IONIZATION OF ATOMS BY AN ELECTROMAGNETIC-WAVE FIELD

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The exact matrix element of the ionization of atoms and systems coupled by short-range and Coulomb forces is represented in a compact form that is convenient for the study of the role of Coulomb forces. The calculation is made for multiquantum ionization, when the matrix element reduces to the product of two factors. The most significant of these has the form of the matrix element of the ionization of a system coupled by short-range forces [1] and depends on the Coulomb forces only via the binding energy I. The other factor takes into account the specific influence of the Coulomb forces. When large distances are effective and $\alpha Z/v \lesssim 1$ (these conditions are satisfied in the case of ionization by means of a circularly polarized wave), the Coulomb factor takes the form of an exponential of the first Coulomb correction, and increases the ionization probability by an amount $\sim (B_0\xi/B\sqrt{1+\xi^2})^{2\nu}$, where B/B_0 is the field amplitude in atomic units, $\xi = 2IB/\omega B_0$, and $\nu = \alpha Zm/\sqrt{2mI}$ (in a purely Coulomb field, i.e., for atoms, ν coincides with the principal quantum number). For ionization by a linearly polarized wave, the effective values are $\alpha Z/v \sim \sqrt{I/\omega} \gg 1$, making a quantitative calculation difficult. A qualitative estimate shows that in this case the Coulomb forces can appreciably increase the ionization, by an amount $\sim \exp(c\sqrt{I/\omega})$, $c \sim 1$. The elimination of the infrared (Coulomb) divergence by a constant field is discussed in passing.

1. INTRODUCTION AND EXPRESSION FOR THE MATRIX ELEMENT

We consider in this paper multiquantum ionization of atoms and systems coupled by short-range and Coulomb forces and by the field of an electromagnetic wave, i.e., ionization that results from the absorption of a large number of photons. The total ionization probability depends on the field amplitude B and the frequency ω in terms of the dimensionless parameters[1)] $\xi = eB/\eta\omega$ and B/B_0, where $\eta = \sqrt{2mI}$ is the characteristic momentum in the coupled state, I is the ionization energy, and $B_0 = \eta^3/em$ is the atomic field intensity. The ionization matrix element is expressed in terms of a product of a universal factor, in the form of the ionization matrix element of a system coupled by short-range forces, and a "form factor" due to the Coulomb forces.

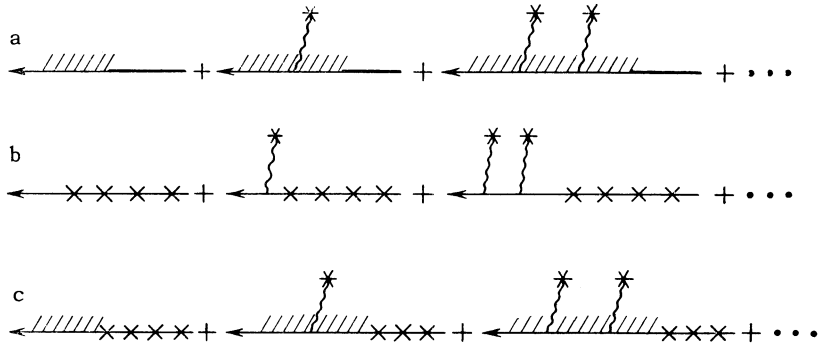
Such a representation is convenient for the analysis of multiquantum ionization, or more accurately when the parameters ξ and B/B_0 satisfy the condition

$$B_0\xi / B\sqrt{1+\xi^2} \equiv 2I/\omega\sqrt{1+\xi^2} \gg 1, \qquad (1)$$

which when $\xi \lesssim 1$ is equivalent to the multiquantum condition, and when $\xi \gg 1$ is equivalent to the weakness of the field compared with the field inside the atom. In this case the first factor determines the most essential features of the process. It was investigated in detail earlier in a paper by the authors [1] [2)]. A characteristic feature of this factor is that it depends on the short-range and Coulomb forces only via the coupling energy I. On the other hand, the Coulomb factor depends on the Coulomb forces via I and the Coulomb parameter $\alpha Z/v$ of the initial and final states, i.e., via $\nu = \alpha Zm/\eta$ and $\kappa = \alpha Zm/p$, where η and p are the momenta of the bound and ionized electron. For systems coupled by short-range and Coulomb forces, ν is determined by the transcendental equation (11). In the particular case of systems coupled only by short-range forces we have $\nu = \kappa = 0$, and we obtain the theory developed in [1] (see also [2]). To the contrary, in the case of pure

[1)] We use the units $\hbar = c = 1$, $e^2/4\pi = \alpha = 1/137$, and the notation of [1].

[2)] We note that in [1] it was necessary to supplement the condition $(\eta r_0)^2 B\sqrt{1+\xi^2}/B_0\xi \ll 1$, which spells out the independence of the ionization of the structure of the short-range potential, by the condition (1), since even when $r_0 \to 0$ a potential having a shallow level changes the wave function of the final state at distances $\lesssim 1/\eta$.



Coulomb systems, i.e., atoms, ν is equal to the principal quantum number: $\nu = n = 1, 2, 3, \ldots$

When condition (1) is satisfied, an important role is played in the ionization by large distances[1]

$$\eta r_{\text{eff}} \sim B_0\xi / B\sqrt{1+\xi^2} \gg 1,$$

and if the characteristic Coulomb parameter $\alpha Z/v$ for the initial and final states is order of unity or less, then the calculation of the Coulomb "form factor" can be reduced to the evaluation of an exponential function whose argument is the first Coulomb correction.

The exact matrix element describing the ionization of the atom by the field of the electromagnetic wave is [3)]

$$M_{pn} = \int \psi_p^{-*}(x) V(x) \psi_n(x) d^4x, \qquad (2)$$

where $\psi_n(x)$ is the wave function of the electron in the n-th state of the atom, $V(x)$ is the operator of interaction between the electron and the wave, and $\psi_p^-(x)$ is the wave function of the electron in the field of the wave and in the Coulomb field (the "minus" sign designates the state in which we have at infinity, when the field of the wave is turned off, a plane wave propagating in the p direction and a converging spherical wave (see[3], Sec. 134)).

We shall henceforth construct the function ψ_p^{-*} using only perturbation theory in terms of the Coulomb field. In such a case, the matrix element M_{pn} is the sum of the diagrams shown in the figure, case a. The thick line represents the bound state of the electron in the atom, the hatched line the state in the wave field, and the wavy line the interaction with the Coulomb field.

Besides the matrix element M_{pn}, we shall consider an auxiliary matrix element $M_{p\nu}$ which differs from M_{pn} in that the bound state $\psi_n(x)$ in the Coulomb field is replaced by a bound state $\psi_\nu(x)$ in the Coulomb field and in the field of the short-range forces of zero radius[4)]. As shown in Sec 2, the function $\psi_\nu(x)$ can be constructed by summing the diagrams of the perturbation theory with respect to the Coulomb field, in which the zeroth approximation (line with crosses in the figure, case b) is chosen to be the function $\psi_\nu^0(x)$, which has for the s-state the form

$$\psi_\nu^0(x) = Nr^{-1} \exp(-\eta r - i\varepsilon_\nu t). \qquad (3)$$

This function coincides in form with the state in the short-range zero-radius potential, but its parameters ϵ_ν, η, and N are determined by the states in the Coulomb field and in the field of the short-range zero-radius forces.

When the energy of the state $\psi_\nu(x)$ tends to the energy of the state in the purely Coulomb field (i.e., when $\epsilon_\nu \to \epsilon_n$), the wave function $\psi_\nu(x)$, as will be shown later, tends to $\psi_n(x)$, so that $M_{pn} = \lim M_{p\nu}$, $\nu \to n$. It is obvious that $M_{p\nu}$ can be represented by the sum of the diagrams c in the figure, in which the Coulomb field is treated as a perturbation. With the aid of these diagrams we can write $M_{p\nu}$ in the form

$$M_{p\nu} = \int \psi_p^{-*}(x)(\Box - m^2)\psi_\nu^0(x) d^4x =$$

$$-4\pi N \int_{-\infty}^{\infty} \psi_p^{-*}(\mathbf{x}=0, t) e^{-i\varepsilon_\nu t} dt, \qquad (4)$$

in which the entire interaction with the Coulomb field is concentrated in the function $\psi_p^-(x)$. In the last equality in (4) we use the relation

$$(\Box - m^2)\psi_\nu^0(x) = -4\pi N \delta(\mathbf{x}) e^{-i\varepsilon_\nu t}.$$

It is easy to verify that (4) actually contains the photoeffect in any order of perturbation theory with respect to the wave. Thus, using in (4) as the first-order approximation

[3)] Although we neglect relativistic effects, it is convenient to use a relativistic form of the matrix elements and a relativistic normalization of the wave functions.

[4)] $M_{p\nu}$ would be the exact matrix element for the ionization of the bound state in the field of the Coulomb and short-range forces if the final function ψ_p^- were to take into account the influence of the short-range forces. When condition (1) is satisfied, this influence can be neglected.

$$\psi_p^{-*}(x) = \psi_{p0}^{-*}(x) + \int d^4x' \psi_{p0}^{-*}(x') V(x') K(x', x),$$

where ψ_{p0}^{-} is the wave function with momentum p in the Coulomb field, V the interaction with the wave, and K(x', x) is the propagation function in the Coulomb field, which is equal to

$$K(x', x) = -i \sum_n \psi_n(\mathbf{x}') \psi_n^*(\mathbf{x}) \exp[-i\varepsilon_n(t' - t)], \quad t' > t,$$
$$K(x', x) = 0, \quad t' < t,$$

and integrating with respect to t, we obtain in the limit as $\epsilon_\nu \to \epsilon_n$

$$M_{pn} = \int d^4x' \psi_{p0}^{-*}(x') V(x') \psi_n(x'),$$

since $4\pi N \psi_n^*(0)(\epsilon_\nu - \epsilon_n)^{-1} \to 1$ (see Sec. 2).

2. BOUND STATE IN THE FIELD OF SHORT-RANGE AND COULOMB FORCES. METHOD OF OBTAINING THE EXPONENTIAL

Let us verify that the wave function $\psi_\nu(x)$ obtained by summing the diagrams b of the figure describes the state in the Coulomb field and in the field of the short-range zero-radius forces. It is easy to see that the summation of these diagrams leads to an integral equation, which takes for the s-states the form

$$\psi_\nu(r) = \psi_\nu^0(r) + \frac{\alpha Zm}{\eta r} \int_0^\infty dr' [e^{-\eta|r-r'|} - e^{-\eta(r+r')}] \psi_\nu(r'). \quad (5)$$

Equation (5) corresponds to the differential equation

$$\left(\frac{\partial^2}{\partial \mathbf{x}^2} - \eta^2 + \frac{2\alpha Zm}{r}\right) \psi_\nu(\mathbf{x}) = -4\pi N \delta(\mathbf{x}), \quad (6)$$

which differs from the Schrödinger equation for the electron in the Coulomb field in the presence of a δ-function in the right hand side. Solution corresponding to the bound state should decrease exponentially at infinity. It follows from this uniquely that it is expressed in terms of the Whittaker function $W_{\nu, 1/2}(2\rho)$ or the confluent-geometric function $\Psi(1 - \nu, 2, 2\rho)$[4] [5]:

$$\psi_\nu(\mathbf{x}) = CW_{\nu, 1/2}(2\rho)/2\rho = Ce^{-\rho}\Psi(1 - \nu, 2, 2\rho), \quad \nu = \alpha Zm/\eta, \quad \rho = \eta r \quad (7)$$

and has the following asymptotic properties:

$$\psi_\nu(\mathbf{x}) = \frac{C}{2\Gamma(1-\nu)\rho} \times \left\{1 - 2\nu\rho\left[\ln 2\gamma^2\rho + \psi(1-\nu) - 1 + \frac{1}{2\nu}\right] + \ldots\right\}, \quad \rho \to 0, \quad (7')$$

$$\psi_\nu(\mathbf{x}) = C(2\rho)^{\nu-1} e^{-\rho}\left\{1 + \frac{(1-\nu)\nu}{\rho} + \ldots\right\}, \quad \rho \to \infty; \quad (7'')$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ-function, $\ln \gamma = 0.577\ldots$ is Euler's constant. Owing to the r^{-1} singularity as $r \to 0$ and the known relation $\Delta r^{-1} = -4\pi\delta(\mathbf{x})$, the constant C is uniquely related to the "bare" constant N:

$$C = 2\Gamma(1 - \nu)\eta N. \quad (8)$$

Equation (6) is inhomogeneous and its solution (7) is valid for any value of η. In order to find the short-range potential and the spectrum of the eigenvalues η, it is necessary to transform (6) into a Schrodinger equation, i.e., represent its right-hand side in the form $2mV(\mathbf{x})\psi_\nu(\mathbf{x})$. This can be done by choosing

$$V(\mathbf{x}) = \frac{2\pi}{m\eta_0}\delta(\mathbf{x})\left(1 + r\frac{\partial}{\partial r} + 2\alpha Zmr \ln 2\gamma^2\alpha Zmr\right), \quad (9)$$

where the parameter η_0 characterizes the short-range potential[6].

Using the asymptotic expression from the function $\psi_\nu(\mathbf{x})$ as $\eta r \to 0$, we easily find that

$$2mV(\mathbf{x})\psi_\nu(\mathbf{x}) = -4\pi N \frac{2\alpha Zm}{\eta_0}\left(\psi(1-\nu) + \frac{1}{2\nu} - \ln \nu\right)\delta(\mathbf{x}). \quad (10)$$

Therefore, if we stipulate that the parameter ν must satisfy the condition

$$\psi(1-\nu) + \frac{1}{2\nu} - \ln \nu \equiv \pi \operatorname{ctg} \pi\nu + \psi(\nu) - \ln \nu + \frac{1}{2\nu} = \frac{\eta_0}{2\alpha Zm}, \quad (11)$$

then the right side of (10) is transformed into $-4\pi N\delta(\mathbf{x})$, and consequently for these values of ν the function (7) will be a solution of the Schrödinger equation for a particle in a Coulomb field and a short-range potential (9) of zero radius.

The roots of (11) are shifted away from integer values, to the right or to the left ($\eta_0 \gtrless 0$), so that

$$\nu = \nu_n = \begin{cases} n + \Delta_n, & 0 < \Delta_n < 0{,}50, \quad n = 0, 1, 2, \ldots \text{ for } \eta_0 > 0 \\ n - \Delta_n, & 0 < \Delta_n < 0{,}53, \quad n = 1, 2, 3, \ldots \text{ for } \eta_0 < 0. \end{cases}$$

Thus, the energy spectrum $\epsilon_\nu = m - \eta^2/2m = m - \alpha^2 Z^2 m/2\nu^2$ constitutes a system of Coulomb levels, deformed upward or downward (depending on the sign of η_0) by a short-range potential, and to this system is added in the case of $\eta_0 > 0$ one more deepest level (ground state). When $|\eta_0/2\,\mathrm{Zm}| \gg 1$ all levels $\epsilon_{\nu n}$ become very close to the Coulomb levels:

[5] The function $\Psi(\alpha, \gamma, z)$ is connected with the function $G(\alpha, \beta, z)$ used in the book of Landau and Lifshitz [3] by the relation $\Psi(\alpha, \gamma, z) = z^{-\alpha} G(\alpha, \alpha - \gamma + 1, -z)$.

[6] When the Coulomb field is turned on, the potential (9) goes over into the Breit pseudopotential [5].

$$\varepsilon_{\nu_n} = m - \frac{\alpha^2 Z^2 m}{2n^2}\left(1 - 4\frac{\alpha Zm}{n\eta_0} + \ldots\right), \quad n = 1, 2, 3, \ldots, \tag{12}$$

except for the ground state level ϵ_{ν_0} in the case $\eta_0 > 0$, the energy of which is much lower than the energy of the ground state in the Coulomb field, and is equal to

$$\varepsilon_{\nu_0} = m - \frac{\eta_0^2}{2m}\left(1 - \frac{4\alpha Zm}{\eta_0}\ln\frac{\eta_0}{\gamma\alpha Zm} + \ldots\right). \tag{12'}$$

We note that for the states of the continuous spectrum condition (11) goes over into

$$\frac{\pi \operatorname{ctg}\delta_0}{e^{-2\pi\varkappa}-1} + \operatorname{Re}\psi(i\varkappa) - \ln\varkappa = \frac{\eta_0}{2\alpha Zm}, \quad \varkappa = \frac{\alpha Zm}{p},$$

where the same constant η_0 relates the momentum p to the additional phase shift δ_0 due to the short-range forces [6].

The constant C in (7) is obtained from the condition for the normalization of the wave function and is found to be[7)]

$$|C|^2 = \frac{\eta^3}{\pi m}\left[\int_0^\infty dz\, W^2_{\nu,1/2}(z)\right]^{-1} = \frac{\eta^3}{2\pi\Gamma^2(1+\nu)m} \times\left\{1 + \left(\frac{\sin\pi\nu}{\pi}\right)^2\left[\frac{2\nu-1}{2\nu^2} - \psi'(1+\nu)\right]\right\}^{-1}. \tag{13}$$

The quantity in the curly brackets lies between $\frac{1}{2}$ and 1 for all $\nu \geq 0$, and is equal to $\frac{1}{2}$ when $\nu = 0$ and is equal to unity when $\nu = 1, 2, 3, \ldots$ Thus, for all $\nu \geq 0$ the constant C differs from zero. Therefore, according to (8), the "bare" constant N is a function of ν, which vanishes when $\nu = 1, 2, 3, \ldots$ When ν tends to an integer number $n = 1, 2, 3, \ldots$ characteristic of the bound state in the purely Coulomb field, we have

$$\Psi(1-\nu, 2, 2\rho) \to (-1)^{1-n}\Gamma(1+n)\Phi(1-n, 2, 2\rho)$$

(see [4], formula 6.7 (7)), so that $\psi_\nu(\mathbf{x})$ tends to the usual solution of the Schrödinger equation in the Coulomb fields, which decreases exponentially at infinity and which is finite at zero:

$$\psi_\nu(\mathbf{x}) \to \psi_n(\mathbf{x}) = \sqrt{\eta^3/2\pi m}\, e^{-\eta r}\Phi(1-n, 2, 2\eta r), \quad \eta = \alpha Zm/n.$$

An important role is played in ionization by a weak field by large distances, $\eta r \gg 1$. In this region $\psi_\nu(\mathbf{x})$ is described by the formula (7″). From the point of view of the diagram technique, the correct dependence of the wave function on $\mathbf{x}$ at large distances can be obtained by taking the exponential of the first Coulomb correction to the function $\psi^0_\nu(\mathbf{x})$. Calculating with the aid of (5) the first Coulomb correction to $\psi^0_\nu(\mathbf{x})$, we get

$$\psi_\nu(x) = \psi_\nu^0(x)\{1 + \nu[\ln 2\gamma\eta r - e^{2\eta r}\mathrm{Ei}(-2\eta r)]\}. \tag{14}$$

When $\eta r \gg 1$, the term $e^{2\eta r}\,\mathrm{Ei}(-2\eta r) \approx -\frac{1}{2}\eta r$ and can be neglected. It can be shown that when $\eta r \gg 1$ the principal terms of the next higher approximations will be of the type $(\nu \ln 2\gamma\eta r)^n/n!$ Indeed, direct calculation of the second Coulomb correction with the aid of (5) yields for $\eta r \gg 1$

$$\psi_\nu(x) = \psi_\nu^0(x)\{1 + \nu\ln 2\gamma\eta r + \nu^2(\tfrac{1}{2}\ln^2 2\gamma\eta r + \pi^2/12)\}. \tag{15}$$

Summation of only the principal logarithmic terms is equivalent to taking the exponential of the first Coulomb correction so that

$$\psi_\nu(\mathbf{x}) = \psi_\nu^0(\mathbf{x})\exp(\nu\ln 2\gamma\eta r) = Nr^{-1}e^{-\eta r}(2\gamma\eta r)^\nu. \tag{16}$$

The function (16) reflects correctly the dependence on r for large ηr, namely for $\eta r \gg \nu(1-\nu)$, but differs from the exact asymptotic expression (7″) in the absence of the factor

$$\gamma^{-\nu}\Gamma(1-\nu), \tag{17}$$

which is equal to $1 + \pi^2\nu^2/12 + \ldots$ when $\nu \ll 1$ and differs noticeably from unity when $\nu \sim 1$. The difference between (16) and (7″) is connected with the fact that in the higher approximations taking the exponential is equivalent to taking into account only the principal terms $(\nu \ln 2\eta r)^n/n!$ and does not take into account the terms $\sim \nu^n$.

It is important in what follows to establish, in momentum space, a procedure corresponding to taking the exponential in coordinate space. Large distances in coordinate space correspond to the vicinity of a pole of the wave function in momentum space (for non-integer ν, it is more accurate to speak not of a pole but of a branch point). The Fourier transform of the exact function (7) in the vicinity of a pole, i.e., at $p^2 + \eta^2 \ll \eta^2$, is of the form

$$\varphi_\nu(\mathbf{p}) = \frac{4\pi N\Gamma(1-\nu)\Gamma(1+\nu)(2\eta)^{2\nu}}{(p^2+\eta^2)^{1+\nu}}. \tag{18}$$

It is obvious that the Fourier transform of the function (16), obtained by taking the exponential of the first Coulomb correction in x-space, differs from (18) in the vicinity of the pole in the absence of the factor $\gamma^{-\nu}\Gamma(1-\nu)$, see (17). If we express the first Coulomb correction in momentum space and take the exponential, then we obtain when $p^2 + \eta^2 \ll \eta^2$ an expression that differs from (18) in the absence of the factor $\Gamma(1-\nu)\Gamma(1+\nu)$. Thus, taking the exponential in momentum space does not correspond to taking the exponential in the coordinate space. In other words, taking the exponential and the Fourier transformation "do not commute."

To establish the procedure in momentum space corresponding to taking the exponential in coordi-

[7)] We obtained the integral of the square of the Whittaker function by expanding $W_{\nu,1/2}$ in Laguerre polynomials.

nate space, we turn to the integral equation for the exact function $\varphi_\nu(\mathbf{p})$:

$$(\mathbf{p}^2+\eta^2)\varphi_\nu(\mathbf{p})=(\mathbf{p}^2+\eta^2)\varphi_\nu^0(\mathbf{p})+\frac{\alpha Zm}{\pi^2}\int\frac{d^3k}{\mathbf{k}^2}\varphi_\nu(\mathbf{k}+\mathbf{p}). \quad (19)$$

Here $\varphi_\nu^0(\mathbf{p})$ is the Fourier transform of the function $\psi_\nu^0(\mathbf{x})$. For the s-state we have $(\mathbf{p}^2+\eta^2)\varphi_\nu^0(\mathbf{p})=4\pi N$. Rewriting (19) with the aid of the shift operator

$$\left[\mathbf{p}^2+\eta^2-\frac{\alpha Zm}{\pi^2}\int\frac{d^3k}{\mathbf{k}^2}\exp\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\right]$$

$$\times\ \varphi_\nu(\mathbf{p})=(\mathbf{p}^2+\eta^2)\varphi_\nu^0(\mathbf{p})=4\pi N,$$

we can represent its solution in the form

$$\varphi_\nu(\mathbf{p})=\left[\mathbf{p}^2+\eta^2-\frac{\alpha Zm}{\pi^2}\int\frac{d^3k}{\mathbf{k}^2}\exp\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\right]^{-1}\cdot 4\pi N$$

$$=i\int_0^\infty d\tau\exp[-i(\mathbf{p}^2+\eta^2)\tau]$$

$$\times T\exp\left\{i\frac{\alpha Zm}{\pi^2}\int_0^\tau d\tau'\int\frac{d^3k}{\mathbf{k}^2}e^{-i(\mathbf{k}^2+2\mathbf{kp})\tau'}e^{\mathbf{k}\partial/\partial\mathbf{p}}\right\}\cdot 4\pi N, \quad (20)$$

where T is a chronological operator which arranges the operators in decreasing order of τ.[8)]

If we expand the T-exponential in terms of the parameter αZm, then in all terms, starting with $(\alpha Zm)^2$, the shift operator gives rise to correlation terms $\mathbf{kk}'$, $\mathbf{kk}''$, $\mathbf{k}'\mathbf{k}''$, ... in the integrals with respect to $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$, ... Neglecting them in (20), we can replace the shift operator by unity. Then the T-exponential becomes an ordinary exponential with argument

$$\frac{i\alpha Zm}{\pi^2}\int_0^\tau d\tau'\int\frac{d^3k}{\mathbf{k}^2}\exp[-i(\mathbf{k}^2+2\mathbf{kp})\tau']=\frac{i\alpha Zm}{p}\int_0^{-\mathbf{p}^2\tau}dx\frac{E_2(\sqrt{ix})}{x};$$

$$E_2(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt. \quad (21)$$

Here $E_2(z)$ is the error function. In the vicinity of the pole, i.e., when $\mathbf{p}^2+\eta^2\ \eta^2$, the values effective in (20) are $\tau\sim(\mathbf{p}^2+\eta^2)^{-1}\gg\eta^{-2}$. Then $-\mathbf{p}^2\tau\ \eta^2\tau\gg 1$ and, by integrating (21) by parts, we get

$$\int_0^{-\mathbf{p}^2\tau}dx\frac{E_2(\sqrt{ix})}{x}\approx\ln(-4\gamma\mathbf{p}^2\tau)+\frac{i\pi}{2},\qquad -\mathbf{p}^2\tau\to\infty. \quad (22)$$

Using (21) as the argument of the T-exponential in (20) and integrating with respect to τ, we obtain for $\varphi_\nu(\mathbf{p})$ an expression that coincides exactly with the Fourier transform of the function obtained by taking the exponential of the first Coulomb correction in x-space. Thus, taking the exponential in coordinate space corresponds to neglecting the correlation terms in momentum space. We note that the smallness of the correlation terms is based on the modified perturbation theory developed by Fradkin [7].

The wave function $\psi_-(\mathbf{x})$ of a particle in the continuous spectrum at large distances when $pr+\mathbf{p}\cdot\mathbf{x}\gg\varkappa^2$, $\varkappa=Zm/p$, can likewise be obtained, accurate except for an insignificant phase factor, by taking the exponential of the first Coulomb correction, see Sec. 4.

The method of taking the exponential in the case when $\nu,\ \varkappa\ll 1$ is applicable for arbitrary distances and includes perturbation theory; for $\nu,\ \varkappa\gtrsim 1$ it is applicable when

$$\eta r\gg\nu^2,\qquad pr+\mathbf{px}\gg\varkappa^2 \quad (23)$$

and is a particular case of the quasiclassical approximation, which is valid when

$$\eta r(\nu^{-1/2}+\nu)\gg 1,\qquad (pr+\mathbf{px})(1+\varkappa)\gg 1. \quad (24)$$

When condition (1) is satisfied, the values effective in ionization are the distances $\eta r_{\text{eff}}\sim B_0\xi/B\sqrt{1+\xi^2}\gg 1$ and the momenta $p_{\text{eff}}\sim\eta\sqrt{1+\xi^2}$ for a circularly polarized wave and $p_{\text{eff}}\sim\eta\sqrt{2B(1+\xi^2)^{3/2}/B_0\xi}$ for a linearly polarized wave (see [1] and Secs. 1 and 3 of the present paper).

3. IONIZATION MATRIX ELEMENT AND PROBABILITY

We turn to the matrix element $M_{p\nu}$ (4) and represent the exact wave function $\psi_p^*(\mathbf{x})$ of a particle with momentum $\mathbf{p}$ in the wave field and the Coulomb field in the form of a perturbation-theory series with respect to the Coulomb field:

$$\psi_{p^-}^*(x)=\psi_p^*(x)+2ie\int d^4x'\,\psi_p^*(x')K_\mu(x')\frac{\partial}{\partial x_{\mu'}}\Delta(x',x)$$

$$+(2ie)^2\int d^4x''\,d^4x'\,\psi_p^*(x'')K_\mu(x'')\frac{\partial}{\partial x_{\mu''}}\Delta(x'',x')$$

$$\times K_\nu(x')\frac{\partial}{\partial x_{\nu'}}\Delta(x',x)+\ldots \quad (25)$$

Here $\psi_p(\mathbf{x})$ is the wave function of a particle with momentum p_μ, $p^2=-m^2$ in the field of a wave with potential A_μ:

[8)] In (20) we have used the formula

$$e^{(a+b)\tau}=e^{a\tau}T\exp\left(\int_0^\tau d\tau' e^{-a\tau'}be^{a\tau'}\right),$$

which is obtained by differentiating the operator $e^{-a\tau}e^{(a+b)\tau}$ with respect to τ.

$$\psi_p(x)=\frac{1}{\sqrt{2q_0}}\exp\left\{-i\int_0^{(kx)}d\varphi\frac{[p-eA(\varphi)]^2+m^2}{2(kp)}+i(px)\right\}. \quad (26)$$

For a periodic potential it is convenient to characterize the function $\psi_p(x)$ also by means of a four-quasimomentum $q_\mu = p_\mu - k_\mu e^2\overline{A^2}/2(kp)$ which enters in the conservation laws. $eK_\mu(x) = (0, -iZ/r)$ is the Coulomb potential and $\Delta(x, x')$ is the propagation function in the wave:

$$\Delta(x,x')=-\frac{1}{(2\pi)^4}\int\frac{d^4f}{f^2+m^2-i\varepsilon}\times\exp\left\{-i\int_{(kx')}^{(kx)}d\varphi\frac{[f-eA(\varphi)]^2-f^2}{2(kf)}+i(f,x-x')\right\}. \quad (27)$$

It is convenient to use also the Fourier transform of $\Delta(x, x')$ with respect to the difference $x - x'$, which depends on x only through a scalar (kx):

$$\Delta(x,f)=-i\int_0^\infty d\tau\times\exp\left\{i\int_0^{-2(kf)\tau}d\varphi'\frac{[f-eA(\varphi+\varphi')]^2+m^2-i\varepsilon}{2(kf)}\right\}, \quad \varphi=(kx). \quad (28)$$

Expressing the functions $\psi_\nu(x)$ and $\Delta(x, x')$ in terms of the corresponding Fourier transforms $\varphi_\nu(\mathbf{Q})$ and $\Delta(x, f)$ and expanding $\psi_p(x)$ and $\Delta(x, f)$ in Fourier series in the periodic variable $\varphi = (kx)$, we obtain after integrating with respect to x, x', ... and making several transformations using the nonrelativistic limit[9]:

$$M_{pv}=-\frac{4\pi N}{\sqrt{2m}}\sum_s\delta(sk_0+\varepsilon_v-q_0)\cdot\times\int_{-\pi}^{\pi}d\varphi\exp\left\{-i\int_0^{\varphi}d\varphi'\frac{[\mathbf{q}-e\mathbf{A}(\varphi')]^2+\eta^2}{2\omega m}\right\}\times\left(1+\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau\int\frac{d^3f}{(\mathbf{f}-\mathbf{q})^2}\times\exp\left\{-i\int_\varphi^{\varphi+\theta}d\varphi'\frac{[\mathbf{f}-e\mathbf{A}(\varphi')]^2+\eta^2}{2\omega m}\right\}+\left(\frac{i\alpha Zm}{\pi^2}\right)^2\int_0^\infty d\tau\int\frac{d^3f}{(\mathbf{f}-\mathbf{q})^2}\times\exp\left\{-i\int_\varphi^{\varphi+\theta}d\varphi'\frac{[\mathbf{f}-e\mathbf{A}(\varphi')]^2+\eta^2}{2\omega m}\right\}\int_0^\infty d\tau'\int\frac{d^3f'}{(\mathbf{f}'-\mathbf{f})^2}\times\exp\left\{-i\int_{\varphi+\theta}^{\varphi+\theta+\theta'}d\varphi'\frac{[\mathbf{f}'-e\mathbf{A}(\varphi')]^2+\eta^2}{2\omega m}\right\}+\ldots\right). \quad (29)$$

Here $\theta = 2\omega m\tau$, $\theta' = 2\omega m\tau'$, etc.

In each of the integrals with respect to φ, except for the principal "zeroth" one, we make the following transformations: in the first we make the substitution $\varphi + \theta \to \varphi$, in the second $\varphi + \theta + \theta' \to \varphi$, etc. With this, the integration limits with respect to the new variable φ can be left the same as before, by virtue of the periodicity of the integrand with respect to φ. We next make the following change of variables: in the first integral $\tau \to \tau$, in the second $\tau' \to \tau'$ and $\tau + \tau' \to \tau$, in the third $t'' \to \tau''$, $\tau' + \tau'' \to \tau'$, $\tau + \tau' + \tau'' \to t$, etc. We finally introduce also the momenta of the Coulomb field $\mathbf{z} = \mathbf{f} - \mathbf{q}$, $\mathbf{z}' = \mathbf{f}' - \mathbf{f}$, $\mathbf{z}'' = \mathbf{f}'' - \mathbf{f}'$, ... and the shift operators $\exp(\mathbf{z}\partial/\partial\mathbf{q})$, $\exp(\mathbf{z}'\partial/\partial\mathbf{q})$, etc. Then, denoting by $V(\tau, \varphi, \mathbf{q})$ the operator[10]

$$V(\tau,\varphi,\mathbf{q})=\int\frac{d^3z}{\mathbf{z}^2}\exp[-i(\mathbf{z}^2+2\mathbf{zq})\tau+2i\mathbf{z}\mathbf{j}(\tau,\varphi)]\,e^{\mathbf{z}\partial/\partial\mathbf{q}}, \quad (30)$$

where

$$\mathbf{j}(\tau,\varphi)=\int_{\varphi-\theta}^{\varphi}\frac{e\mathbf{A}(\varphi')d\varphi'}{2\omega m},\qquad \theta=2\omega m\tau, \quad (31)$$

we obtain

$$M_{pv}=-\frac{4\pi N}{\sqrt{2m}}\sum_s\delta(sk_0+\varepsilon_v-q_0)\times\int_{-\pi}^{\pi}d\varphi e^{f(\varphi)}\left\{1+\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau\,V(\tau,\varphi,\mathbf{q})+\left(\frac{i\alpha Zm}{\pi^2}\right)^2\int_0^\infty d\tau\int_0^\tau d\tau' V(\tau,\varphi,\mathbf{q})V(\tau',\varphi,\mathbf{q})+\ldots\right\}\cdot 1$$
$$=-\frac{1}{\sqrt{2m}}\sum_{s>s_0}^{\infty}\delta(sk_0+\varepsilon_v-q_0)\int_{-\pi}^{\pi}d\varphi e^{f(\varphi)}F(\varphi,\mathbf{q}). \quad (32)$$

Here

$$f(\varphi)=-i\int_0^{\varphi}\frac{[\mathbf{q}-e\mathbf{A}(\varphi')]^2+\eta^2}{2\omega m}d\varphi', \quad (33)$$

$$F(\varphi,\mathbf{q})=4\pi NT\exp\left[\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau\,V(\tau,\varphi,\mathbf{q})\right]\cdot 1, \quad (34)$$

T is a chronological operator which arranges $V(\tau, \varphi, \mathbf{q})$ in decreasing order of τ. The unity on the right side denotes that the differential operator acts on unity, so that F is a function.

The expression (32) obtained for the matrix element M_{pv} differs from the matrix element M_0 for the ionization of a system coupled by short-range forces (see [1], formulas (11'), (17), (24)) in that the constant $4\pi N$ is replaced by the function

[9] We note that in the nonrelativistic limit $\mathbf{q} = \mathbf{p}$ and $q_0 = m_* + \mathbf{q}^2/2m$, where $m_* = m + e^2\overline{A^2}/2m$.

[10] The more compact form $V = 2\pi^2/\sqrt{(2\mathbf{q}\tau - 2\mathbf{j} + i\partial/\partial\mathbf{q})^2}$ follows from the relation $e^a \cdot e^b = e^{a+b+[a,b]/2}$, which is valid if [a, b] is a c-number.

$F(\varphi, \mathbf{q})$ under the integral sign with respect to ρ[11]. This function contains the entire effect of the Coulomb forces and has resonances for the values of ω corresponding to the differences in the levels of the system. In addition, after integrating with respect to φ in (32), this function should separate the singular factor $\Gamma(1-\nu)$, which compensates for the zeros of N at the points $\nu = 1, 2, 3, \ldots$

According to the conservation laws, the contributions to the sum in (32) are made by the terms $s > s_0$, where

$$s_0 = \frac{m_* - \varepsilon_v}{\omega}\Big|_{\text{nonreal}} = \frac{I}{\omega}\left(1 + \frac{e^2\bar{\mathbf{A}}^2}{\eta^2}\right) = \begin{cases} I\omega^{-1}(1+\xi^2/2), & \text{linear polarization} \\ I\omega^{-1}(1+\xi^2), & \text{circular polarization} \end{cases} \quad (35)$$

$I = m - \epsilon_\nu|_{\text{nonrel}} = \eta^2/2m$ is the ionization energy.

We shall consider a multiquantum ionization such that $s_0 \gg (1+\xi^2)^{3/2}$, i.e., the condition (1) is satisfied. This enables us to simplify the problem in two respects.

First, when (1) is satisfied, as shown in [1], large distances $\sim B_0\xi/B\sqrt{1+\xi^2}$ are significant for the ionization of states coupled by short-range forces. It is clear that the additional long-range Coulomb forces, which distinguish the present problem, retain the effectiveness of the large distances. This makes it possible, in accord with Sec. 2, to use in lieu of the exact matrix element an approximate one obtained by neglecting the correlation terms in p-space. This neglect is equivalent to replacing the shift operator in (30) by unity, so that the function $F(\varphi, \mathbf{q})$ takes the form

$$\bar{F}(\varphi, \mathbf{q}) = 4\pi N e^{L(\varphi,\mathbf{q})}, \quad (36)$$

where L is the first Coulomb correction:

$$L(\varphi, \mathbf{q}) = \frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau \int \frac{d^3z}{z^2}\exp\{-i(z^2 + 2\mathbf{zq})\tau + 2i\,\mathbf{zj}(\tau,\varphi)\} = i\alpha Zm\int_0^\infty d\tau \frac{E_2(\sqrt{G(\tau)/i\tau})}{\sqrt{G(\tau)}}. \quad (37)$$

Here $E_2(x)$ is the same error function as in (21), and

$$G(\tau) = (\mathbf{q}\tau - \mathbf{j})^2 = \left[\int_{\varphi-\theta}^{\varphi} \frac{\mathbf{\Pi}(\varphi')\,d\varphi'}{2\omega m}\right]^2, \quad \mathbf{\Pi}(\varphi) = \mathbf{q} - e\mathbf{A}(\varphi). \quad (38)$$

We note that inasmuch as the function F was obtained by taking the exponential (this is designated by the tilde symbol in (36)), it requires, in accord with Sec. 2, a correction term which after integration with respect to φ should turn into $\gamma^{-\nu}\Gamma(1-\nu)$ (see (17)).

Second, when $s_0 \gg 1$, the exponential $e^{f(\varphi)}$ in (32) is a function strongly dependent on φ, and if $F(\varphi, \mathbf{q})$ is not as strongly dependent, then the integral with respect to φ in (32) can be calculated by the saddle-point method. The saddle points φ_i are determined by the condition $f'(\varphi) = 0$ or

$$[\mathbf{q} - e\mathbf{A}(\varphi)]^2 + \eta^2 = 0. \quad (39)$$

Denoting by f_i and f_i'' the values of the function $f(\varphi)$ and its second derivative at the saddle points, we obtain from (32)

$$M_{pv} = -\frac{1}{\sqrt{2m}}\sum_s \delta(sk_0 + \varepsilon_v - q_0)\sum_i \left(\frac{2\pi}{-f_i''}\right)^{1/2} e^{f_i} F(\varphi_i, \mathbf{q}). \quad (40)$$

Expression (37), generally speaking, diverges logarithmically at the upper limit of integration with respect to τ, owing to the specific infrared divergence inherent in terms of the perturbation-theory series in the Coulomb field [8]. It is usually eliminated introducing the mass λ of the Coulomb photon, i.e., by replacing the propagation function $1/\mathbf{z}^2$ by $1/(\mathbf{z}^2 + \lambda^2)$. In summing the perturbation theory which has be regularized in this manner, all the terms that diverge when $\lambda \to 0$ are gathered into a phase factor with a diverging phase, and this factor does not influence the physical quantities [9,10].

To regularize expression (37), however, it is simpler to introduce a finite upper limit τ_0 for the integration with respect to τ, and carry out the usual subtraction procedure

$$L = i\alpha Zm\int_0^\infty d\tau\left[\frac{E_2(\sqrt{G(\tau)/i\tau})}{\sqrt{G(\tau)}} - \frac{1}{q\tau + q/2\omega m}\right] + i\varkappa \ln 2\omega m\tau_0, \quad \tau_0 \to \infty. \quad (41)$$

The imaginary part of L, which does not depend on φ, does not enter in the probability.

It is further convenient to represent L in the form $L = L_0 + (L - L_0)$, where L_0 differs from (37) in that the function G is replaced by the polynomial

$$G_0(\tau) = (-x^2 + i\beta_1 x^3 + \beta_2 x^4)/\eta^2, \qquad x = \eta^2\tau, \quad (42)$$

which represents the three first terms of the expansion of the function $G(\tau)$ taken at the saddle point, near $\tau = 0$. Inasmuch as $\beta_1 \sim \sqrt{\beta_2} \sim B\sqrt{1+\xi^2}/B_0\xi \ll 1$, it is easy to calculate L_0 explicitly, accurate to terms that vanish when $B\sqrt{1+\xi^2}/B_0\xi \to 0$:

[11] For a constant field, the sum over s in (32) should be replaced by an integral, q_0 should be replaced by p_0, and the integration with respect to φ should be carried out from $-\infty$ to $+\infty$.

$$L_0 = i\alpha Zm\int_0^\infty d\tau \frac{E_2(\sqrt{G_0(\tau)/i\tau})}{\sqrt{G_0(\tau)}} \approx \nu\left(\ln\frac{16\gamma}{\beta_1+2\sqrt{\beta_2}}+i\pi\right). \tag{43}$$

We represent the difference $L - L_0$ in the form

$$L - L_0 = i\varkappa J + i\varkappa \ln\theta_0,$$

$$J = \int_0^\infty d\theta\left[\frac{E_2(\sqrt{(s-s_0)g(\theta)/i\theta})}{\sqrt{g(\theta)}} - \frac{E_2(\sqrt{(s-s_0)g_0(\theta)/i\theta})}{\sqrt{g_0(\theta)}} \times -\frac{1}{1+\theta}\right], \tag{44}$$

where $g(\theta) = (\eta^2\omega/Iq)^2 G(\tau)$ and $g_0(\theta) = (\eta^2\omega\psi Iq)^2 G_0(\tau)$. Supplementing now the relation (36) with the factor (17) and going over from the constant N to C, in accord with (8), we obtain

$$F(\varphi_0, \mathbf{q}) = \frac{2\pi C}{\eta}\left(\frac{16}{\beta_1+2\sqrt{\beta_2}}\right)^\nu \exp(i\varkappa J + i\varkappa\ln\theta_0 + i\pi\nu). \tag{45}$$

Constant field. In this case $\mathbf{A} = a\varphi$, and $\mathbf{B} = \omega\mathbf{a}$; the function $G(\tau)$ at the saddle point coincides with the polynomial $G_0(\tau)$, for which

$$\beta_1 = 2B\sqrt{\zeta}/B_0,\ \beta_2 = (B/B_0)^2,\ \zeta = 1 + (q_2^2 + q_3^2)/\eta^2; \tag{46}$$

the constant field eliminates the Coulomb divergence, and no regularization of the function L is required. Thus, $L = L_0$ and

$$F(\varphi_0, \mathbf{q}) = \frac{2\pi C}{\eta}\left[\frac{8B_0}{(1+\sqrt{\zeta})B}\right]^\nu e^{i\pi\nu}. \tag{47}$$

Using now the results of [1] on the ionization of a system bound by short-range forces, we obtain for the differential and total ionization probabilities per unit time in a weak field ($B/B_0 \ll 1$) expressions of the type

$$W = \frac{|C|^2}{2\eta^3}\int dq_2 dq_3 \frac{\exp(-2B_0\zeta^{3/2}/3B)}{\zeta^{1/2}}\left[\frac{8B_0}{(1+\sqrt{\zeta})B}\right]^{2\nu} = \frac{2\pi|C|^2}{\eta}\left(\frac{4B_0}{B}\right)^{2\nu-1}\exp\left(-\frac{2B_0}{3B}\right). \tag{48}$$

For $\nu = 0$ this formula describes the ionization of a system coupled by short-range forces, and coincides with (8′) of [1], while for $\nu = n$, where $n = 1, 2, 3, \ldots$, it describes ionization of ns-states of hydrogenlike atoms (see the problem of Sec. 77 of [3] and [11]).

Circularly polarized wave. $\mathbf{A} = \mathbf{a}_1\cos\varphi + \mathbf{a}_2\sin\varphi$, $\mathbf{a}_i^2 = a^2$, $\mathbf{a}_i\cdot\mathbf{k} = 0$, $\mathbf{a}_1\cdot\mathbf{a}_2 = 0$. The coefficients β_1 and β_2 of the polynomial G_0 are equal to

$$\beta_1 = \frac{B[(\mathbf{u}^2+1+\xi^2)^2 - 4\xi^2\mathbf{u}_\perp^2]^{1/2}}{B_0\xi},\qquad \beta_2 = \frac{B^2(2\mathbf{u}^2+2+\xi^2)}{3B_0^2\xi^2}, \tag{49}$$

where

$$u = \frac{q}{\eta} = \left[\frac{(s-s_0)(1+\xi^2)}{s_0}\right]^{1/2},\qquad s_0 = \frac{B_0\xi(1+\xi^2)}{2B} = \frac{I(1+\xi^2)}{\omega},$$

$u_\perp = u\sin\theta'$, and θ' is the angle between $\mathbf{q}$ and $\mathbf{k}$.

The function $g(\theta)$ at the saddle point is equal to

$$g(\theta) = \theta^2 + \left\{\frac{i[(\mathbf{u}^2+1+\xi^2)^2 - 4\xi^2\mathbf{u}_\perp^2]^{1/2}}{\mathbf{u}^2}(1-\cos\theta) - \frac{\mathbf{u}^2+1+\xi^2}{\mathbf{u}^2}\sin\theta\right\}\theta + \frac{2\xi^2}{\mathbf{u}^2}(1-\cos\theta). \tag{50}$$

The main contribution to the probability of ionization by a circularly polarized wave is made by those values of s and θ' for which $s - s_0 \sim s_0 \gg 1$ and $(\theta' - \pi/2)^2 \sim \sqrt{1+\xi^2}/s \ll 1$ (see below and [1]). Therefore the significant momenta $u \sim u_\perp \sim \sqrt{1+\xi^2}$ are those for which the Coulomb parameter is $\kappa \equiv \alpha Zm/2 \sim \nu/\sqrt{1+\xi^2} \lesssim \nu$.

When $\xi \lesssim 1$, the effective values in the integral J of (44) will be $\theta \sim 1$, for which $g(\theta) \sim 1$, and the arguments of the E_2 functions are of the order of $\sqrt{s - s_0} \sim \sqrt{s_0} \gg 1$ and these functions can be replaced by unity. Then

$$J \approx \int_0^\infty d\theta\left[\frac{1}{\sqrt{g(\theta)}} - \frac{1}{\sqrt{g_0(\theta)}} - \frac{1}{1+\theta}\right]; \tag{51}$$

J is a complex function of s, θ', s_0, and ξ, and is of the order of unity in the effective region of s and θ' for all $\xi \lesssim 1$, so that $i\varkappa J \sim \varkappa \sim \nu$. Thus, when $\xi \lesssim 1$ it is necessary to use (45) with β_1 and β_2 from (49), and to obtain from (51) the imaginary part of J, which gives a correction of the order of unity (Re J does not enter into the probability).

For $\xi \gg 1$, we have $\kappa J \sim \nu\ln\xi/\xi \ll 1$ and can be neglected, and since in this case $\beta_1 \approx 2B(1+\mathbf{u}^2\cos^2\theta')^{1/2}/B_0$ and $\beta_2 \approx (B/B_0)^2$, we obtain the same function F as for a constant field, but with the replacement $q_2^2 + q_3^2 \to q^2\cos^2\theta'$.

Thus, using the results of [1], we obtain for the ionization probability the expression

$$W = \frac{|C|^2}{2\eta}\sum_{s>s_0} u\int_0^\pi d\theta'\sin\theta'\frac{e^{-2s(\alpha-\mathrm{th}\alpha)}}{s\,\mathrm{th}\,\alpha}\left(\frac{16}{\beta_1+2\sqrt{\beta_2}}\right)^{2\nu} e^{-2\varkappa \mathrm{Im}J}, \tag{52}$$

where $\cosh\alpha = (s/2\xi\sin\theta')[(1+\xi^2)/s_0(s-s_0)]^{1/2}$. The angular distribution has a sharp maximum at $\theta' = \pi/2$ with a width $\Delta\theta' \sim (s\tanh\alpha)^{-1/2}|_{\theta'=\pi/2} \ll 1$. Integrating with respect to the angle θ' by the saddle-point method, we obtain

$$W = \frac{\sqrt{\pi}|C|^2}{2\eta}\sum_{s>s_0} u\frac{e^{-2s(\alpha-\mathrm{th}\alpha)}}{(s\,\mathrm{th}\,\alpha)^{3/2}}\left(\frac{16}{\beta_1+2\sqrt{\beta_2}}\right)^{2\nu} e^{-2\varkappa \mathrm{Im}J}; \tag{53}$$

it is understood that $\theta' = \pi/2$ in all the quantities. The distribution over s has the form of a Gaussian curve with a maximum at $s = s_c$, where s_c is determined from the equation

$$\frac{2\alpha_c}{\operatorname{th}\alpha_c} = \frac{s_c}{s_c - s_0}, \quad \operatorname{ch}\alpha_c = \frac{s_c}{2\xi}\left[\frac{1+\xi^2}{s_0(s_c-s_0)}\right]^{1/2}$$

$$s_c - s_0 = \begin{cases} s_0/\ln\xi^{-2} + \ldots, & \ln\xi^{-2} \gg 1 \\ s_0(1 - 2/3\xi^2 + \ldots), & \xi \gg 1 \end{cases} \quad (54)$$

and a width

$$(s-s_c)_{\text{eff}} \sim \left[\frac{s_c \operatorname{th}\alpha_c}{\alpha_c^2 + (\alpha_c/\operatorname{th}\alpha_c - 1)^2}\right]^{1/2}$$

$$= \begin{cases} \sqrt{2s_0}/\ln\xi^{-2}, & \ln\xi^{-2} \gg 1 \\ \sqrt{2s_0\xi}, & \xi \gg 1 \end{cases} \quad (55)$$

Thus, for all values of ξ (except the exponentially small ones, when $\ln\xi^{-2} \gtrsim \sqrt{2s_0} \gg 1$), the width of the distribution over s is large compared with unity, i.e., ionization proceeds with absorption of a large number of photons in excess of the number necessary to overcome the threshold. At the same time, this width is small compared with s_0, and the maximum of the distribution is located a distance $s_c - s_0 \sim s_0$ from the threshold, so that in units of s_0 the distribution has the form of a narrow peak.

Assuming that ξ is not exponentially small, and replacing summation with integration in (53), we obtain ultimately

$$W = \frac{\pi|c|^2}{2\eta}\left[\frac{s_c - s_0}{s_0}(1+\xi^2)\right]^{1/2} \times \frac{\exp[-2s_c(\alpha_c - \operatorname{th}\alpha_c)]}{[\alpha_c^2 + (\alpha_c/\operatorname{th}\alpha_c - 1)^2]^{1/2} s_c \operatorname{th}\alpha_c} \times \left(\frac{16}{\beta_{1c} + 2\sqrt{\beta_{2c}}}\right)^{2\nu} \exp[-2\varkappa_c \operatorname{Im} J_c] \quad (56)$$

(the subscript c denotes that the corresponding quantity is taken at the point $s = s_c$, $\theta' = \pi/2$). The complex Coulomb factor in formula (56) can be approximated, in order of magnitude, by the formula

$$\left(\frac{16}{\beta_{1c} + 2\sqrt{\beta_{2c}}}\right)^{2\nu} \exp(-2\varkappa_c \operatorname{Im} J_c) \sim \left(\frac{B_0\xi}{B\sqrt{1+\xi^2}}\right)^{2\nu}.$$

When $\xi \gg 1$ formula (56) goes over into (48), except that the argument of the exponential is $-(2B_0/3B)(1 - 1/15\xi^2)$. At small values of ξ it simplifies if $\xi^2 \ll s_0^{-1} \ll 1$, when we have throughout $\tanh\alpha_c \approx 1$, $\alpha_c \approx s_c/2(s_c - s_0)$, where s_c is determined from the equation $\ln[s_c^2/\xi^2 s_0(s_c - s_0)] = s_c/(s_c - s_0)$.

Linearly polarized wave. The main contribution to the probability of ionization by a linearly polarized wave is made by those values of s for which $s - s_0 \lesssim (1+\xi^2)^{3/2}$ [1]. Therefore the significant momenta are

$$q = \eta[2B(s-s_0)/B_0\xi]^{1/2} \lesssim \eta[2B(1+\xi^2)^{3/2}/B_0\xi)^{1/2}.$$

When $q_{\text{eff}} \gtrsim \eta$, i.e. when $\xi \gtrsim \sqrt{B_0/B} \gg 1$, the form factors $F(\varphi_\mp, \mathbf{q})$ corresponding to the two limiting points

$$\varphi_\mp = \mp\psi - i\varepsilon, \quad \cos\varphi_\mp = (u_1 \mp i\sqrt{\zeta})/\xi, \zeta = 1 + u_2^2 + u_3^2.$$

coincide and have the same form (47) as for a constant field. Therefore, when $\xi \gtrsim \sqrt{B_0/B} \gg 1$ for example, the total ionization probability is (cf. (23') of [1]):

$$W = \frac{4\sqrt{3\pi}|C|^2}{\eta}\left(\frac{4B_0}{B}\right)^{2\nu-3/2} \exp\left[-\frac{2B_0}{3B}\left(1 - \frac{1}{10\xi^2}\right)\right]. \quad (57)$$

When $\xi \lesssim 1$ the problem becomes greatly more complicated by the fact that $(s - s_0)_{\text{eff}} \sim 1$, and the effective momenta become small compared with η: $q = \eta\sqrt{(s-s_0)\omega/I} \sim \eta\sqrt{\omega/I} \ll \eta$ and the Coulomb parameter κ becomes large: $\kappa \equiv \alpha Zm/q \sim \nu\sqrt{I/\omega} \gg \nu$. Therefore when $\nu \sim 1$ (the case of greatest interest) we have $\kappa \sim q r_{\text{eff}} \gg 1$ and condition (23) for the applicability of the method of taking the exponential is not satisfied.

However, the quasiclassical condition (24) is satisfied, as before, and therefore it would be possible to calculate the matrix element by means of formula (4), using for $\psi_p^-(x)$ the quasiclassical function $e^{iS(x)}$ in the field of the wave and in the Coulomb field. Unfortunately, the determination of the action function S, which satisfies the Hamilton-Jacoby equation, is in this case a complicated and independent problem. Therefore we shall attempt in Sec. 4 to present a qualitative estimate of the form factor F when $\xi \lesssim 1$, making use of the exact expression for F.

4. WAVE FUNCTION OF FINAL STATE AND PROPERTIES OF THE FORM FACTOR F

The procedure presented at the start of Sec. 3 gives for $\psi_q^{-*}(x)$ (see (25)) the following expression ($\varphi = -\omega t$)

$$\psi_q^{-*}(x) = \frac{1}{\sqrt{2q_0}} \exp\left\{-iqx + iq_0 t - i\int_0^\varphi d\varphi' \frac{[\mathbf{q} - e\mathbf{A}(\varphi')]^2 - \mathbf{q}^2 - e^2\overline{\mathbf{A}^2}}{2\omega m}\right\} \times T\exp\left\{\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau \int \frac{d^3z}{z^2} e^{-i(z^2+2zq)\tau + 2izj - izx} e^{z\partial/\partial q}\right\} \cdot 1. \quad (58)$$

When the field of the wave is turned off ($A = j = 0$) this function goes over into the wave function of a particle with momentum q in a Coulomb field, and

should therefore differ from the known solution (see [3], formula (134.9))

$$\psi_{\mathbf{q}}^{-*}(x)=\frac{1}{\sqrt{2q_0}}\exp\left(-i\mathbf{qx}+iq_0t+\frac{\pi\varkappa}{2}\right)\times\Gamma(1-i\varkappa)\Phi(i\varkappa,1,ib),$$

$$\varkappa=\alpha Zm/q,\quad b=qr+\mathbf{qx} \tag{59}$$

only in the infrared phase factor. Since this factor does not depend on $\mathbf{x}$, it can be readily obtained by using the limiting case $b\gg 1$, when (58) can be calculated, neglecting the correlation terms, i.e., replacing the shift operator by unity (cf. Sec. 2). We then obtain from (58) for $\mathbf{A}=0$

$$\psi_{\mathbf{q}}^{-*}(x)\approx\frac{1}{\sqrt{2q_0}}\exp\left[-i\mathbf{qx}+iq_0t-i\varkappa\ln b+i\varkappa\ln\frac{2q}{\gamma\lambda}\right].$$

This function differs from the asymptotically exact function (59) only in the infrared phase factor $\exp[i\varkappa\ln(2q/\gamma\lambda)]$.

We have thus established the following important relation:

$$T\exp\left\{\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau\int\frac{d^3z}{z^2+\lambda^2}e^{-i(z^2+2z\mathbf{q}-i\varepsilon)\tau-i\mathbf{zx}}e^{\mathbf{z}\partial/\partial\mathbf{q}}\right\}\cdot 1$$

$$=\exp\left[i\varkappa\ln\frac{2q}{\gamma\lambda}+\frac{\pi\varkappa}{2}\right]\Gamma(1-i\varkappa)\Phi(i\varkappa,1,ib),$$

$$\lambda\to 0,\quad \varkappa=\alpha Zm/q,\quad b=qr+\mathbf{qx}. \tag{60}$$

Relation (60) gives a certain idea of the complexity of the Coulomb "form factor"

$$F(\varphi,\mathbf{q})=4\pi NT\exp\left\{\frac{i\alpha Zm}{\pi^2}\int_0^\infty d\tau\int\frac{d^3z}{z^2+\lambda^2}\times e^{-i(z^2+2\mathbf{qz}-i\varepsilon)\tau+2i\mathbf{z}\mathbf{j}(\tau,\varphi)}e^{\mathbf{z}\partial/\partial\mathbf{q}}\right\}\cdot 1, \tag{61}$$

which differs from the left side of (60) in the fact that the constant vector $\mathbf{x}$ is replaced by the vector $-2\mathbf{j}(\tau,\varphi)$, which depends on τ and represents the change in the coordinate of a classical particle from the instant $t=-\varphi/\omega$ to $t=-\varphi/\omega+2m\tau$ in a coordinate frame where the particle is on the average at rest.

Constant field. $\mathbf{A}(\varphi)=\mathbf{a}\varphi$, $\mathbf{B}=\omega\mathbf{a}$, and (see (31))

$$\mathbf{j}(\tau,\varphi)=e\mathbf{A}(\varphi)\tau-e\mathbf{B}m\tau^2. \tag{62}$$

The term that is linear in τ, added to the term $-2i\mathbf{z}\cdot\mathbf{q}\tau$, leads to a replacement of the momentum $\mathbf{q}$ by the vector $\mathbf{\Pi}=\mathbf{q}-e\mathbf{A}(\varphi)$, so that F depends on the two vectors $\mathbf{\Pi}$ and $em\mathbf{B}$. It is easy to see that these vectors enter in F only in the form of the dimensionless invariant combinations

$$\alpha Zm/\Pi,\qquad emB/\Pi^3,\qquad em\mathbf{B\Pi}/\Pi^4. \tag{63}$$

Owing to the term that is quadratic in τ, the function F does not contain an infrared Coulomb divergence and therefore needs no regularization. Thus, the constant field plays the same role for the elimination of the Coulomb divergence as the photon mass λ. In this connection, the factor F has the point $B=0$ a singularity, and in a sufficiently small vicinity of this point its functional dependence on the field should be the same as the dependence of the expression in (60) on λ as $\lambda\to 0$.

Thus, for a sufficiently weak constant field the function F is equivalent to expression (60) in which $\mathbf{x}=0$, $\mathbf{q}$ is replaced by $\mathbf{\Pi}=\mathbf{q}-e\mathbf{a}\varphi$, and λ is replaced by $\lambda_B=(emB/\Pi^2)f(\mathbf{B}\cdot\mathbf{\Pi}/B\Pi)$:

$$F(\varphi,\mathbf{q})=4\pi N\exp\left(i\frac{\alpha Zm}{\Pi}\ln\frac{2\Pi}{\gamma\lambda_B}+\frac{\pi\alpha Zm}{2\Pi}\right)\Gamma\left(1-i\frac{\alpha Zm}{\Pi}\right). \tag{64}$$

An exact expression for λ_B can be obtained by calculating the first term of the expansion (61) in powers of $\alpha Zm/\Pi$ and comparing it with the corresponding term of the expansion (64). We then obtain

$$\lambda_B=em(B\Pi+\mathbf{B\Pi})/4\gamma\Pi^3. \tag{65}$$

At the saddle point $\varphi=\varphi_0=(q_1-i\eta\sqrt{\xi})/ea$ we have

$$\Pi=i\eta,\quad i\frac{\alpha Zm}{\Pi}=\nu,$$

$$\frac{em(B\Pi+\mathbf{B\Pi})}{\Pi^4}=i\frac{(1+\sqrt{\xi})B}{B_0}, \tag{66}$$

and $F(\varphi,\mathbf{q})$ goes over into (47).

Unlike in the method of taking the exponential, the factor $\Gamma(1-\nu)$ arises here in natural fashion and, together with the coefficient N, makes the matrix element finite for a purely Coulomb field (i.e., for $\nu=1,2,3,\ldots$).

Alternating field. If ξ is sufficiently large, then the effective values in (61) will be $\eta^2\tau\sim B_0/B\gg 1$, so that the variable θ will be effectively small: $\theta_{\text{eff}}\sim 1/\xi\ll 1$. Expanding (31) in powers of θ and retaining only the linear and quadratic terms, we obtain for $\mathbf{j}(\tau,\varphi)$ the expression (62), in which $\mathbf{A}(\varphi)$ and $\mathbf{B}(\varphi)$ now stand for the potential and field of the wave corresponding to the phase φ.[12] By the same token, we essentially reduce the problem to a calculation of $F(\varphi,\mathbf{q})$ in a constant field, thus leading to the result (64), where $\mathbf{\Pi}=\mathbf{q}-e\mathbf{A}(\varphi)$. For the function F at the saddle point we obtain expression (47).

[12] For example, for a linearly polarized wave $\mathbf{A}(\varphi)=\mathbf{a}\cos\varphi$ and $\mathbf{B}(\varphi)=-\omega\mathbf{a}\sin\varphi$.

A rough qualitative idea of the value of F when $\xi \lesssim 1$ can be obtained by assuming that $\mathbf{j}(\tau, \varphi)$ does not depend on τ, and that at the saddle point $qj \sim q \cdot j \sim \sqrt{I/\omega} \gg 1$ (we have in mind a linearly polarized wave, since the case of a circularly polarized wave was considered more accurately in Sec. 3). Then

$$F(\varphi, q) \sim 4\pi N \exp\left(i\varkappa \ln \frac{2q}{\gamma\lambda} + \frac{\pi\varkappa}{2} \right) \Gamma(1 - i\varkappa)\Phi(i\varkappa, 1, z),$$
$$z = i2(qj - qj), \tag{67}$$

and we arrive at the need for determining the asymptotic behavior of $\Phi(i\kappa, 1, z)$ when $\kappa \sim |z| \sim \sqrt{I/\omega} \gg 1$.

Let us replace the sum

$$\sum_{k=0}^{\infty} \frac{\Gamma(i\varkappa + k) z^k}{\Gamma(i\varkappa)\Gamma^2(k+1)} \equiv \Phi(i\varkappa, 1, z)$$

by an integral and the Γ functions by their asymptotic values, and let us evaluate the integral by the saddle-point method. Then

$$\Phi(i\varkappa, 1, z) = \frac{e^{f(k_0)}}{\Gamma(i\varkappa)[k_0(k_0 + 2i\varkappa)]^{1/2}},$$

$$f(k) = -i\varkappa + k + i\varkappa \ln (k + i\varkappa) + k \ln zk^{-2}(k + i\varkappa), \tag{68}$$
$$k_0 = z/2 + \sqrt{(z/2)^2 + i\varkappa z}.$$

The obtained asymptotic expression is valid for those z and κ for which $\mathrm{Re}\, k_0$ and $\mathrm{Re}\, f(k_0) \gg 1$. At the saddle points $\varphi_\pm = \mp\psi - i\epsilon$, and $\cos\varphi_\mp = (u_1 \mp i\sqrt{\xi})/\xi$ the value of z at $\xi < 1$, when $u_{eff} \sim \sqrt{\omega/I} \ll 1$, is determined by the formula

$$z_\mp = 2i(qj - qj)_\mp,$$

$$j_{1\mp} = \mp \frac{\eta}{2\omega m} [\sqrt{1 + \xi^2}\,(1 - \cos\theta) + i \sin\theta], \quad j_{2\mp} = j_{3\mp} = 0.$$

The effective values of the variable θ can be such that $z_{\mp eff}$ satisfy the conditions of applicability of the asymptotic relation (68). In this case $F(\varphi_\mp, q) \sim \exp(c_\pm \sqrt{I/\omega})$, $c_\pm \sim 1$, i.e., the Coulomb field greatly intensifies the ionization.

We have considered in this paper the Coulomb forces both in the initial and in the final states on par, and have shown that if the momentum of the ionized electron is comparable with the intra-atomic one or exceeds it: $q \gtrsim \eta$, i.e, $\alpha Z/v \lesssim 1$ (as is the case for the effective momentum in the ionization by a circularly polarized wave), then the Coulomb field increases the probability of ionization by a factor $\sim (B_0 \xi / B\sqrt{1 + \xi^2})^{2\nu}$, owing to the fact that it increases the density of the electrons at large distances; the role of the Coulomb forces is then equally important in the initial and in the final states (2), and were they to be neglected, say, in the final state, the exponent of this factor would be decreased by one half. Thus, the effect of the Coulomb forces in the final state reduces to a factor $\sim (B_0 \xi / T\sqrt{1 + \xi^2})^\nu$, which depends strongly on the initial state via the exponent ν (for a purely Coulomb field ν coincides with the principal quantum number n).

On the other hand, if the momentum of the ionized electron is small: $q \ll \eta$, i.e., $\alpha Z/v \gg 1$ (as is the case for the effective momentum in the case of a linearly polarized wave), then the Coulomb forces in the final state lead to an appreciably stronger intensification of the ionization by an amount $\sim \exp(c\sqrt{I/\omega})$, due to the decrease in the effective distances by a factor $1 - c'\sqrt{\omega/I}$ ($c' \sim 1$).

Keldysh [12] calculated the ionization of an atom by a linearly polarized wave without taking into account the Coulomb forces in the final state, and introduced in the final result heuristically a correction factor $\sim B_0 \xi / B\sqrt{1 + \xi^2}$, which, as shown by our calculation, coincides in order of magnitude with ours for a circularly polarized wave and $\nu = 1$.

Although the exact matrix element (32) has resonances, these are not taken into account in the present approximation, since we have assumed that the frequency ω does not satisfy the resonance condition. Resonance effects can be taken into account separately, as was done in [12]. We note that Gold and Bebb [13,14] developed a standard perturbation theory for multiquantum ionization. This approach, which is valid when $\xi \ll 1$, accounts for resonance effects, but leads to a very cumbersome formula for the ionization probability, and the corresponding approximation formula in the multiquantum case is subject to a large error, since it is highly sensitive to the adjustment parameter Ω.

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