MACROSCOPIC DESCRIPTION OF A COLLISION PLASMA IN A STRONG MAGNETIC FIELD IN STABILITY PROBLEMS

A. B. MIKHAĬLOVSKIĬ

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A new method is developed for solving Boltzmann's kinetic equation for a fully ionized plasma situated in a magnetic field. The method is based on assumptions that are characteristic of a collision plasma and differ from the assumption of the ordinary theory of transport phenomena. A system of macroscopic equations is obtained for a zero-pressure plasma in a straight magnetic field; these equations describe "drift" processes in the plasma more correctly than Braginskiĭ's system of equations [2].

1. INTRODUCTION

We present in this paper a method of solving Boltzmann's kinetic equations for a fully ionized plasma situated in a magnetic field, under the following main assumptions:

A. The characteristic time τ of the problem is of the order of the "drift" time, i.e.

$$1/\tau \approx \omega_{dr} \sim \rho v_T/a_\perp^2. \quad (1.1)$$

Here v_T is the thermal velocity of the particles, ρ their Larmor radius ($\rho \sim v_T/\omega_B$, where $\omega_B = eB/mc$ is the cyclotron frequency, e and m the charge and mass of the particles, B the magnetic field, and c the speed of light), and $a_\perp$ is the characteristic transverse dimension of the plasma.

B. The frequency ν of pair collisions greatly exceeds the frequency (1.1):

$$\nu \gg \omega_{dr}. \quad (1.2)$$

C. The magnetic field is sufficiently strong:

$$\omega_B \gg \nu. \quad (1.3)$$

Assumptions (1.1)—(1.3) are typical of problems involving the stability of a collision plasma in a strong magnetic field (cf., e.g., the review[1]). The initial equations used in investigations of such problems are as a rule the system of transport equations derived by Braginskiĭ.[2] The sufficient condition for the applicability of this system of equations is assumed to be smallness of the frequency of the investigated process compared with the frequency of the pair collisions and the relative smallness of the spatial gradients. These are the very assumptions on which Braginskiĭ's derivation is based.[2] However, when these equations are used, sight is frequently lost of the fact that they were derived neglecting the second-approximation distribution function. We shall show in this paper that this assumption, which at first glance is natural, is not universal (this is also indicated in [3]). In particular, it is precisely in the case of processes having a characteristic time (1.1) that certain second-order terms can be comparable with first-order terms. Consequently the use of Braginskiĭ's equations in problems where the corresponding first- and second-order terms are significant may lead to physically incorrect results.

Without using additional assumptions, in which account is taken of the specific features of the concrete problem, it would be very difficult to calculate the first-order corrections. Introduction of the limitations (1.1) and (1.3), and of a few others typical of stability problems (see Sec. 2), leads to appreciable simplification, and the determination of the necessary number of higher approximations becomes practically realizable. At the same time, in the presence of the foregoing additional assumptions we can construct a system of approximations that differs from [2], in particular, in the fact that terms making comparable contributions appear at frequencies of the order of (1.1) only in the same order of approximation. The main purpose of the present paper is to construct such a system of approximations. The method developed here is similar in principle to that used in one of Braginskii's papers [4] to calculate the particle and heat fluxes transverse to a strong magnetic field.

2. APPROXIMATIONS USED IN THE PRESENT PAPER

1. Basic approximations. The starting point in this paper, as in Braginskiĭ's [2], is Boltzmann's

kinetic equation for electrons and ions with a collision term in the Landau form [5]:

$$\frac{\partial f_a}{\partial t}+\mathbf{v}\nabla f_a+\frac{e_a}{m_a}\left(\mathbf{E}+\frac{[\mathbf{vB}]}{c}\right)\frac{\partial f_a}{\partial \mathbf{v}}=S_a, \quad (2.1)^*$$

$$S_a=\sum_b S_{ab}, \quad (2.2)$$

$$S_{ab}=-\frac{2\pi\lambda e_a^2 e_b^2}{m_a}\frac{\partial}{\partial v_\alpha}\int\left\{\frac{f_a(\mathbf{v})}{m_b}\frac{\partial f_b(\mathbf{v}')}{\partial v_\beta'}-\frac{f_b(\mathbf{v}')}{m_a}\frac{\partial f_a(\mathbf{v})}{\partial v_\beta}\right\}U_{\alpha\beta}d\mathbf{v}', \quad (2.3)$$

$$U_{\alpha\beta}=u^{-3}(u^2\delta_{\alpha\beta}-u_\alpha u_\beta), \quad u_\beta=v_\beta-v_\beta'. \quad (2.4)$$

The indices a and b denote here electrons (e) or ions (i), f is the distribution function, λ the "Coulomb logarithm," and **E** the electric field. The remaining symbols have already been defined. Where possible, we shall omit the electron and ion subscripts.

We shall solve (2.1) under the assumption that the following parameters are small quantities and have the same order of smallness ϵ:

$$\left\{\frac{\nu}{\omega_B}, \frac{\omega}{\nu}, \frac{\rho}{a_\perp}, \frac{a_\perp}{a_\parallel}, \frac{V_{\parallel e}-V_{\parallel i}}{v_T}, \frac{V_\perp}{v_T}, \frac{eE_\perp}{m\omega_B v_T}, \frac{eE_\parallel}{m\nu v_T}\right\}$$

$$\sim \varepsilon \ll 1. \quad (2.5)$$

Here ν is the frequency of the pair collisions between particles of the same kind; $E_\parallel$ and $E_\perp$ are the components of the electric field **E** parallel and perpendicular to the field **B**; $V_\parallel$ and $V_\perp$ are the mean velocities of the plasma components parallel and perpendicular to **B** (they will be defined rigorously later), and $a_\parallel$ is the dimension of the longitudinal inhomogeneity of the plasma.

2. Simplifying assumptions. We shall assume that the plasma pressure p is negligibly small compared with the magnetic field pressure $B^2/8\pi$, $\beta \equiv 8\pi p/B^2 \to 0$, and that the electric field is almost potential, $\mathbf{E} \approx -\nabla\psi$. We neglect the curvature and inhomogeneity of the magnetic field and put $\mathbf{B} \parallel z$, and $\partial B_z/\partial x = \partial B_z/\partial y = 0$. We note that similar approximations are frequently employed in the theory of stability of a collisionless plasma [6].

3. Transformation of kinetic equation. Assuming that $\mathbf{B} \parallel z$, we introduce a Cartesian coordinate system in ordinary space, and a cylindrical system in velocity space, such that

$$\mathbf{r}=(x,y,z), \quad \mathbf{v}=(v_\perp\cos\alpha, v_\perp\sin\alpha, v_z). \quad (2.6)$$

Expressing in (2.1) all the velocity functions in terms of $v_\perp$, v_z, and α we write out separately the parts of this equation which do and do not depend on α. Putting

$$f=\bar{f}+\tilde{f}, \quad S=\bar{S}+\tilde{S}, \quad (2.7)$$

where the superior bar and tilde denote respectively the functions that depend and do not depend on α, we get

$$L_\parallel\tilde{f}+L_\perp\bar{f}+L_\perp\tilde{f}-\langle L_\perp\tilde{f}\rangle-\tilde{S}=\omega_B\partial\tilde{f}/\partial\alpha, \quad (2.8)$$

$$L_\parallel\bar{f}+\langle L_\perp\tilde{f}\rangle=\bar{S}. \quad (2.9)$$

We have introduced here the operators

$$L_\parallel=\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}+\frac{e}{m}E_z\frac{\partial}{\partial v_z}, \quad L_\perp=\mathbf{v}_\perp\nabla+\frac{e}{m}\mathbf{E}_\perp\frac{\partial}{\partial \mathbf{v}_\perp}, \quad (2.10)$$

and the symbol $\langle\,\rangle$ denotes averaging with respect to α.

According to assumption (2.5), the terms $\tilde{S}$, $L_\parallel\bar{f}$, and $L_\perp\tilde{f}$ in (2.8) are small quantities of first order, while $L_\parallel\bar{f}$ is of second order compared with the right-hand side. As to Eq. (2.9), the term $L_\perp\tilde{f}$ is of the same order as the right-hand part, and $L_\parallel\bar{f}$ is of the order of smallness of ϵ.

These estimates pertain only to the higher-order terms of each of the components. In order to develop a perturbation theory in terms of the parameter ϵ, it is necessary first to separate in each of the terms of (2.8) and (2.9) the terms of different orders.

We introduce the density n, the mean velocity V, and the temperature of each kind of particle, defining them by means of the relations

$$n=\int f\,d\mathbf{v}\equiv\int\bar{f}\,d\mathbf{v},$$

$$n\mathbf{V}_\perp=\int\mathbf{v}_\perp f d\mathbf{v}\equiv\int\mathbf{v}_\perp\bar{f}\,d\mathbf{v},$$

$$nV_z\equiv nV_\parallel=\int v_z f d\mathbf{v}\equiv\int v_z\bar{f}\,d\mathbf{v},$$

$$\frac{3}{2}nT=\int\frac{m(\mathbf{v}-\mathbf{V})^2}{2}f\,d\mathbf{v}\equiv\int\frac{mv_a^2}{2}\bar{f}\,d\mathbf{v}-\frac{mV_\perp^2}{2}n. \quad (2.11)$$

In the last of the formulas we have introduced

$$\mathbf{v}_a=\mathbf{v}-\mathbf{e}_z V_z, \quad (2.12)$$

where $\mathbf{e}_z$ is a unit vector in the z direction. According to (2.12) we have $\mathbf{v}_{0\perp}=\mathbf{v}_\perp$ and $v_{az}=v_z-V_z$, so that v_{az} has the meaning of the "random" component of the longitudinal particle velocity [2].

We shall henceforth express all velocity functions in terms of the variables v_a. The operators $L_\parallel$ and $L_\perp$ then take the form

*$[\mathbf{vB}] \equiv \mathbf{v}\times\mathbf{B}$.

$$L_{\parallel}=\frac{\partial_a}{\partial t}+v_{z\alpha}\frac{\partial}{\partial z}+\left(\frac{e}{m}E_z-\frac{\partial_a V_z}{\partial t}-\frac{\partial V_z}{\partial z}v_{az}\right)\frac{\partial}{\partial v_{az}},$$

$$L_{\perp}=\mathbf{v}_{\perp}\nabla+\frac{e}{m}\mathbf{E}_{\perp}\frac{\partial}{\partial \mathbf{v}_{\perp}}-\mathbf{v}_{\perp}\nabla V_z\frac{\partial}{\partial v_{az}},$$

$$\frac{\partial_a}{\partial t}=\frac{\partial}{\partial t}+V_z\frac{\partial}{\partial z}. \qquad (2.13)$$

Using the smallness of the ratio m_e/m_i, we separate the small terms in the crossing collision integrals, S_{ei} and S_{ie}. Following [2], we represent the electronic integral in the approximate form

$$S_{ei}=S_{ei}'+S_{ei}'', \qquad (2.14)$$

where

$$S_{ei}'=\frac{3\sqrt{\pi}}{8\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2}\frac{\partial}{\partial v_\alpha}\left(C_{\alpha\beta}^{*}\frac{\partial f_e}{\partial v_\beta}\right),$$

$$S_{ei}''=\frac{3\sqrt{\pi}}{8\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2}\frac{m_e}{m_i}\frac{\partial}{\partial v_\alpha}\left(\frac{2v_\alpha'}{v'^3}f_e\right.$$

$$\left.+\frac{T_e}{m_i}\frac{3v_\alpha'v_\beta'-v'^2\delta_{\alpha\beta}}{v'^5}\frac{\partial f_e}{\partial v_\beta}\right), \quad C_{\alpha\beta}^{*}=\frac{v'^2\delta_{\alpha\beta}-v_\alpha'v_\beta'}{v'^3}$$

$$v'=v-V. \qquad \tau=\frac{3\sqrt{m_e}T_e^{3/2}}{4\sqrt{\pi}\lambda(e_ee_i)^2n_i}. \qquad (2.15)$$

We expand the function $C_{\alpha\beta}*$ in powers of $V_{\perp i}$ and $V_{ze}-V_{zi}$. According to (2.5), each of these quantities, when reduced to dimensionless form, is of the order of ϵ. Retaining only the first three terms of the expansion, we get

$$C_{\alpha\beta}^{*}=C_{\alpha\beta}-V_{i\gamma}\frac{\partial C_{\alpha\beta}}{\partial v_\gamma}+(V_{ze}-V_{zi})\frac{\partial C_{\alpha\beta}}{\partial v_z}$$

$$+\frac{1}{2}V_{i\gamma}V_{i\delta}\frac{\partial^2 C_{\alpha\beta}}{\partial v_\gamma\partial v_\delta}+\frac{1}{2}(V_{ze}-V_{zi})^2\frac{\partial^2 C_{\alpha\beta}}{\partial v_z^2}. \qquad (2.16)$$

Here $C_{\alpha\beta}=(v_e^2\delta_{\alpha\beta}-v_{e\alpha}v_{e\beta})/v_e^3$, and the indices γ and δ take on only the values x and y.

Using (2.2) and (2.14)—(2.16), recognizing that τ_e is of the order of the time of the electron-electron collisions, and assuming the ratio m_e/m_i to be of the order of ϵ^2, we obtain the following series for S_e:

$$S_e(\varepsilon^0,f_e)=S_{ee}[f_e,f_e]+S_{ei}'(\varepsilon^0,f_e), \quad S_e(\varepsilon,f_e)=S_{ei}'(\varepsilon,f_e),$$
$$S_e(\varepsilon^2,f_e)=S_{ei}'(\varepsilon^2,f_e)+S_{ei}''(f_e), \qquad (2.17)$$

where the expressions for $S_{ei}'(\epsilon^n)$ are obtained in obvious fashion from (2.15) and (2.16). At the assumed degree of accuracy, we can replace the quantity $\mathbf{v}'$ in the expression for S_{ei}' by $\mathbf{v}_e$.

The expression for S_{ie} will be simplified in exactly the same manner as in [2]. We represent the result in the form

$$S_{ie}=S_{ie}(\varepsilon,f_i)+S_{ie}(\varepsilon^2,f_i)+\ldots, \qquad (2.18)$$

where

$$S_{ie}(\varepsilon,f_i)=-\frac{\mathbf{R}_i(\varepsilon)}{m_in_i}\frac{\partial f_i}{\partial \mathbf{v}}$$

$$S_{ie}(\varepsilon^2,f_i)=\frac{m_en_e}{m_in_i}\frac{1}{\tau_e}\frac{\partial}{\partial v_\alpha}\left(v_{i\alpha}f_i+\frac{T_e}{m_i}\frac{\partial f_i}{\partial v_\alpha}\right)-\frac{\mathbf{R}_i(\varepsilon^2)}{m_in_i}\frac{\partial f_i}{\partial \mathbf{v}},$$

$$\mathbf{R}_i(\varepsilon^n)=-\int m_e\mathbf{v}_eS_{ei}'(\varepsilon^n,f_e)\,dv_e\equiv-\mathbf{R}_e(\varepsilon^n). \qquad (2.19)$$

We represent the distribution function in the form of a series in powers of ϵ:

$$f=\sum_{n=0}^{\infty}f^{(n)}\equiv\sum_{n=0}^{\infty}(\bar{f}^{(n)}+\tilde{f}^{(n)}). \qquad (2.20)$$

Following [7], we assume that the zeroth approximation $f^{(0)}$ gives the exact values of the zero-order magnitude—the density, average longitudinal velocity, and temperature:

$$\int\bar{f}^{(0)}d\mathbf{v}=n, \quad \int\bar{f}^{(0)}v_z d\mathbf{v}=nV_z,$$
$$\int\frac{mv_a^2}{2}\bar{f}^{(0)}d\mathbf{v}=\frac{3}{2}nT. \qquad (2.21)$$

Then, as follows from (2.11), we shall have for the n-th order functions the following additional conditions:

$$\int f^{(n)}d\mathbf{v}=0, \quad \int v_{az}f^{(n)}d\mathbf{v}=0,$$

$$\int\frac{mv_a^2}{2}f^{(n)}\,d\mathbf{v}-\frac{m(V_\perp^2)^{(n)}}{2}n=0. \qquad (2.22)$$

The last expression has from the point of view of classical theory [5] an unusual form. The reason is that the mean velocity $V_\perp$ transverse to the magnetic field, was assumed to be a small quantity [see (2.5)] and does not pertain to the zeroth approximation. This quantity can be represented in series form:

$$\mathbf{V}_\perp=\sum_{n=1}^{\infty}\mathbf{V}_\perp^{(n)}, \qquad (2.23)$$

where

$$\mathbf{V}_\perp^{(n)}=\int\mathbf{v}_\perp\bar{f}^{(n)}d\mathbf{v}. \qquad (2.24)$$

The absence of a term with n = 0 from the sum in (2.23) is equivalent to the assumption that

$$\int\mathbf{v}_\perp\bar{f}^{(0)}dv=0, \qquad (2.25)$$

which must be verified later.

According to (2.21) and (2.22), the quantities n, V_z, and T, determined in the zeroth approximation, actually contains terms of all orders in ϵ. Therefore the derivatives $\partial n/\partial t$, $\partial V_z/\partial t$, and $\partial T/\partial t$ will also be series in powers of ϵ. In analogy with [5], we shall retain in these series the

full values of n, V_z, and T. Then we get, say for $\partial n/\partial t$,

$$\frac{\partial_a n}{\partial t} = -n\frac{\partial V_z}{\partial z} - \sum_{k=1}^{\infty} \operatorname{div}(nV_\perp^{(k)}). \qquad (2.26)$$

Following [5], we shall denote by $(\partial n/\partial t)^{(k)}$, $(\partial V_z/\partial t)^{(k)}$ and $(\partial T/\partial t)^{(k)}$ the sum of the terms of k-th order of the right side of equations such as (2.26), and introduce the symbol

$$\frac{\partial^{(k)}}{\partial t} = \left(\frac{\partial n}{\partial t}\right)^{(k)}\frac{\partial}{\partial n} + \left(\frac{\partial V_z}{\partial t}\right)^{(k)}\frac{\partial}{\partial V_z} + \left(\frac{\partial T}{\partial t}\right)^{(k)}\frac{\partial}{\partial T}, \quad k \geqslant 1. \qquad (2.27)$$

Recognizing that the operator $\partial/\partial t$ in (2.1) acts on a function f that depends on n, V_z, and T (and also, in particular, on E), and using the definition (2.27), we shall represent the operator $L_{||}$ in the form of the following series in ϵ:

$$L_{||}(\varepsilon^0) = \frac{\partial_a^{(0)}}{\partial t} + v_{az}\frac{\partial}{\partial z} + \left(\frac{e}{m}E_z - \frac{\partial_a^{(0)}V_z}{\partial t} - \frac{\partial V_z}{\partial z}v_{az}\right)\frac{\partial}{\partial v_{az}},$$
$$L_{||}(\varepsilon^n) = \frac{\partial^{(n)}}{\partial t} + \frac{\partial^{(n)}V_z}{\partial t}\frac{\partial}{\partial v_{az}}, \quad n \geqslant 1, \qquad (2.28)$$

where

$$\frac{\partial_a^{(0)}}{\partial t} = \left(\frac{\partial_a}{\partial t}\right)_{n, V_z, T} + \left(\frac{\partial n}{\partial t}\right)^{(0)}\frac{\partial}{\partial n} + \left(\frac{\partial V_z}{\partial t}\right)^{(0)}\frac{\partial}{\partial V_z} + \left(\frac{\partial T}{\partial t}\right)^{(0)}\frac{\partial}{\partial T}. \qquad (2.29)$$

Using Eqs. (2.8) and (2.9), in which we can separate in unique fashion the terms of any prescribed order in ϵ, using the foregoing relations, we can readily obtain equations for any term of the series (2.20).

3. SOLUTION OF EQUATIONS (2.8) AND (2.9) ACCURATE TO ϵ^2 INCLUSIVE

In the zeroth order in ϵ we get from (2.8)

$$\partial \tilde{f}^{(0)}/\partial\alpha = 0. \qquad (3.1)$$

this means that

$$\tilde{f}^{(0)} = 0. \qquad (3.2)$$

The condition (2.25) is thereby automatically satisfied.

Taking (3.2) into account, the zeroth approximations of (2.9) for the electrons and ions respectively are

$$S_{ee}(\bar{f}_e^{(0)}, \bar{f}_e^{(0)}) + S_{ei}'(\varepsilon^0, \bar{f}_e^{(0)}) = 0,$$
$$S_{ii}(\bar{f}_i^{(0)}, \bar{f}_i^{(0)}) = 0. \qquad (3.3)$$

Each of these equations has a solution satisfying the conditions (2.22) in the form

$$\bar{f}_a^{(0)} = \left(\frac{m_a}{2\pi T_a}\right)^{3/2}\exp\left(-\frac{mv_a^2}{2T_a}\right) \equiv F_a, \qquad (3.4)$$

Let us consider the higher-order equations.

1. First approximation for $\tilde{f}$. In first order in ϵ, we get from (2.8)

$$\frac{\partial \tilde{f}^{(1)}}{\partial\alpha} = \frac{1}{\omega_B}L_\perp F. \qquad (3.5)$$

From this we get

$$\tilde{f}^{(1)} = F\mathbf{v}_\perp\left[\frac{\mathbf{e}_z}{\omega_B}, \nabla\ln F - \frac{e\mathbf{E}_\perp}{T} + \frac{mv_{az}}{T}\nabla V_z\right]. \qquad (3.6)$$

By integrating (3.6) with respect to the velocities we find that the following quantities differ from zero in first order: the transverse velocity $\mathbf{V}_\perp^{(1)}$, the transverse heat flux $\mathbf{q}_\perp^{(1)}$, and the components of the viscosity tensor $\pi_{\alpha\beta}$ with indices (x, z) and (y, z). (For a definition of $\mathbf{q}$ and $\pi_{\alpha\beta}$ see, e.g., [2,3,7].)

2. First approximation for $\bar{f}$. Substituting (3.6) in (2.9), averaging the expression for $L_\perp\tilde{f}^{(1)}$ with respect to α, and taking into account the assumptions of Item 2 of Sec. 2, we get in first order in ϵ:

$$\left(\frac{\partial^{(0)}}{\partial t} + V_z\frac{\partial}{\partial z} + \mathbf{V}_E\nabla\right)F + \frac{mv_{az}}{T}F\left(\frac{\partial^0}{\partial t} + V_z\frac{\partial}{\partial z} + \mathbf{V}_E\nabla\right)V_z + v_{az}\left(\frac{\partial F}{\partial z} - \frac{eF}{T}E_z\right) + \frac{mv_{az}^2}{T}F\frac{\partial V_z}{\partial z} = \bar{S}^{(1)}. \qquad (3.7)$$

Here

$$\mathbf{V}_E = \frac{e}{m\omega_B}[\mathbf{E}\mathbf{e}_z] \equiv \frac{c}{B^2}[\mathbf{E}\mathbf{B}], \qquad (3.8)$$

and $\bar{S}^{(1)}$ for the corresponding type of charge takes the form

$$\bar{S}_e^{(1)} = S_{ee}[F_e, \bar{f}_e^{(1)}] + S_{ei}'(\varepsilon^0, \bar{f}_e^{(1)}) + S_{ei}'(\varepsilon, F_e),$$
$$\bar{S}_i^{(1)} = S_{ii}[F_i, \bar{f}_i^{(1)}] + S_{ie}(\varepsilon, F_i). \qquad (3.9)$$

Integrating (3.7) over the velocities with weight 1, v_{az}, and $mv_\alpha^2/2$ we obtain the relations between the derivatives $\partial^0(n, V_z, T)/\partial t$ which result from the differentiation of F with respect to t (see (3.4)) on the one hand, and the remaining quantities of order ϵ, on the other. Carrying out this integration, we obtain

$$\left(\frac{\partial^{(0)}}{\partial t} + V_z\frac{\partial}{\partial z} + \mathbf{V}_E\nabla\right)n + n\frac{\partial V_z}{\partial z} = 0,$$
$$\left(\frac{\partial^{(0)}}{\partial t} + V_z\frac{\partial}{\partial z} + \mathbf{V}_E\nabla\right)V_z = \frac{e}{m}E_z - \frac{1}{mn}\frac{\partial p}{\partial z} + \frac{R_z(\varepsilon)}{mn},$$
$$\frac{3}{2}n\left(\frac{\partial^{(0)}}{\partial t} + V_z\frac{\partial}{\partial z} + \mathbf{V}_E\nabla\right)T + p\frac{\partial V_z}{\partial z} = 0. \qquad (3.10)$$

The expressions for the quantities $R_{ze}(\epsilon)$

$= -R_{zi}(\epsilon)$, which stand for the friction forces between the corresponding plasma components, has according to (2.15), (2.16) and (2.19) the following form:

$$R_{ze}(\varepsilon) = -\frac{n_e m_e}{\tau}(V_{ez} - V_{iz}) + R_{ze}'(\varepsilon),$$

$$R_{ze}'(\varepsilon) = -\frac{3\sqrt{\pi}}{8\tau_e}\left(\frac{2T_e}{m_e}\right)^{3/2} m_e \int C_{zz}\frac{\partial \bar{f}_e^{(1)}}{\partial v_{ez}}\,dv. \quad (3.11)$$

The friction force contains the function $\bar{f}^{(1)}$, as a result of which it can be determined only after solving Eq. (3.7) for $\bar{f}^{(1)}$ (see also [3]).

If Eqs. (10) are represented in the form used in the classical theory of transport phenomena[7,2], then we get, accurate to terms of order ϵ:

$$\frac{d_a n}{dt} + n\,\mathrm{div}\,\mathbf{V} = 0,$$

$$mn\frac{d_a V_z}{dt} = enE_z - \frac{\partial p}{\partial z} - (\nabla\pi)_z + R_z,$$

$$\frac{3}{2}n\frac{d_a T}{dt} + p\,\mathrm{div}\,\mathbf{V} = -\mathrm{div}\,\mathbf{q}, \quad (3.12)$$

where $d_a/dt = \partial/\partial t + (\mathbf{V}\cdot\nabla)$.

Comparing (3.12) with the zeroth- and first-approximation equations of [7,2], we arrive at the conclusion that terms which are referred to different orders ($\mathbf{V}$, $\mathbf{q}$, π) in the usual scheme are included in our scheme in the same order in ϵ. When these terms are summed, some of their components cancel one another. Therefore when equations of the type given in [7,2] are used in problems where these components are appreciable, the latter must be taken into account not only in the largest but also in the higher-order terms. As to Eqs. (3.12) they contain in accordance with Braginskiĭ's scheme terms of zeroth and first order. The calculations made in [2] take both orders into account. Therefore, by reclassifying the orders of magnitude of the quantities in the final results of [2] and retaining in them only terms of order not higher than ϵ, we arrive at Eqs. (3.10). Since our approximations correspond, according to (2.6), to processes having frequencies on the order of the drift velocity, it follows from the foregoing that Braginskiĭ's equations [2] can be used to describe drift processes, at least accurate to ϵ. The degree to which these equations take correct account of the higher orders in ϵ will be discussed later.

With the aid of (3.10) we exclude from (3.7) the derivatives $\partial^0/\partial t$, after which the equation for each kind of charge takes the form

$$F_e\left\{\left[\left(\frac{m_e v_e^2}{2T_e} - \frac{5}{2}\right)\frac{\partial \ln T_e}{\partial z} + \frac{m_e}{T_e\tau_e}\left(\frac{3\sqrt{\pi}}{\sqrt{2}}\left(\frac{T_e}{m_e}\right)^{3/2}\frac{1}{v_e^3} - 1\right)(V_{ez} - V_{iz}) + \frac{R_z'(\varepsilon)}{n_e T_e}\right]v_{ez} + \frac{m_e}{T_e}\left(v_{ez}^2 - \frac{v_e^2}{3}\right)\frac{\partial V_{ez}}{\partial z}\right\} = S_{ee}[F_e, \bar{f}_e^{(1)}] + S_{ei}'(\varepsilon_0, \bar{f}_e^{(1)});$$

$$F_i\left[\left(\frac{m_i v_i^2}{2T_i} - \frac{5}{2}\right)\frac{\partial \ln T_e}{\partial z}v_{iz} + \frac{m_i}{T_i}\left(v_{iz}^2 - \frac{v_i^2}{3}\right)\frac{\partial V_{iz}}{\partial z}\right] = S_{ii}[F_i, \bar{f}_i^{(1)}]. \quad (3.13)$$

The difference between (3.13) and the similar equations of Braginskiĭ [2] for the first-order correction (which does not depend on the angle in transverse-velocity space) consists in the fact that the left sides of our equations do not contain the divergence of the transverse velocity, $\mathrm{div}\,\mathbf{V}_\perp^{(1)}$. This result is connected with the fact that in the case considered by us, that of a straight and homogeneous magnetic field and a nonsolenoidal electric field (see the assumptions of Item 2 of Sec. 2), the contribution made by the second-approximation terms that were unaccounted for in [2] is exactly offset by the contribution connected with $\mathrm{div}\,\mathbf{V}_\perp^{(1)}$. This is proved in the Appendix of the present paper.

The solution of Eqs. (3.13) reduces to the expression obtained in [2] for the first-approximation function. It is only necessary to make in the corresponding results of [2] the substitution

$$\mathrm{div}\,\mathbf{V} \to \partial V_z/\partial z. \quad (3.14)$$

The expression for $R_{ze}'(\epsilon)$ (see (3.11) turns out to be exactly the same as in [2]:

$$R_{ze}' = -0.71 n_e \frac{\partial T_e}{\partial z}. \quad (3.15)$$

3. Second approximation for $\tilde{f}$. Substituting in (2.8) the results of the zeroth and first approximations, we get

$$\tilde{f}^{(2)} = \tilde{f}_1^{(2)} + \tilde{f}_2^{(2)}, \quad (3.16)$$

where

$$\tilde{f}_1^{(2)} = -\frac{1}{\omega_B}\frac{\partial}{\partial\alpha}(L_\perp \bar{f}^{(1)} - \bar{S}^{(1)}), \quad (3.17)$$

and the function $\tilde{f}_2^{(2)}$ contains $\cos 2\alpha$ and $\sin 2\alpha$, and will be of no use to us in the present work.

The function $f^{(2)}$ is a correction to $\tilde{f}^{(1)}$ of order $\rho/a_\perp$ and ν/ω_B. Since

$$\rho_i \gg \rho_e, \quad (\nu/\omega_B)_i \gg (\nu/\omega_B)_e, \quad (3.18)$$

we can confine ourselves to calculation of the

ionic correction only. The expression $\widetilde{S}_i^{(1)}$ can be replaced with the same degree of accuracy by $\widetilde{S}^{(1)}$.

4. Second-approximation equation for $\bar{f}$. From (2.9) we have in the second-order approximation in:

$$L_{\|}(\varepsilon^0)\bar{f}^{(1)}+\left(\frac{\partial^{(1)}}{\partial t}-\frac{\partial^{(1)}V_z}{\partial t}\frac{mv_{az}}{T}\right)F+\langle L_\perp \tilde{f}_1^{(2)}\rangle=S^{(2)}. \tag{3.19}$$

The operator $L_{\|}(\epsilon^0)$ is defined by (2.28). The mean value $\langle L_\perp \tilde{f}_1^{(2)}\rangle$ denotes, according to (2.13) and (3.17)

$$\langle L_\perp \tilde{f}_1^{(2)}\rangle = \mathbf{V}_E\left(\nabla\bar{f}^{(1)}-\nabla V_z\frac{\partial\bar{f}^{(1)}}{\partial v_{az}}\right)$$
$$+\left[\frac{\mathbf{e}_z}{\omega_B},\ \nabla+\frac{e}{m}\frac{\partial}{\partial\varepsilon_\perp}-\nabla V_z\frac{\partial}{\partial v_{az}}\right]\langle \mathbf{v}_\perp\bar{S}^{(1)}\rangle. \tag{3.20}$$

For the electrons and the ions, the expression for $\bar{S}^{(2)}$ has, in accord with (2.14)—(2.17) and (2.20), the form

$$\bar{S}_e = S_{ee}[F_e,\bar{f}_e^{(2)}]+\langle S_{ee}[\tilde{f}_e^{(1)},\tilde{f}_e^{(1)}]\rangle+S_{ee}[\bar{f}_e,\bar{f}_e^{(1)}]$$
$$+S_{ei}'(\varepsilon^0,\bar{f}_e^{(2)})+S_{ei}'(\varepsilon,\bar{f}_e^{(1)})+S_{ei}'(\varepsilon^2,F_e)+S_{ei}''(F_e),$$
$$\bar{S}_i = S_{ii}[F_i,\bar{f}_i^{(2)}]+\langle S_{ii}[\tilde{f}_i^{(1)},\tilde{f}_i^{(1)}]\rangle+S_{ii}[\bar{f}_i^{(1)},\bar{f}_i^{(1)}]$$
$$+S_{ie}(\varepsilon,\bar{f}_i^{(1)})+S_{ie}(\varepsilon^2,F_i). \tag{3.21}$$

5. Summary of the results of the first two approximations. With the aid of (3.19) we can find expressions for $\partial^{(1)}(n, V_z, T)$. Combining this result with (3.10), we arrive at the following system of macroscopic equations

$$\frac{D_e n_e}{Dt}+n_e\frac{\partial V_{ze}}{\partial z}=0,\quad \frac{D_i n_i}{Dt}+n_i\frac{\partial V_{zi}}{\partial z}=0,$$
$$m_e n_e\frac{D_e V_{ze}}{Dt}=e_e n_e E_z-\frac{\partial p_e}{\partial z}-\frac{n_e m_e}{\tau_e}(V_{ze}-V_{zi})$$
$$-0{,}71 n_e\frac{\partial T_e}{\partial z},$$
$$m_i n_i\frac{D_i V_{zi}}{Dt}=e_i n_i E_z-\frac{\partial p_i}{\partial z}+\frac{n_e m_e}{\tau_e}(V_{ze}-V_{zi})+0{,}71 n_e\frac{\partial T_e}{\partial z}$$
$$+\frac{4}{3}\frac{\partial}{\partial z}\left(\eta_{0i}\frac{\partial V_{zi}}{\partial z}\right)+\operatorname{div}(\eta_{2i}\nabla_\perp V_{zi}),$$
$$\frac{3}{2}n_e\frac{D_e T_e}{Dt}+p_e\frac{\partial V_{ze}}{\partial z}=\frac{\partial}{\partial z}\left(\varkappa_{\|e}\frac{\partial T_e}{\partial z}\right)-\frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_e-T_i),$$
$$\frac{3}{2}n_i\frac{D_i T_i}{Dt}+p_i\frac{\partial V_{zi}}{\partial z}=\operatorname{div}(\varkappa_{\perp i}\nabla_\perp T_i)+\frac{4}{3}\eta_{0i}\left(\frac{\partial V_{zi}}{\partial z}\right)^3$$
$$+\eta_{2i}(\nabla_\perp V_{zi})^2+\frac{\partial}{\partial z}\left(\varkappa_{\|i}\frac{\partial T_i}{\partial z}\right)+\frac{3m_e}{m_i}\frac{n_e}{\tau_e}(T_e-T_i), \tag{3.22}$$

where

$$\frac{D_\alpha}{Dt}=\frac{\partial}{\partial t}+V_{z\alpha}\frac{\partial}{\partial z}+\mathbf{V}_E\nabla,$$

and the quantities η_{0i}, η_{2i}, $\kappa_{\|i}$, and $\kappa_{\perp i}$ were calculated by Braginskiĭ [2]:

$$\eta_{0i}=0.96 n_i T_i\tau_i \qquad \eta_{2i}=\frac{6}{5}\frac{n_i T_i}{\omega_{Bi}^2\tau_i},$$
$$\varkappa_{\|e}=3.16\frac{n_e T_e\tau_e}{m_e}, \qquad \varkappa_{\|i}=3{,}9\frac{n_i T_i\tau_i}{m_i},$$
$$\varkappa_{\perp i}=2\frac{n_i T_i}{m_i\omega_{Bi}^2\tau_i}, \qquad \tau_i=\frac{3\sqrt{m_i}T_i^{3/2}}{4\sqrt{\pi}\lambda e_i^4 n_i}. \tag{3.23}$$

In deriving (3.22) we neglected small terms of order ν_e/ω_{Be}. In particular, the equations for $\partial T/\partial t$ do not contain, for the same reasons, the terms corresponding to the heat transfer due to the transverse friction force between the components (these terms and their like do not differ from those given in [2]).

4. DISCUSSION OF RESULTS

Equations (3.22) differ from Braginskiĭ's [2] in the following respects:

1. The component of the viscosity tensor does not contain the divergence of the transverse velocity. This is caused by the presence of a compensating term in the second-order approximation, a term discarded by Braginskiĭ [2][1)]. In addition, part of div $\mathbf{V}_\perp$ drops out by virtue of the assumptions of Item 2 of Sec. 2). For the same reason, the equations for the heat contain, in lieu of the full product $\pi_{\alpha\beta}\partial V_\alpha/\partial x_\beta$, only the terms containing the derivatives of the longitudinal velocity, but not the transverse one.

Our analysis leads to the following prescription for using the equations of [2] in problems corresponding to the assumptions of Items 1 and 2 of Sec. 2. If the investigated effects are only of order ϵ, the equations of [2] can be regarded as fully corresponding to the conditions of the problem. When account is taken of effects of order ϵ^2 it is necessary and sufficient to omit from the quantities $\pi_{\alpha\beta}$ and $\pi_{\alpha\beta}\partial V_\alpha/\partial x_\beta$ contained in these equations the derivatives of the transverse velocity which make up the combination div $\mathbf{V}_\perp$.

2. Equations (3.22) do not take into account terms of order ϵ^3 and higher, whereas in [2], owing to the use of a different set of expansion parameters, some of these terms are retained. These terms describe such effects as, say, the transverse inertia and transverse viscosity of the ions, and can be significant in problems where the ratio $a_\perp/a_{\|}$ is assumed to be very small [1].

[1)] The second approximation is taken into account in [8,9].

$$\frac{a_\perp}{a_\parallel} \ll \left(\frac{m_e}{m_i}\right)^{1/2} \frac{\nu_i}{\omega_{Bi}} \qquad (4.1)$$

In order to obtain macroscopic equations for a comprehensive set of problems (within the scope of the assumptions of Item 2 of Sec. 2) it is necessary to supplement (3.22) with higher-order terms. To this end it is necessary to find the solution of the kinetic equation in higher orders, namely ϵ^3 and ϵ^4. Such a solution has been obtained. It turns out that all the physically interesting effects taken into account in orders ϵ^3 and ϵ^4 are contained in Braginskiĭ's equations [2]. (Details of the calculations are not presented in this article.) Therefore the prescription for using the system of equations of [2], with due allowance for the terms in ϵ^3 and ϵ^4, remains the same as when allowance is made for terms of order ϵ^2 only.

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APPENDIX

The result (3.18) was obtained under the assumption that the transverse velocity is a quantity of first order of smallness. We shall show that the very same result can be obtained also in the usual manner [2,3] when $\mathbf{V}_\perp$ is a quantity of zeroth order.

Making in (2.1) the substitution $\mathbf{v} = \mathbf{c} + \mathbf{V}$, we transform it to Eq. (4.2) of Braginskiĭ's paper [3]:

$$S - [\mathbf{c}\omega_B]\frac{\partial f_a}{\partial \mathbf{c}} = \frac{d_a f_a}{dt} + \mathbf{c}\nabla f_a + \left(\frac{e_a}{m_a}\mathbf{E}_a^* - \frac{d_a \mathbf{V}_a}{dt}\right)\frac{\partial f_a}{\partial \mathbf{c}} - c_\alpha \frac{\partial V_{a\beta}}{\partial x_\alpha}\frac{\partial f_a}{\partial c_\beta}, \qquad (A.1)$$

where

$$\frac{d_a}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_a\nabla, \quad \mathbf{E}_a^* = \mathbf{E} + \frac{1}{c}[\mathbf{V}_a\mathbf{B}].$$

We shall assume for simplicity that (A.1) pertains to ions, and S does not contain S_{ie}. Just as in [2], we assume as the zeroth approximation for $f(\mathbf{c})$ the Maxwellian distribution $F(\mathbf{c})$. For that part of the first-order correction which oscillates with respect to the angle in the $\mathbf{c}_\perp$ space ($\tilde{f}^{(1)} = F\tilde{\Phi}^{(1)}$) we get from (A.1):

$$\tilde{\Phi}^{(1)} = \mathbf{c}_\perp \frac{m}{T}\mathbf{a}_\perp, \qquad (A.2)$$

where

$$\mathbf{a} = \frac{\mathbf{q}_\perp}{p}\left(\frac{mc^2}{5T} - 1\right) + \frac{c_z}{p}\pi_{z\perp}, \qquad (A.3)$$

and $\mathbf{q}_2$ and $\pi_{z\perp}$ are defined by

$$\mathbf{q}_\perp = \frac{5}{2}\frac{p}{m\omega_B}[\mathbf{e}_z\nabla T],$$

$$\pi_{z\perp} \equiv (\pi_{zx}, \pi_{zy}) = \frac{p}{\omega_B}[\mathbf{e}_z, \nabla V_z]. \qquad (A.4)$$

Substituting in the right side of (A.1) in lieu of f_α the sum $F(1 + \tilde{\Phi}^{(1)})$ we arrive at the following equation for the distribution-function increment that does not depend on the angles in $\mathbf{c}_\perp$-space:

$$S[F, F\tilde{\Phi}] = F\left\{c_z\left(\frac{mc^2}{2T} - \frac{5}{2}\right)\frac{\partial \ln T}{\partial z} + \left(\frac{mc_z^2}{T} - \frac{mc^2}{3T}\right)\frac{\partial V_z}{\partial z} + \Delta\right\}, \qquad (A.5)$$

where

$$\Delta = \frac{m}{T}\left(\frac{c_\perp^2}{2} - \frac{c^2}{3}\right)\operatorname{div}\mathbf{V}_\perp + \nabla_\perp\left(\frac{mc_\perp^2}{2T}\mathbf{a}F\right) + e\mathbf{E}_\perp^* \frac{\partial}{\partial(c_\perp^2/2)}\left(\frac{c_\perp^2}{2}\mathbf{a}_\perp F\right) - \nabla_\perp V_z \frac{\partial}{\partial c_z}\left(F\frac{mc_\perp^2}{2T}\mathbf{a}\right). \qquad (A.6)$$

The expression for Δ contains terms of the first order ($\sim \operatorname{div}\mathbf{V}_\perp$) and of the second; the latter are connected with the vector $\mathbf{a}$, which, according to (A.3) is expressed in terms of $\mathbf{q}$ and π. According to this attribute, equation (A.5) can be represented as a set of two equations, for $\tilde{\Phi}^{(1)}$ and $\tilde{\Phi}^{(2)}$ (the number of the approximation corresponds to the scheme adopted in [2]). The first term of the right side should be assigned here to the first approximation, and all others to the second. On the other hand, substituting in (A.5) the explicit forms of $q_\perp$ and $\pi_{z\perp}$ recognizing that according to [2]

$$V_\perp = \frac{1}{m\omega_B}\left[\mathbf{e}_z, \frac{\nabla p}{n} - e\mathbf{E}\right], \qquad (A.7)$$

and taking into consideration the conditions of Item 2 of Sec. 2, we get

$$\Delta = 0. \qquad (A.8)$$

We have shown by the same token that the "excess" first-approximation terms of the scheme adopted in [2] actually are cancelled by the second-approximation terms that were not accounted for there.

[1] A. A. Galeev, S. S. Moiseev, and R. Z. Sagdeev, Atomnaya énergiya **15**, 451 (1963).

[2] S. I. Braginskiĭ, JETP **33**, 459 (1957), Soviet Phys. JETP **6**, 358 (1958).

[3] S. I. Braginskiĭ, Voprosy teorii plazmy (Problems in Plasma Theory) 1, 183 (1963).

[4] S. I. Braginskiĭ, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problems of Controlled Thermonuclear Reactions), v. 1, AN SSSR, 1958.

[5] L. D. Landau, JETP 7, 203 (1937).
[6] A. B. Mikhaĭlovskiĭ, op. cit.[3], 3, 141 (1963).
[7] S. Chapman and T. G. Cowling, The Mathematical Theory of Nonuniform Gases, Cambridge, 1939.
[8] E. S. Fradkin, JETP 32, 1176 (1957), Soviet Phys. JETP 5, 956 (1957).
[9] R. Herdan and B. S. Lilly, Revs. Modern Phys. 32, 731 (1960).

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