POSSIBILITY OF FINITENESS OF ORDINARY QUANTUM ELECTRODYNAMICS

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The photon wave-function constant Z_3 is calculated under the assumption of absence of infinities in quantum electrodynamics. The result is that Z_3^{-1} diverges as the first power of the logarithm of Λ (ultraviolet cut-off parameter). The result is discussed within the framework of perturbation theory.

1. In our previous paper [1] we have developed a scheme for the solution of the asymptotic Schwinger-Dyson equations for the unrenormalized electron propagator $S(p)$, on the assumption that

$$\lim_{k^2\to\infty} D(k^2) = k^{-2}, \tag{1}$$

where $D(k^2)$ is the unrenormalized invariant photon propagator. A finite nontrivial solution for $S(p)$ was obtained, which behaves like $1/\gamma p$ for large p under the condition that the bare electron mass vanishes. In order that the hypothesis (1) be self consistent, the photon wave function renormalization constant, calculated under the conditions of the same hypothesis, must be finite. If Eq. (1) is valid, then one may set $D(k^2) = k^{-2}$ in calculating the main contributions to Z_3 (i.e., that part of Z_3 which diverges in perturbation theory).

We have carried out such calculations and obtained the following result: if

$$D(k^2) = k^{-2}, \tag{2}$$

then

$$Z_3^{-1} - 1 = f(\alpha_0) \int \frac{dp^2}{p^2} + \text{finite terms}, \tag{3}$$

where $f(\alpha_0)$ is a function of the unrenormalized fine-structure constant. In other words, we find that if $D(k^2) = k^{-2}$, then Z_3^{-1} diverges as the first power of the logarithm of the ultraviolet cut-off parameter. Consequently, if a nonvanishing value of α_0 exists, for which $f(\alpha_0) = 0$, then the quantity Z_3^{-1} should be finite for this value of α_0. This would mean that hypothesis (1) is self consistent.[4] In that case the unrenormalized equations of quantum electrodynamics have solutions, which in general contain no infinities[5]. All conventional infinities would in that case be simply the result of an improper use of perturbation theory.

The analysis carried out by us shows that the function $f(\alpha_0)$ may be expressed in the form

$$f(\alpha_0) = \frac{\alpha_0}{2\pi}\left[\frac{2}{3} + g(\alpha_0)\right], \tag{4}$$

where

$$g(\alpha_0) = \frac{f_1(\alpha_0) + 2f_2(\alpha_0) + f_2^2(\alpha_0)}{1 - f_1(\alpha_0)} + f_3(\alpha_0). \tag{5}$$

The functions $f_1(\alpha_0)$, $f_2(\alpha_0)$ and $f_3(\alpha_0)$ are expressed here in terms of the asymptotic form of the kernel of the Bethe-Salpeter equation

$$K^a = K^a(p + k/2,\ p - k/2;\ p' + k/2,\ p' - k/2)$$

for electron-positron scattering with the help of the following equations

$$f_1(\alpha_0) = \frac{1}{48}\int \frac{d^4p'}{(2\pi)^4} \times \operatorname{Sp}\frac{(\gamma_\alpha(\gamma p')\gamma_\mu - \gamma_\mu(\gamma p')\gamma_\alpha)}{2p'^4} K^a(p', p)\,(\gamma_\mu(\gamma p)\gamma_\alpha - \gamma_\alpha(\gamma p)\gamma_\mu), \tag{6}$$

$$f_2(\alpha_0) = \frac{1}{48}\int \frac{d^4p'}{(2\pi)^4}\operatorname{Sp}\frac{1}{(\gamma p')}\gamma_\mu\frac{1}{(\gamma p')}K_\alpha{}^a(p', p)\,(\gamma_\mu(\gamma p)\gamma_\alpha - \gamma_\alpha(\gamma p)\gamma_\mu), \tag{7}$$

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[4]In practice, to obtain self-consistency one also needs the condition $f'(\alpha_0) < 0$.

[5]Making use of general arguments of the renormalization group, Gell-Mann and Low [2] arrive at analogous conclusions on the existence of an equation for the proper value of α_0. However, our explicit result (Eq. (3)) for $Z_3^{-1} - 1$ shows directly how a theory may arise which contains no divergences.

$$f_3(\alpha_0) = \frac{1}{48}\int \frac{d^4p'}{(2\pi)^4} \mathrm{Sp} \frac{1}{(\gamma p')}\gamma_\mu \frac{1}{(\gamma p')} \times K_{\alpha\alpha}{}^a(p',p)(\gamma p)\gamma_\mu(\gamma p), \quad (8)$$

where

$$K^a(p',p) \equiv K^a|_{k=0}, \quad (9)$$

$$K_\alpha{}^a(p',p) \equiv \frac{\partial}{\partial k_\alpha}K^a|_{k=0}, \quad (10)$$

$$K_{\alpha\beta}{}^a(p',p) \equiv \frac{\partial^2}{\partial k_\alpha \partial k_\beta}K^a|_{k=0}. \quad (11)$$

The symptotic value of the kernel of the Bethe-Salpeter equation K^a may be obtained from its exact value by replacing all propagators S(p) of internal electron lines by $1/(\gamma p)$, and all propagators $D_{\mu\nu}$ of internal photon lines by $D^0_{\mu\nu}$, where

$$D_{\mu\nu}{}^0 = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + b\frac{k_\mu k_\nu}{k^4}, \quad (12)$$

with the gauge constant b fixed by the requirement that the function Γ be finite.[6)] The convergence of the integrals (6), (7), and (8) is the essential point in the proof of Eq. (6). We have shown that the integrals indeed converge in the case when K^a may be expressed in the form of a power series in α_0 up to an arbitrary order of perturbation theory.

Thus the generality of our proof of Eq. (3) is limited by considerations of K^a in perturbation theory.

The Feynman graphs corresponding to some of the first terms in the perturbation-theory series for K are shown in Fig. 1.

The term $2/3$ in the brackets of Eq. (4) corresponds to the weak coupling limit for the quantity Z_3^{-1} (the Landau approximation)[3]. Our results show that if $D(k^2)$ is taken equal to k^{-2}, then in the exact asymptotic form for Z_3^{-1} there appear no higher powers of ln Λ. Moreover, the coefficient of the logarithm of Λ is obtained by replacing $2/3$ by $2/3 + g(\alpha_0)$. The requirement of internal consistency for the theory

$$2/3 + g(\alpha_0) = 0, \quad (13)$$

means, that the sum of all the terms of higher order precisely cancels the effect obtained in the first approximation in the coupling constant. We can calculate $g(\alpha_0)$ with the help of Eq. (5) and Eqs. (6)—(8) for f_1, f_2, and f_3.

If lowest order perturbation theory is used for calculation of K^a, then expression (5) for $g(\alpha_0)$ amounts to a sum of the contributions to Z_3^{-1} due

6)We have obtained an explicit formula expressing b in terms of $K^a(0, p')$.

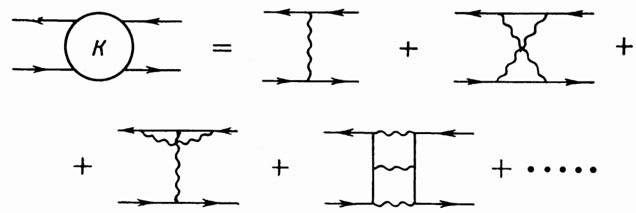

FIG. 1. Perturbation-theory diagrams for K.

to "uncrossed" ladder diagrams. In that case we find $f_1 = \alpha_0/2\pi$ and $f_2 = f_3 = 0$. It follows hence that

$$g(\alpha_0) = \frac{\alpha_0/2\pi}{1-\alpha_0/2\pi},$$

and that no positive α_0 exists for which Eq. (13) is satisfied.

Let us calculate now higher order corrections to f_1, f_2, and f_3. Since we are making use here of perturbation theory the results are valid only for small α_0. In that region, generally speaking, it should be true that $|g(\alpha_0)| \ll 2/3$. However it may turn out that the sum of diagrams of a certain class may give a large value of $g(\alpha_0)$ for small α_0. But if that is not the case, then for the study of Eq. (13), in particular for the study of the question whether it has at all a solution, it is essential to know how to calculate $g(\alpha_0)$ outside the framework of the perturbation theory.

In conclusion of this section let us emphasize that the calculation of the basic function of quantum electrodynamics $g(\alpha_0)$ has not yet been performed. We hope that we have made it sufficiently clear how important it is to obtain this function.

Below, in Sec. 2, we shall study some ideas which lead to our main result—Eq. (3)—within the framework of perturbation theory. A detailed derivation of Eqs. (3)—(8) will be given in a separate article.

2. The quantity Z_3^{-1} is defined as follows in terms of the photon operator $\Pi_{\mu\nu}(k)$:

$$Z_3^{-1} = 1 + \frac{1}{24}\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\alpha}\Pi_{\mu\mu}(k)|_{k=0}, \quad (14)$$

where

$$\Pi_{\mu\nu}(k) = -ie_0^2\int\frac{d^4p}{(2\pi)^4} \times \mathrm{Sp}\,\gamma_\mu S\left(p+\frac{k}{2}\right)\Gamma_\nu\left(p+\frac{k}{2}, p-\frac{k}{2}\right)S\left(p-\frac{k}{2}\right). \quad (15)$$

Let us rewrite Eq. (15) in a Euclidean system of coordinates,

$$\int d^4p f(p) = i\int p^3 dp d\Omega f(p) \equiv i\int p^3 dp 2\pi^2 \langle f(p)\rangle_p.$$

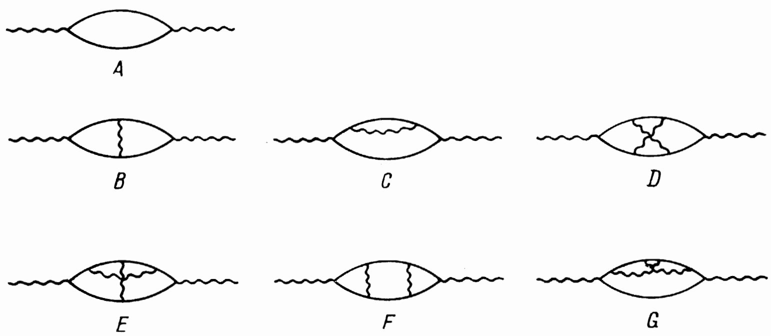

FIG. 2. Perturbation-theory diagrams for $\Pi_{\mu\nu}(k)$.

Then making use of Eq. (14), we obtain the following expression for Z_3^{-1}:

$$Z_3^{-1} = 1 + \int_0^\infty dp^2 p^2 \sigma(p^2), \qquad (16)$$

where

$$\sigma(p^2) = \frac{\alpha_0}{96\pi} \mathrm{Sp} \left\langle \frac{\partial}{\partial k_\alpha} \frac{\partial}{\partial k_\alpha} \gamma_\mu S\left(p + \frac{k}{2}\right) \times \Gamma_\nu \left(p + \frac{k}{2}, p - \frac{k}{2}\right) S\left(p - \frac{k}{2}\right) \right\rangle_{p_k=0}. \qquad (17)$$

We consider a theory in which $m_0 = 0$. This means that the complete electron propagator $S(p)$ will be finite for an appropriate choice of gauge. The complete electron propagator $S(p)$ differs from $1/(\gamma p)$ by terms which depend on the physical electron mass m; these terms give a finite contribution to Z_3^{-1}. Below, in the discussion of Z_3^{-1} on the basis of perturbation theory, we neglect the mass dependence of all such internal electron propagators, and therefore in the end the integrals automatically give the asymptotic form for $p^2\sigma(p^2)$.

Certain characteristic perturbation-theory graphs for $\Pi_{\mu\nu}(k)$ are shown in Fig. 2. Since $D = 1/k^2$ and the photon propagator $D^0_{\mu\nu}$, appearing in these diagrams, is defined by Eq. (12), it follows that $\Pi_{\mu\nu}(k)$ does not depend on the value of the constant b. We shall choose b so that Γ_μ is finite. This will substantially simplify the calculations.

The diagram in Fig. 3A does not enter into $\Pi_{\mu\nu}(k)$, since it corresponds to the photon self-energy correction to the internal photon propagator. Should Z_3^{-1} turn out to be a finite quantity, then the use of the exact Green's function (Fig. 3B) for the internal photon line in this diagram makes no contribution to the asymptotic value of $p^2\sigma(p^2)$. Therefore in the calculation of $\Pi_{\mu\nu}(k)$ one should not take into account the contribution from the graphs of the form Fig. 3A.

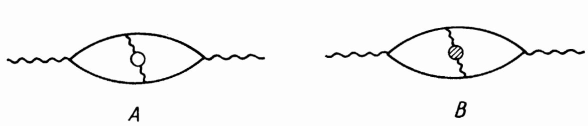

FIG. 3. Photon self-energy corrections to $\Pi_{\mu\nu}(k)$.

We see from Eqs. (16) and (17) that to calculate Z_3^{-1} or $\sigma(p^2)$, all internal lines with photon momentum k, should be twice differentiated. In Fig. 4 are given certain contributions to Z_3^{-1} arising in differentiation of the diagrams of Fig. 2. The line with one (two) strokes represents the electron propagator differentiated once (twice). In Fig. 4A is shown the result of conventional perturbation theory in lowest order:

$$\lim_{p^2\to\infty} p^2\sigma(p^2) = \frac{\alpha_0}{3\pi}\frac{1}{p^2}. \qquad (18)$$

In order to calculate $p^2\sigma(p^2)$ to order α_0^2, we take $b = 0$ (the Landau gauge)[7]. The graph $4B_1$ gives a contribution of the form

$$\frac{1}{p}\left\langle \int \frac{d^4p'}{(2\pi)^4} \frac{1}{(p-p')^2} \frac{1}{p'^3} \right\rangle \qquad (19)$$

to the quantity $p^2\sigma(p^2)$. The quantity (19) is the vertex diagram differentiated once; it is written out with only the powers of p and p' taken into account since common factors, spinor and vector indices are inessential for our purposes. It is clearly seen that the Eq. (19) has neither infrared divergences (in the region $p' \to 0$), nor ultraviolet ones (in the region $p' \to \infty$). It then follows from dimensional considerations (from expression (19)) that

$$p^2\sigma(p^2) = \mathrm{const}\cdot p^{-2}. \qquad (20)$$

The graph $4B_2$ gives an integral of the form

$$\frac{1}{p^2}\int \frac{d^4p'}{(p-p')^2}\frac{1}{p'^2} \qquad (21)$$

for $p^2\sigma(p^2)$. The integral (21) is a correction to the vertex, is finite in the ultraviolet region in the

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[7]The choice $b = 0$ corresponds to the fact that in the Landau gauge the correction to Γ_μ in lowest order is given by a convergent integral.

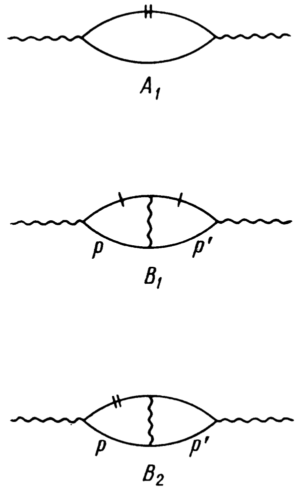

FIG. 4. Second- and fourth-order diagrams for Z_3^{-1}.

Landau gauge and, as can be seen, finite in the infrared region in an arbitrary gauge. It follows therefore from dimensional considerations that in the Landau gauge the integral (21), like the graph $4B_1$, also has the required form.[8]

The integral over the internal photon line in the graph of Fig. 2C converges in the Landau gauge when $m_0 = 0$. From dimensional considerations it follows that in that case the final correction to the internal electron propagator has the form const $\cdot (\gamma p)^{-1}$. It follows hence that the contribution of the diagram Fig. 2C is proportional to the contribution from the diagram of Fig. 2A and when differentiated gives for the quantity $p^2\sigma(p^2)$ an expression of the desired form (20).

Thus, in the Landau gauge, the separate contributions of the diagrams B_1, B_2, and C have the desired form (20). In any other gauge the contribution of each of the diagrams B_2 and C have divergences of higher order, which must cancel each other as a consequence of the gauge invariance of the theory.

In precisely the same way in order to simplify the sixth order calculation of Z_3^{-1}, we choose the gauge

$$b = b^{(2)} = \frac{3}{2}\frac{\alpha_0}{4\pi},$$

[8]As a consequence of the Ward identity, this contribution to $p^2\sigma(p^2)$ exactly cancels the contribution to the diagram 2C. The contribution to the divergent part of Z_3^{-1} of order α_0^2 is due, consequently, entirely to the diagrams $4B_1$. A simple calculation of the contribution to second order gives the Yost-Luttinger result.

$$\lim_{p^2\to\infty} p^2\sigma(p^2) = \frac{1}{p^2}\left(\frac{\alpha_0}{2\pi}\right)^2.$$

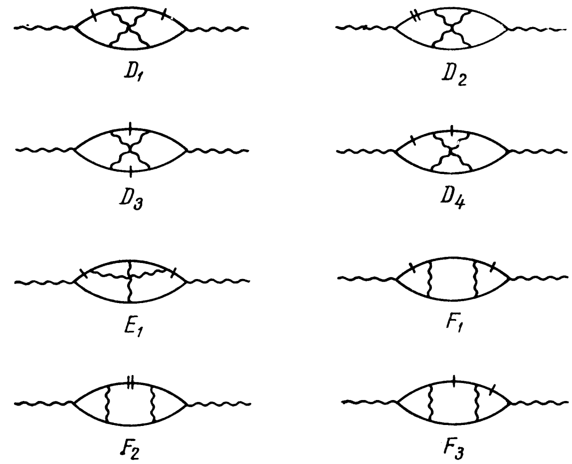

FIG. 5. Sixth-order diagrams for Z_3^{-1}.

in which, as can be shown, Γ_μ is finite. Then contributions of sixth order to $\Pi_{\mu\nu}(k)$ arise not only from graphs of the type D, E. F, and G of Fig. 2, calculated in the Landau gauge, but also from gauge corrections to B and C, calculated for a photon propagator taken equal to

$$\frac{3}{2}\frac{\alpha_0}{4\pi}\frac{k_\mu k_\nu}{k^4}.$$

Some of the sixth-order contributions resulting from differentiation of the diagrams D, E, and F are shown in Fig. 5. Counting the powers of the momentum in the integrands for $\sigma(p^2)$ from the contributions of the graphs D_1, D_3, D_4, and F_1 we find that these integrals (similar to the integral (19) corresponding to the graph B_1) contain neither ultraviolet nor infrared divergences. It follows therefore from dimensional considerations that they will also contribute to $p^2\sigma(p^2)$ in the form const $\cdot\, p^{-2}$.

The same considerations apply to the graph E_1, if we take into account the fact that the second-order vertex parts contained in them are finite in the Landau gauge. The graph D_2 contains a divergent vertex part of fourth order. However, the sum of the graph D_2 and the gauge correction to the graph B_2 have a finite vertex correction. This sum gives a contribution to $p^2\sigma(p^2)$ of the desired form (20). In precisely the same way the gauge correction to the graph C cancels the infinite electron self-energy part, contained in the graph G.

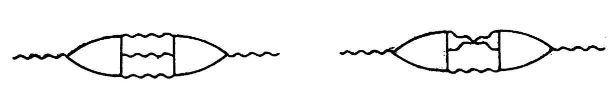

FIG. 6. Eighth-order diagrams with three photon lines for $\Pi_{\mu\nu}(k)$.

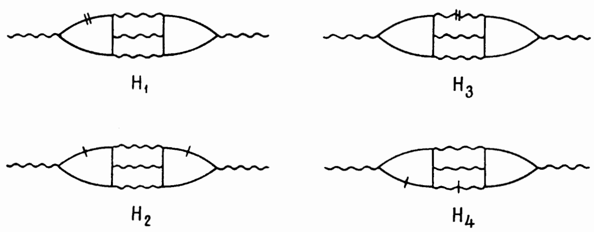

FIG. 7. Eighth-order diagrams with three photon lines for Z_3^{-1}.

Let us summarize the results of perturbation theory. In order to calculate $p^2\sigma(p^2)$ with an accuracy to order 2n, we choose a gauge in which Γ_μ is finite up to $2(n-1)$st order inclusive. In our expressions for $p^2\sigma(p^2)$ are contained contributions from the graphs of type B_1, D_1, and D_3 which have no nondifferentiable vertex or self-energy parts. With the help of arguments based on simple counting of powers, it can be shown that the contributions of these graphs have no divergences. Therefore they give a contribution to $p^2\sigma(p^2)$ of the form const $\cdot\, p^{-2}$. The diagrams of the type B_2, D_2, and F_2, containing nondifferentiable vertex parts, are cancelled by a choice of gauge.

However, the above mentioned arguments, based on simple counting of powers, are not sufficiently general. It can be shown that a class of diagrams containing three photon intermediate states is an exception (and the only exception), i.e., it gives rise to divergences in $p^2\sigma(p^2)$. It can be shown, however, that even these diagrams give a contribution of the form Eq. (20) if one makes use of gauge invariance considerations.

Let us consider the simplest example of such a case—the graphs H and I, Fig. 6. Certain characteristic contributions to Z_3^{-1}, arising from differentiation of the graph H, are shown in Fig. 7. To these graphs we should add the corresponding contributions obtained from differentiation of the graph I. By simple counting of powers it is easy to see that the graph H also gives a contribution of the form (20) to $p^2\sigma(p^2)$.

However, such simple arguments cannot be used for the analysis of the graphs H_1 and H_3. The integral of the twice differentiated photon line in the graph H_3, for example, is manifestly divergent in the infrared region. However, each of the graphs H_1, H_3, and H_4 contains insertions of the type of a photon-photon scattering amplitude, where one of the photons (external) has zero momentum. It follows from gauge invariance that such an amplitude should vanish. From here it follows that if the photon-photon scattering amplitude appearing in the graphs H_1, H_3, and H_4 is calculated with gauge invariance taken into account, then the contributions of these graphs should vanish[9)].

Let us note the correspondence between our discussion of perturbation theory and the functions f_1, f_2, and f_3 (which describe the exact behavior of $p^2\sigma(p^2)$ for large p^2). The graphs B_1, D_1, E_2, and H_2 contain the nondifferentiated Bethe-Salpeter kernel and thus contribute to f_1 (see Eq. (6)). The graphs D_4 (D_3) contain the kernel differentiated once (twice), and thus give contributions to f_2 (f_3) (see Eqs. (7) and (8)). The contribution of the diagram of type F_1, which contains the iteration of the nondifferentiated kernel, enters into $g(\alpha_0)$ through f_1 in the denominator of Eq. (5). As a consequence of the Ward identity, the graphs B_2, D_2, F_2 and F_3, which contain the nondifferentiated external vertex corrections, give contributions to $g(\alpha_0)$ which are exactly cancelled by the graphs containing the electron self-energy parts. Therefore all electron propagators and two external vertices enter into the equations for f_1, f_2, and f_3 without corrections. Moreover, the arguments based on simple counting of powers, mentioned before in the discussion of the perturbation theory diagrams, serve as a basis for the proof that the integrals defining f_1, f_2, and f_3 converge.

In conclusion we note that the function $g'(\alpha_0)$ is related to the asymptotic behavior of the Bethe-Salpeter kernel for scattering of light by light. This circumstance may turn out to be useful for devising methods for calculation of $g(\alpha_0)$. However, we have not studied this possibility in detail as yet.

In order to understand whether the nonrenormalized quantum electrodynamics is or is not a finite theory, it is necessary to determine the function $g(\alpha_0)$ with sufficient accuracy. If it should turn out that Eq. (13) has some nonvanishing α_0 as a solution, then quantum electrodynamics can be thought of as a fully internally consistent theory.

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9)For example, the use of the correct gauge-invariant current gives rise to the appearance of subtraction terms in the photon-photon scattering amplitude, which ensure its vanishing for $k = 0$.

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