = e\hat{\mathbf{v}}\psi^{+\prime}\psi$ is the current operator ($2m\hat{\mathbf{v}} = \hat{\mathbf{p}} - \hat{\mathbf{p}}' - 2e\mathbf{A}$, $\mathbf{p} = -i\nabla$; ψ and ψ^+ are the electron annihilation and creation operators at the given point); $\tau_{ik} = m\hat{v}_i\hat{v}_k\psi^{+\prime}\psi$ is the stress tensor, $\epsilon_{ik} = \frac{1}{2}(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$ is the deformation tensor, $\mathbf{u}$ is the vector displacement of the ionic lattice; $\mathbf{E} = -\dot{\mathbf{A}}$ is the effective electric field in the system of coordinates connected with the lattice. The dot denotes differentiation with respect to time, the superior bar indicates the time average, and the angular brackets denote thermodynamic averaging.

In the mixed state, the sound wave generally produces oscillations of the vortex lines relative to the ionic lattice, which is accompanied by great changes in the magnetic and electric fields. For low-frequency sound ($\omega\tau \ll 1$), these changes in the field take place slowly. It is not difficult to generalize the Kubo technique[11] to the case of large fields that change slowly with time. Then, with account of (1), we obtain the expression for the thermodynamic means of the current and the stress tensor:[2)]

$$\langle j_i\rangle = \langle j_i\rangle_0 + \sigma_{ik}\{\mathbf{v}_s\}E_k, \quad \langle\tau_{ik}\rangle = \langle\tau_{ik}\rangle_0 - \eta_{iklm}\{\mathbf{v}_s\}\dot{\varepsilon}_{lm}, \quad (2)$$

where the conductivity tensor σ_{ik} and the viscosity coefficient tensor η_{iklm} are defined by the following expressions

$$\sigma_{ik} = -i\frac{d}{d\omega}\langle j_i; j_k\rangle(\omega) \text{ for } \omega\to 0,$$
$$\eta_{iklm} = -i\frac{d}{d\omega}\langle\tau_{ik}; \tau_{lm}\rangle(\omega) \quad \text{for } \omega\to 0. \quad (3)$$

In Eqs. (2), the expressions $\langle j_i\rangle_0$ and $\langle\tau_{ik}\rangle_0$ denote averages over the equilibrium Gibbs distribution with the instantaneous value of the Hamiltonian of the electrons, describing the given distribution of the vortex currents and the magnetic field in the mixed state at the given instant of time. The expressions $\langle j_i; j_k\rangle$ and $\langle\tau_{ik}; \tau_{lm}\rangle$ are the well-known time correlation functions in the Fourier representation (ω is the frequency) computed by means of the same equilibrium Gibbs distribution with the instantaneous value of the Hamiltonian.

In the general case, the correlation functions $\langle j_i; j_k\rangle$ and $\langle\tau_{ik}; \tau_{lm}\rangle$ are themselves integral operators of the conductivity and viscosity; these operators are local in the time, and nonlocal in the space, coordinates. However, in the limit of long-wave sound, coordinates that are nonlocal in space can be neglected. Moreover, these correlators are also nonlocal functions of the velocity $\mathbf{v}_S(\mathbf{r})$ of the superconducting condensate, but, in the local approximation used in this research, one can consider them as functions of the local value of $\mathbf{v}_S(\mathbf{r})$.

With the aid of Eqs. (2), we transform Eq. (1) for the dissipation of energy to the following:

$$\overline{\langle\dot{\mathscr{H}}\rangle}^t = \int dV[\overline{\sigma_{ik}E_iE_k}^t + \overline{\eta_{iklm}\dot{\varepsilon}_{ik}\dot{\varepsilon}_{lm}}^t]. \quad (4)$$

Physically, the first term describes the possible ohmic losses due to the incomplete attraction of the electrons by the ionic lattice, and the noncompensation of the electron and ion currents. The second term is the loss of sound energy in the electron gas.

1. We first consider the contribution to the absorption made by the viscosity of the electron gas. To calculate the tensor of viscosity coefficients

[2)]In the approximation considered in this paper, the mixed terms $\langle j_i; \tau_{kl}\rangle\,\epsilon_{kl}$ and $\langle\tau_{ik}; j_l\rangle A_l$ in Eq. (2) are equal to zero.

η_{iklm}, one can use a technique analogous to the techniques of [12, 4]. The correlation function $\langle\tau_{ik};\tau_{lm}\rangle(\omega)$ can be found by analytic continuation of the temperature correlation function $\langle\tau_{ik};\tau_{lm}\rangle^T(\nu_n)$ with discrete frequencies $\nu_n = 2\pi Tn$ ($n = 0, \pm 1, \pm 2, \ldots$) to the physical frequency ω:

$$\langle\tau_{ik};\tau_{lm}\rangle(\omega) = \langle\tau_{ik};\tau_{lm}\rangle^T(\nu_n), \qquad i\nu_n \to \omega + i0.$$

We determine the temperature correlation function as previously[4] by means of the variational derivative:

$$\langle\tau_{ik};\tau_{lm}\rangle^T(x_1,x_2) = \frac{\delta\langle\tau_{ik}\rangle(x_1)}{\delta U_{lm}(x_2)},$$

$$\langle\tau_{ik}\rangle(x) = -2m\hat{v}_i\hat{v}_k\cdot\frac{1}{2}\operatorname{Sp}(\sigma_z K(x,x'))\,|_{x'=x} \tag{5}$$

Here $U_{lm}(x)$ is the elastic field of external sources; $K(x_1, x_2)$ is the matrix Green's function:

$$K(x_1,x_2) = \begin{pmatrix} G & F \\ F^+ & \tilde{G}\end{pmatrix}(x_1,x_2);$$

G, F, F^+, and $\tilde{G}$ are the functions introduced by Gor'kov,[13] and σ_z is the Pauli matrix.[3)]

In the presence of a current state given by the superconducting velocity $\mathbf{v}_s$, the Gor'kov equation[13] for the Green's function $K(x_1, x_2)$ in the field of external sources can be written in the form

$$\begin{aligned} K^{-1}(x_1,x_2;U) = \Big[\frac{\partial}{\partial\tau_1} &+ \sigma_z\xi(\hat{\mathbf{p}}_1 + \sigma_z m\mathbf{v}_s) - \sigma_x\Delta(x_1;U) \\ &- \sigma_z U_{ik}(x_1)m\hat{v}_{1i}\hat{v}_{1k}\Big]\delta(x_1 - x_2) \\ &- nw(\mathbf{r}_1 - \mathbf{r}_2)\sigma_z K(x_1,x_2;U)\sigma_z. \end{aligned} \tag{6}$$

where $w(\mathbf{r}) = (2\pi)^{-3}\int d^3\mathbf{p}\,|v(\mathbf{p})|^2 e^{i\mathbf{p}\cdot\mathbf{r}}$, $K^{-1}(x_1, x_2)$ is the inverse Green's function, $\xi(p) = p^2/2m - \mu$ (μ is the chemical potential), σ_x the Pauli matrix, Δ the parameter of superconducting ordering ($\Delta = gF(x, x)$, $g < 0$ is the coupling constant). The last term in Eq. (6) describes the scattering of the electrons by the randomly distributed impurities (in the volume of the metal), over the positions of which the averaging is made; $v(\mathbf{p})$ is the Fourier transform of the scattering potential by the impurities and n is the concentration of the impurities.

The form of the Green's function $K(\mathbf{p}; \omega_n)$ in the Fourier representation (see the Appendix) for the case of a current state in a superconductor with impurities was found by Maki.[15] According to [15], it follows from Eq. (6) that, for the case $U_{ik} = 0$ (as $i\omega_n \to z$):

$$K(\mathbf{p},z) = \frac{\sigma_z\xi - \sigma_x\tilde{\Delta} + \tilde{z}}{\xi^2 + \tilde{\Delta}^2 - \tilde{z}^2}. \tag{7}$$

The functions $\tilde{z}(z)$ and $\tilde{\Delta}(z)$ in the equations for the correlators enter in the form of the combinations $u = \tilde{z}/\tilde{\Delta}$ and $\tilde{\epsilon} = \sqrt{\Delta^2 - z^2}$. According to [15], for an "impure" alloy in the principal approximation in the small parameter $\tau T_c \ll 1$ ($\tau p_0 v_s \lesssim \sqrt{\tau T_c} \ll 1$, p_0 being the Fermi momentum) the explicit z dependence of the introduced function is determined by the following relations:

$$\tilde{\varepsilon} \approx \frac{1}{2\tau},\quad \frac{z}{\Delta} = u\Big(1 - \frac{\zeta}{\sqrt{1-u^2}}\Big),\quad \zeta = \frac{2}{3}\tau_{tr}\frac{(p_0 v_s)^2}{\Delta},$$

$$\frac{1}{\tau_{tr}} = \frac{1}{\tau} - \frac{1}{\tau_1},\quad \frac{1}{\tau_n} = \frac{nmp_0}{\pi}\int\frac{do}{4\pi}|v(\theta)|^2\cos^n\theta\quad(\tau_0 \equiv \tau). \tag{8}$$

In the last line we give the usual definitions of the free path times of the electrons (do is the element of solid angle).

Transforming in (5) to the Fourier representation for the long-wave sound ($q \to 0$), we get:

$$\langle\tau_{ik};\tau_{lm}\rangle(\nu_n) = -\frac{3N}{v_0^2}2\pi T\sum_{\omega_n}\int\frac{do}{4\pi}v_{0i}v_{0k}\cdot \times\frac{1}{2}\operatorname{Sp}(\sigma_z L_{lm}(\mathbf{p}_0;\omega_+,\omega_-)), \tag{9}$$

$$L_{ik}(\mathbf{p}_0;\omega_+,\omega_-) = \int\frac{d\xi}{2\pi}\frac{\delta K}{\delta U_{ik}}(\mathbf{p},\mathbf{q};\omega_+,\omega_-)_{\substack{q\to 0\\ U\to 0}}. \tag{10}$$

Calculating the variational derivative $\delta K(x_1, x_2)/\delta U_{ik}(x_3)$ by means of Eq. (6) and the identity $\int dx K(x_1, x)K^{-1}(x, x_2) = \delta(x_1 - x_2)$, substituting its Fourier representation in the definition (10), and neglecting the variations of the ordering parameter Δ (i.e., the contribution from the collective excitations,[16] we find the equation for L_{ik}:

$$L_{ik} = \int\frac{d\xi}{2\pi}K_+\Big[mv_{0i}v_{0k}\sigma_z + \frac{nmp_0}{\pi} \times\int\frac{do}{4\pi}|v(\theta)|^2\sigma_z L_{ik}\sigma_z\Big]K_-, \tag{11}$$

where for brevity we put $K_\pm = K(\omega_\pm)$.

The matrix tensor L_{ik} can be represented in the form of the following expansion:

$$L_{ik} = mv_{0i}v_{0k}(\sigma_z\Pi + \sigma_z\sigma_x\Pi_1) + mv_0^2\delta_{ik}(\sigma_z\Omega + \sigma_z\sigma_x\Omega_1). \tag{12}$$

Substitution of this expansion in Eq. (11) gives a set of algebraic equations for the scalar functions $\Pi(\omega_+, \omega_-)$, $\Pi_1(\omega_+, \omega_-)$, $\Omega(\omega_+, \omega_-)$, and $\Omega_1(\omega_+, \omega_-)$,

[3)]We note that in the given case (nonparamagnetic impurity) the spin matrices have already been separated in the Green's functions and G, F, F^+, and G do not depend on the spin variables. The Pauli matrices, as also the matrix form of $K(x_1, x_2)$, are used for the sake of convenience.

the solution of which, with account of Eqs. (7) and (8), has the form

$$\Pi = \frac{\tau_{tr}'}{2}\,\frac{u_+u_- + \sqrt{(1-u_+^2)(1-u_-^2)} - 1}{\sqrt{(1-u_+^2)(1-u_-^2)}} \qquad \Omega = -\frac{1}{3}\Pi$$

$$\frac{1}{\tau_{tr}'} = \frac{3}{2}\left(\frac{1}{\tau} - \frac{1}{\tau_2}\right). \tag{13}$$

Combining Eqs. (3), (9), (12) and (13), we find (after performing the analytic continuation) the final expression for the tensor of the viscosity coefficients:

$$\eta_{iklm} = \overset{n}{\eta}_{iklm} f(T, v_s),$$

$$f(T, v_s) = \int_{z_m}^{\infty} dz \frac{\mathrm{sh}^2\gamma}{\mathrm{ch}^2\gamma - \sin^2\alpha}\frac{d}{dz}\,\mathrm{th}\frac{z}{2T}$$

$$\begin{gathered} z/\Delta = \sin\alpha(\mathrm{ch}^2\gamma - \cos^2\alpha)/\mathrm{ch}\,\gamma \\ \cos\alpha(\mathrm{ch}^2\gamma - \sin^2\alpha) = \zeta\,\mathrm{ch}\,\gamma, \end{gathered} \quad z_m = \begin{cases} \Delta(1-\zeta^{2/3})^{3/2} & \zeta < 1 \\ 0 & \zeta > 1 \end{cases} \tag{14}$$

where $\overset{n}{\eta}_{iklm}$ is the tensor of the viscosity coefficients of the normal electron gas:

$$\overset{n}{\eta}_{iklm} = {}^1/_5 N p_0 v_0 \tau_{tr}'(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl} - {}^2/_3\delta_{ik}\delta_{lm}).$$

The absorption coefficient for a plane sound wave $\mathbf{u}_t = \mathbf{u}_{q,\,\omega}\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ is determined from the dissipative function (4) by the well-known relation $\alpha = \langle\bar{\mathcal{H}}\rangle^t/2\bar{E}^t$, $\bar{E}^t = {}^1/_2 V\rho_{\mathrm{ion}}\omega^2|\mathbf{u}|^2$ is the mean energy of the sound wave, V the volume of the metal, ρ_{ion} the ion density. It then follows that for the viscous part of the ultrasonic absorption coefficient in a mixed state, independently of the polarization and direction of propagation of the sound wave, the following relation is valid:[4)]

$$\left(\frac{\alpha_s}{\alpha_n}\right)_{\mathrm{vis}} = \overline{f(T, v_s)} = \frac{1}{S}\int dS\cdot f(T, v_s) \tag{15}$$

Here the superior bar denotes averaging over the spatial distribution of the vortex currents, which actually means averaging over the vortex-lattice cross section perpendicular to the external magnetic field $\mathbf{H}_0$.

Proceeding to the discussion of the results obtained, we first note that the function $f(T, v_s)$(15) does not coincide, as one would expect, with the local value of the function $2(e^{\Delta/T} + 1)^{-1}$ which, in the BCS theory, represents the ratio of the coefficients of ultrasound absorption in the superconducting and normal states. Using Eqs. (14), (15) and the corresponding equation for the parameter of ordering as a function of v_s,[15] it is not difficult to find the asymptotic values of the function $f(T, v_s)$:

$$f(T, v_s) \to \begin{cases} 1, \quad v_s \to v_{\mathrm{cr}}\ (\Delta(v_{\mathrm{cr}}) = 0,\ \zeta \to \infty),\ \zeta_0 = {}^2/_3\tau_{tr}(p_0 v_s)^2/\Delta_0, \\ \dfrac{2}{e^{\Delta_0/T}+1} + \dfrac{9}{20}\left(\Delta_0/2T\,\mathrm{ch}^2\dfrac{\Delta_0}{2T}\right)\zeta_0^{2/3}, \quad v_s \ll v_{\mathrm{cr}}\ (\zeta \ll 1), \end{cases} \tag{16}$$

where $\Delta_0(T)$ is the BCS gap.[1]

As is well known,[5] for isolated vortex lines, the superconducting velocity $\mathbf{v}_s(r)$ behaves in the following fashion as a function of the distance r from the axis of the vortex: Far from the center of the vortex $(r \gg a \sim \delta_0/\chi)$

$$v_s(r) \approx \frac{1}{2m\delta_0}K_1\left(\frac{r}{\delta_0}\right)$$

($K_1(x)$ is a Bessel function of the second kind); at small distances, $v_s(r) \approx 1/2mr$. Then, in accord with Eqs. (15) and (16), it follows that at the center of the vortex ($v_s \gtrsim v_{\mathrm{cr}}$), the ultrasonic viscous absorption is identical with the absorption of normal electrons.[5)] At zero temperature, as is seen from the asymptotic values of (16), the normal excitations outside the center of the vortex "freeze," and all the remaining absorption is entirely determined by the normal cores of the vortices. With account of these remarks, we find from Eq. (15)

$$T = 0, \quad \left(\frac{\alpha_s}{\alpha_n}\right)_{\mathrm{vis}} = k(0)\frac{B}{H_{c2}}$$

$$\left(B = \bar{H} = n_v\Phi_0 = n_v\frac{\pi}{e},\quad H_{c2} \approx \frac{1}{ea^2}\right), \tag{17}$$

where B is the magnetic induction in the superconductor, n_v is the density of the vortical lines, Φ_0 the quantum of magnetic flux of the vortex, and k(0) a numerical factor of the order of unity. The accurate value of k(0), however, cannot be obtained in the local approximation used here, since the latter is not applicable at distances $a \sim \delta_0/\kappa$.

[5)]Close to the field H_{c2}, when the density of lines is great, $(\alpha_s/\alpha_n)_{\mathrm{vis}} = 1$.

[4)]In the time averaging of the viscous term in Eq. (4) for the dissipative function, one can assume the vortex lattice to be in equilibrium and fixed, with accuracy up to terms of very high order of smallness in the deformations.

At a temperature different from zero, the situation is different. As is seen from Eqs. (16), in this case the thermal excitations outside the center of the vortex line begin to play a principal role. Actually, the function $\zeta_0^{2/3} \propto v_S^{4/3}$ falls off too slowly as $v_S \to 0$, $r \to \infty$, so that the characteristic distances are seen to be of the order of δ_0. This makes valid the use of the local approximation for the quantitative calculation and, moreover, as is easy to see, leads to a sharp increase in the coefficient of proportionality f(T) in the magnetic dependence of the absorption for weak fields. We substitute the asymptotic formula (16) in the expression (15) and estimate its mean value $\overline{\zeta_0^{2/3}}$ for sufficiently weak fields, when the distance between the vortices is larger than δ_0:

$$T \neq 0,\ \left(\frac{\alpha_s}{\alpha_n}\right)_{\text{vis}} = \frac{2}{e^{\Delta_0/T}+1} + \frac{9}{20}\left(\Delta_0/2T\,\mathrm{ch}^2\frac{\Delta_0}{2T}\right)\overline{\zeta_0^{2/3}},$$

$$\overline{\zeta_0^{2/3}} = 2Be\delta_0^2\left(\frac{\tau_{tr}v_0^2}{6\Delta_0\delta_0^2}\right)^{2/3}\int\limits_0^\infty dx\cdot xK_1^{4/3}(x) \sim \varkappa^{2/3}\frac{B}{H_{c2}}$$

$$\times\ (n_B\delta_0^2 \sim Be\delta_0^2 \ll 1) \tag{18}$$

Thus, combining Eqs. (17) and (18), it can be stated that the following relation is valid for viscous absorption in a weak field at any temperature

$$\left(\frac{\alpha_s}{\alpha_n}\right)_{\text{vis}} = \left(\frac{\alpha_s}{\alpha_n}\right)_{\text{BCS}} + k(T)\frac{B}{H_{c2}} \quad (Be\delta_0^2 \ll 1), \tag{19}$$

while the limiting value

$$k(0) \ll k(T \neq 0) \sim \varkappa^{2/3}$$

and the temperature dependence

$$k(T) \sim \varkappa^{2/3}\Delta_0/2T\,\mathrm{ch}^2\frac{\Delta_0}{2T} \tag{20}$$

are determined principally by the factor $\Delta_0/2\tau \cosh^2(\Delta_0/2T)$. In the general case, the dependence on the magnetic field is a complicated nonlinear one.

2. We proceed to the consideration of the ohmic part of the losses in Eq. (4) or the dissipative function. Usually, both in the normal metal and in the superconductor in the absence of a magnetic field, this term gives a zero contribution in the form of complete compensation of the electron and ion currents for long-wave sound ($ql \ll 1$). In a type II superconductor in a mixed state, the passage of the sound wave through the metal leads to a displacement of the ionic lattice relative to the system of vortex lines. As is well known,[17] the motion of the vortex through the lattice is accompanied by the appearance of a force of viscous friction, which acts on the line. Therefore, the vortex lines must be attracted by the vibrating lattice. However, this attraction cannot be complete, because of the magnetic interaction between the vortices. This interaction plays the role of elastic interaction for the vortex lattice.[6)] Thus, in the set of coordinates connected with the metal, motion of vortex lines arises and brings about the appearance of induction electric fields and normal currents, and correspondingly, ohmic loss. According to the mechanism proposed by Bardeen and Stephen,[17] these losses are precisely the losses from the viscous motion of the vortex lines themselves. Therefore, one can write down the following equality:

$$\int dV\overline{\sigma_{ik}E_iE_k}^t = Vn_B\eta\overline{v_L^2}^t = V\frac{Be}{\pi}\eta\overline{v_L^2}^t, \tag{21}$$

where η is the coefficient of viscosity of the vortex lines per unit length, $\mathbf{v}_L = \dot{\mathbf{u}}^V - \dot{\mathbf{u}}_\perp$ is the velocity of the line relative to the lattice, $\mathbf{u}^V$ is the vector displacement of the vortices, $\mathbf{u}_\perp$ the projection of the displacement of the ionic lattice on a plane perpendicular to the magnetic field. The calculation of the viscosity coefficient of the vortices did not enter into the problem of the present work. Its value can be found, for example, from the experiments of Kim and co-workers on the dissipative current states previously mentioned.[8] According to these experiments, η is represented in the form

$$\eta = \pi\sigma_n H_{c2}/e\beta(T) \qquad (\sigma_n = Ne^2\tau_{tr}/m), \tag{22}$$

where $\beta(T)$ depends weakly on the temperature ($\beta \sim 1$), and σ_n is the conductivity of the normal metal.

To find the velocity $\mathbf{v}_L$, it is necessary to write down the equations of motion of the vortex lines. For the case of an "impure" alloy, neglecting inertial vortices, we have (see, for example, [18]):

$$\eta\mathbf{v}_L(\mathbf{r}_n) = -\partial U/\partial\mathbf{u}^v(\mathbf{r}_n), \tag{23}$$

where

$$U = \frac{1}{16(e\delta_0)^2}\sum_{i\neq k}K_0\left(\frac{|\mathbf{r}_i - \mathbf{r}_k|}{\delta_0}\right) \quad (\mathbf{r}_i = \mathbf{r}_i^0 + \mathbf{u}^v(\mathbf{r}_i^0)) \tag{24}$$

is the potential energy of the interaction of the vortex lines,[5] $\mathbf{r}_i$ is the radius vector which determines the position of the center of the vortex line in the plane perpendicular to the magnetic fields; $\mathbf{r}_i^0$ is the equilibrium position of the center of the line.

6)We assume that the forces (connected with various inhomogeneties) which fix the position of the vortex lines relative to the ionic lattice are absent. In practice, this means that the intensity of the sound must be rather large in order to include the considered mechanism of absorption in the presence of such forces.

For long-wave oscillations ($q\delta_0 \ll 1$ or $qd \sim q\sqrt{n_V} \ll 1$, if $\delta_0\sqrt{n_V} \ll 1$) the potential energy (24) can be written in the "elastic" approximation:

$$U - U_0 = \frac{B}{32\pi e\delta_0^2}\int dS\lambda_{\alpha\beta\gamma\delta}\frac{\partial u_\alpha^v}{\partial x^\gamma}\frac{\partial u_\beta^v}{\partial x^\delta}. \qquad (25)$$

Here U_0 is the potential energy of the equilibrium vortex lattice; the operations of integration and differentiation, like the vectors $\mathbf{r}$ and $\mathbf{u}^V$, lie in the plane perpendicular to the magnetic field. The tensor of the moduli of magnetic "elasticity" is defined here in the following way:

$$\lambda_{\alpha\beta\gamma\delta} = \sum_{r_k\neq 0}\Big[(\delta_{\alpha\beta}n_{k\gamma}n_{k\delta} - n_{k\alpha}n_{k\beta}n_{k\gamma}n_{k\delta})\frac{r_k}{\delta_0}K_0'\Big(\frac{r_k}{\delta_0}\Big) + n_{k\alpha}n_{k\beta}n_{k\gamma}n_{k\delta}\Big(\frac{r_k}{\delta_0}\Big)^2 K_0''\Big(\frac{r_k}{\delta_0}\Big)\Big], \quad \mathbf{n}_k = \frac{\mathbf{r}_k}{r_k} \qquad (26)$$

In the isotropic case[7)] the tensor $\lambda_{\alpha\beta\gamma\delta}$ contains only two independent components, which can be expressed in terms of the equations of [5] through the function $H_0(B)$. Omitting the simple calculations, we obtain the final expression for the potential energy U:

$$U - U_0 = \frac{1}{2}\int dS\Big\{\lambda_0\Big[\Big(\frac{\partial u_\alpha^v}{\partial x_\alpha}\Big)^2 + \Big(\frac{\partial u_\alpha^v}{\partial x_\beta}\frac{\partial u_\beta^v}{\partial x_\alpha}\Big)\Big] + \lambda_1\Big(\frac{\partial u_\alpha^v}{\partial x_\beta}\Big)^2\Big\}, \qquad (27)$$

$$\lambda_0 = \frac{B^2}{8\pi}\frac{dH_0}{dB}, \quad \lambda_1 = \int_0^B \frac{B^2}{8\pi}\frac{d^2H_0}{dB^2}dB. \qquad (28)$$

It follows from Eqs. (23) and (27) that

$$\frac{Be}{\pi}\eta\mathbf{v}_L = 2\lambda_0\nabla\operatorname{div}\mathbf{u}^v + \lambda_1\Delta\mathbf{u}^v.$$

Neglecting the difference in the displacements of the vortices $\mathbf{u}^V$ and the lattice $\mathbf{u}_\perp$ on the right side of this equation, and considering only a plane sound wave $\mathbf{u} \propto \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$, we get

$$\frac{Be}{\pi}\eta\mathbf{v}_L = -[2\lambda_0\mathbf{q}_\perp(\mathbf{q}_\perp\mathbf{u}_\perp) + \lambda_1 q_\perp^2\mathbf{u}_\perp], \qquad (29)$$

where $\mathbf{q}_\perp$ is the projection of the wave vector on the plane perpendicular to the magnetic field.

We introduce the angle θ between the directions of the vectors $\mathbf{q}$ and $\mathbf{H}_0$. Substituting Eq. (29) in Eqs. (21) and (4), we find the following for the coefficients of absorption of longitudinal and transverse sound due to the viscosity of the vortex lines:

$$\Big(\frac{\alpha_s^L}{\alpha_n^L}\Big)_{\text{vis}} = \frac{15}{64\pi^2}\Big(\frac{\tau_{tr}}{\tau_{tr}'}\beta(T)\Big)\frac{B}{H_{c2}}\Big(\frac{Be}{\sigma_n p_0 s_L}\Big)^2 \times\Big[\frac{dH_0}{dB}\Big(1 + \frac{\lambda_1}{2\lambda_0}\Big)\Big]^2\sin^6\theta, \qquad (30)$$

$$\Big(\frac{\alpha_s^T}{\alpha_n^T}\Big)_{\text{vis}} = \frac{5}{16\pi^2}\Big(\frac{\tau_{tr}}{\tau_{tr}'}\beta(T)\Big)\frac{B}{H_{c2}}\Big(\frac{Be}{\sigma_n p_0 s_L}\Big)^2\Big(\frac{dH_0}{dB}\Big)^2 \times\Big[\Big(1 + \frac{\lambda_1}{2\lambda_0}\Big)^2\cos^2\theta\cos^2\varphi + \Big(\frac{\lambda_1}{2\lambda_0}\Big)^2\sin^2\varphi\Big]\sin^4\theta, \qquad (31)$$

where λ_0 and λ_1 are defined above in (28), s_L is the velocity of longitudinal sound and s_T the velocity of transverse sound; φ is the angle between the vector $\mathbf{u}$ of the polarization of the sound wave and the plane formed by the vectors $\mathbf{H}_0$ and $\mathbf{q}$.

The strong anisotropy of the absorption described by Eqs. (30) and (31) is of interest. In particular, for waves traveling along the magnetic field ($\theta = 0$), both for longitudinal and transverse sound, this absorption is equal to zero. The longitudinal wave generally does not deform the vortex lattice in this case, while in a transverse wave, the individual cross sections of the vortex lattice are displaced as a whole without deformation, being absorbed completely following the ionic lattice. The maximum absorption of longitudinal sound is attained, naturally, for waves traveling perpendicular to the magnetic field.

The dependence of the absorption, (30) and (31), on the weak magnetic field is seen to be nonlinear and is determined principally by the factor dH_0/dB. According to Abrikosov,[5] as $B \to 0$,

$$dH_0/dB \sim B^{-2}\exp(-\text{const}/\sqrt{B}).$$

Taking Eqs. (30) and (31) into account, we obtain

$$(\alpha_s/\alpha_n)_{\text{vis}} \sim B\exp(-\text{const}/\sqrt{B}) \quad (B \to 0).$$

It must be noted that since we have $B < H_{c2} \to 0$ as $T \to T_c$, the contribution of the considered mechanism to the absorption as $T \to T_c$ is relatively small, thanks to the factor $Be/\sigma_n p_0 s$. The effective field, in the denominator of the last expression, is equal, in order of magnitude, to

$$\frac{\sigma_n p_0 s}{e} \sim \Big(\frac{s}{v_0}\Big)\Big(\frac{\mu}{T_c}\Big)\frac{H_{c2}(0)}{\varkappa^2}.$$

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APPENDIX

The Fourier representations of the various quantities are determined in the following fashion:

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7)Aiming at obtaining sensible results, we have neglected weak interactions in the plane perpendicular to the magnetic field. For a triangular lattice, [5] the results obtained are exact.

$$K(x_1, x_2) = T \sum_{\omega_n} \int \frac{d^3\mathbf{p}}{(2\pi)^3} K(\mathbf{p}; \omega_n) \exp[i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2) - i\omega_n(\tau_1 - \tau_2)],$$

$$\omega_n = \pi T(2n+1), \quad n = 0, \pm 1, \pm 2, \ldots,$$

$$\langle \tau_{ik}; \tau_{lm} \rangle (x_1, x_2) = T \sum_{\nu_n} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \langle \tau_{ik}; \tau_{lm} \rangle (\mathbf{q}; \nu_n) \times \exp[i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2) - i\nu_n(\tau_1 - \tau_2)], \quad \nu_n = \pi T \cdot 2n,$$

$$n = 0, \pm 1, \pm 2, \ldots;$$

$$\frac{\delta K(x_1, x_2)}{\delta U_{ik}(x_3)} = T^2 \sum_{\omega_+} \sum_{\omega_-} \int\int \frac{d^3\mathbf{p}\, d^3\mathbf{q}}{(2\pi)^6} \frac{\delta K}{\delta U_{ik}} (\mathbf{p}, \mathbf{q}; \omega_+, \omega_-) \times \exp\left[i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2) + i\mathbf{q}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2} - \mathbf{r}_3\right) - i\omega_+(\tau_1 - \tau_3) - i\omega_-(\tau_3 - \tau_2) \right], \quad \omega_+ = \omega_n, \quad \omega_- = \omega_n - \nu_{n'}.$$

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