FLUTE INSTABILITY OF A CYLINDER OF STRONGLY INHOMOGENEOUS RAREFIED PLASMA

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We consider flute instability of a maximally inhomogeneous plasma cylinder, in which the thickness of the transition layer between the plasma and the vacuum amounts to two ionic Larmor radii. We show that the mode $|m| = 1$ is not stabilized even with a large Larmor radius (comparable with the radius of the cylinder). We present an example of a distribution which is unstable against perturbations of higher modes with $|m| > 1$, at an arbitrarily small value of a gravitational force imitating the curvature of the magnetic-field force line.

1. It is known that a finite Larmor radius suppresses flute instability in a plasma with sufficiently hot ions. This stabilizing effect was investigated in the case of a weakly inhomogeneous plasma, in which the characteristic inhomogeneity dimension L is much larger than the average ion Larmor radius R_i. It has turned out that in this case all the modes can be stabilized in a cylinder of the inhomogeneous plasma, except the mode $|m| = 1$ [1,2]. Since the transverse dimension of some experimental installations, such as "Ogra," is comparable with R_i, it is of interest to consider the flute instability of the cylinder without assuming weak inhomogeneity $R_i \ll L$. In particular, the question arises whether the mode $|m| = 1$ is stabilized by a large Larmor radius (comparable with the radius of the cylinder.

We consider in this paper the flute instability of a cylinder with maximally strong inhomogeneity. We choose as the ionic distribution function the following function of the integrals of motion ($z \parallel H_0$)

$$f_{0i} = \frac{N_0}{2\pi v_0}\,\delta(v_\perp - v_0)\,Q_i([(x + v_y/\omega_i)^2 + (y - v_x/\omega_i)^2]^{1/2})\,f_{\parallel}(v_z), \tag{1}$$

where $v_\perp = \sqrt{v_x^2 + v_y^2}$, $\omega_i = eH_0/m_i c$ is the ion-cyclotron frequency, and Q_i is a function characterizing the inhomogeneity:

$$Q_i = \begin{cases} 1, & r < a \\ 0, & r > a. \end{cases} \tag{2}$$

The distribution $f_{\parallel}(v_z)$ with respect to the longitudinal velocities drops out of the equations in the flute-instability problem.

Let $R_i \ll d_{i\perp}$, where $d_{i\perp} = [m_i v_0^2/4\pi e^2 N_0]^{1/2}$ is the "transverse" ionic Deybe length. Then we can retain in the right-hand side of the Poisson equation $\nabla^2\varphi = -4\pi(\rho_i + \rho_e)$ ($\varphi = \exp(im\theta - i\omega t)$, θ is the azimuthal angle and m is the number of the mode) only the terms connected with the inhomogeneity and containing the derivative dQ_i/dr. We assume that the electrons are cold and neglect the finite length of the electron Larmor radius R_e ($R_e = 0$). In the calculations of the charge densities ρ_e and ρ_i we can carry out the integration over the velocities by using the Fourier-Bessel integral representation for the perturbed quantities (in lieu of the usual Fourier representation in the plane case [3]). Under these assumptions, the equation for the oscillations of the cylinder (in dimensionless quantities) takes the form ($|\omega| \ll \omega_i$) [4]

$$\frac{\Phi(\varkappa)}{\Omega_0^2} = \left(\frac{1}{\Omega + \Omega^*} - \frac{1}{\Omega}\right) \times \int_0^\infty K_0^{(m)}(\varkappa, \lambda)\Phi(\lambda)\,d\lambda - \frac{1}{\Omega}\int_0^\infty K_1^{(m)}(\varkappa, \lambda)\Phi(\lambda)\,d\lambda.$$

$$\Phi(\varkappa) = \varkappa^{3/2}\psi(\varkappa), \quad \psi(\varkappa) = \int_0^\infty \psi(r) J_m(\varkappa r/a)\,r\,dr \tag{3}$$

is the Fourier-Bessel integral of the potential $\psi(r)$

$$K_0^{(m)}(\varkappa, \lambda) = \frac{m}{\sqrt{\varkappa\lambda}} J_m(\varkappa) J_0(\varepsilon\varkappa) J_m(\lambda) J_0(\varepsilon\lambda), \tag{4}$$

$$K_1^{(m)}(\varkappa, \lambda) = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{m+s}{\sqrt{\varkappa\lambda}} J_{m+s}(\varkappa) J_s(\varepsilon\varkappa) J_{m+s}(\lambda) J_s(\varepsilon\lambda), \tag{5}$$

$\epsilon = v_0/\omega_i a$ is the dimensionless Larmor radius, $\Omega_0 = \omega_i^{-1}(4\pi e^2 N_0/m_i)^{1/2}$, $\Omega = \omega/\omega_i$, $\Omega^* = m\omega^*/\omega_i$, and ω^* is the precession frequency in a field with curved force lines. We shall henceforth assume for concreteness that $m > 0$.

2. Neglecting the finite length of the ionic Larmor radius ($\epsilon = 0$), Eq. (3) has a unique solu-

tion $\Phi = \kappa^{-1/2} J_m(\kappa)$ (in the coordinate representation it corresponds to a solution $\psi(r)$ equal to $(r/a)^m$ when $r < a$ and $(r/a)^{-m}$ when $r > a$), corresponding to the dimensionless frequencies

$$\Omega = -{}^1/_2\Omega^* \pm {}^1/_2(\Omega^{*2} - 2\Omega_0^2\Omega^*)^{1/2}. \tag{6}$$

The perturbations are unstable if $\Omega^* < 2\Omega_0^2$. We note that when $\epsilon = 0$ instability sets in for arbitrarily small Ω^*.

Let us consider now the oscillations of the cylinder when $\epsilon \neq 0$. In a weakly inhomogeneous plasma, the finite Larmor radius exerts a stabilizing influence: when $\Omega^* \lesssim \epsilon^4\Omega_0^2$ the perturbations are stable (except the mode $m = 1$). We shall show that in the case of the maximal inhomogeneity (2), not only is the mode $m = 1$ not stabilized, but the higher modes $m > 1$ are also unstable, at least for one value of ϵ.

We seek the solutions of (3) in the form of the sum

$$a_0\Phi_0^{(m)} + \sum_{q=1}^{\infty} a_q\Phi_q^{(m)}, \tag{7}$$

where $\Phi_0^{(m)} = (m/\kappa)^{1/2} J_m(\kappa) J_0(\epsilon\kappa)$, and $\Phi_q^{(m)}$ are mutually orthogonal eigenfunctions of the kernel K_1^m:

$$\Phi_q^{(m)}(\varkappa) = \mu_q^{(m)} \int_0^{\infty} K_1^{(m)}(\varkappa, \lambda)\Phi_q^{(m)}(\lambda)\,d\lambda. \tag{8}$$

Let us substitute (7) in (3). Recognizing that $K_0^{(m)}(\kappa, \lambda) = \Phi_0^{(m)}(\kappa)\,\Phi_0^{(m)}(\lambda)$, and assuming that the functions $\Phi_0^{(m)}$ and $\Phi_q^{(m)}$ are linearly independent, we equate the coefficients of these functions in the right and left sides of (3). Eliminating a_q ($q = 1, 2, 3, \ldots$) from the resultant system of equations, we obtain the following equation for the frequency:

$$\left(1 - \sum_{q=1}^{\infty} \frac{b_{0q}{}^2}{b_{00}'b_{qq}'}\right) b_{00}' = 0, \tag{9}$$

where

$$b_{ik} = \int_0^{\infty} \Phi_i\Phi_k\,d\varkappa, \quad b_{00}' = b_{00} + \frac{\Omega(\Omega + \Omega^*)}{\Omega_0{}^2\Omega^*},$$

$$b_{qq}' = b_{qq}(1 + \Omega\mu_q^{(m)}/\Omega_0{}^2).$$

In the region of interest to us, $|\Omega| \ll \Omega_0^2$, we have

$$\sum b_{0q}{}^2/b_{qq}' \approx \sum b_{0q}{}^2/b_{qq} \equiv S,$$

and (9) has an unstable root ($\mathrm{Im}\,\Omega > 0$), determined from the equation

$$b_{00} - S + \frac{\Omega(\Omega + \Omega^*)}{\Omega_0{}^2\Omega^*} = 0, \tag{10}$$

if $S < b_{00}$ and $\Omega^* < 4\Omega_0^2(b_{00} - S)$. (The case $\epsilon = 0$ considered above corresponds to $S = 0$ and $b_{00} = {}^1/_2$). The stability problem reduces thus to a check on the completeness of the system of functions $\Phi_q^{(m)}$ in the "class" (containing one function) $\Phi_0^{(m)}$. If the system $\Phi_q^{(m)}$ is not complete in the class $\Phi_0^{(m)}$, then $S < b_{00}$ [5] and the instability exists at arbitrarily small Ω^*.

3. We consider first the mode $m = 1$. We shall show that in this case the system $\Phi_q^{(1)}$ is not complete in the "class" $\Phi_0^{(1)} = J_1(\kappa) J_1(\epsilon\kappa)/\sqrt{\kappa}$. Let us assume the contrary, and then the Fourier series

$$\sum_q b_{0q}\Phi_q^{(1)}/b_{qq} \sim \Phi_0^{(1)}$$

can be integrated in the interval $[0, \infty]$ term by term with weight ξ, where ξ is any function with integrable square [5]:

$$\sum_q \frac{b_{0q}}{b_{qq}} \int_0^{\infty} \xi(\varkappa)\Phi_q^{(1)}(\varkappa)\,d\varkappa = \int_0^{\infty} \xi(\varkappa)\Phi_0^{(1)}(\varkappa)\,d\varkappa. \tag{11}$$

We choose the function ξ in the form $\xi^{(1)} = J_1(\alpha\kappa)/\sqrt{\kappa}$, where $\alpha > 1 + \epsilon$. This function is orthogonal to the kernel $K^{(1)}$ ([6], formula 6.578), so that the left side of (11) vanishes. At the same time $\int_0^{\infty} \Phi_0(\kappa)\,\xi^{(1)}(\kappa)\,dn = {}^1/_2\alpha$. The resultant contradiction proves the incompleteness of the system $\Phi_q^{(1)}$ in the class $\Phi_0^{(1)}$. This result obviously does not depend on the value of ϵ, and consequently in our case of a plasma having a sharp boundary the mode $m = 1$ does not become stabilized even by an arbitrarily large ion Larmor radius (comparable with the radius of the cylinder).

4. We consider now the modes $m > 1$. As an example we take the particular case $\epsilon = 1$, for which the kernel $K_1^{(m)}(\kappa, \lambda)$ simplifies to

$$K_1^{(m)}(\varkappa, \lambda)_{\epsilon=1} = \sum_{s=-(m-1)}^{-1} \frac{m+s}{\sqrt{\varkappa\lambda}} J_{m+s}(\varkappa)J_s(\varkappa)J_{m+s}(\lambda)J_s(\lambda)$$
$$+ \sum_{s=1}^{\infty} \frac{b_{0q}}{\sqrt{\varkappa\lambda}} J_{m+s}(\varkappa)J_s(\varkappa)J_{m+s}(\lambda)J_s(\lambda). \tag{12}$$

Let $m = 3$. Assuming the system $\Phi_q^{(3)}$ to be complete, we have

$$\sum \frac{b_{0q}}{b_{qq}} \int_0^{\infty} \xi(\varkappa)\Phi_q^{(3)}(\varkappa)\,d\varkappa = \int_0^{\infty} \xi(\varkappa)\Phi_0^{(3)}(\varkappa)\,d\varkappa. \tag{13}$$

we take ξ in the form

$$\xi^{(3)} = \frac{1}{\varkappa^{3/2}} + \frac{2\alpha(1-\beta^2)}{\beta^2 - \alpha^2}\frac{J_1(\alpha\varkappa)}{\varkappa^{5/2}} + \frac{2\beta(1-\alpha^2)}{\alpha^2 - \beta^2}\frac{J_1(\beta\varkappa)}{\varkappa^{5/2}};$$

$$\alpha, \beta > 2, \quad \alpha \neq \beta.$$

Then the right side of (13) is equal to $-\sqrt{3/24}$, and the left side vanishes, since it is easy to see that

$\xi^{(3)}$ is orthogonal to the kernel $K^{(3)}(\kappa, \lambda)_{\epsilon=1}$. This contradiction means that actually the system $\Phi_q^{1(3)}$ is not complete in the class $\Phi_0^{(3)}$. Therefore instability for arbitrarily small Ω^* takes place also for the mode $m = 3$ in the case $\epsilon = 1$. The instability of the remaining modes at $\epsilon = 1$ is proved similarly.

5. In a strongly inhomogeneous plasma cylinder whose radius is comparable with the ion Larmor radius, the electron and ion drift velocities in the electric field of the perturbation are markedly different, owing to averaging of the field acting on the ion, so that one could expect a finite R_i to have a stabilizing effect. However, in our case of a maximally strong inhomogeneity, this averaging not only does not stabilize the mode $m = 1$, but, as we have already seen in one example, it can lead (unlike the case of a weakly inhomogeneous plasma) also to instability of the higher modes. We note that in the frequency region $|\Omega| \ll \Omega_0^2$ under consideration we have for unstable solutions Φ, defined by formula (7), $a_q \approx -a_0 b_{0q}/b_{qq}$, so that

$$\frac{1}{\Omega}\int_0^\infty K_1(\varkappa, \lambda)\Phi(\lambda)\,d\lambda \approx 0. \qquad (14)$$

Since the difference between the electronic and ionic charge densities ρ_e and ρ_i is characterized precisely by the quantity $\Omega^{-1}\int K_1\Phi d\lambda$, it follows that (14) signifies that $\Omega\rho_e \approx (\Omega + \Omega^*)\rho_i$ in the unstable oscillations found by us. A similar situation obtains, as is well known, in the oscillations of the mode $m = 1$ of a weakly inhomogeneous plasma, which cannot be stabilized by a finite Larmor radius.

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[1] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nucl. Fusion 1962, Suppl. I, p. 143.

[2] A. B. Mikhailovskiĭ, Nuclear Fusion 4, 108 (1964). L. V. Mikhailovskaya and A. B. Mikhailovskiĭ, ibid. 5, 249 (1965).

[3] V. V. Arsenin, ibid. 3, 190 (1963).

[4] V. V. Arsenin, ZhTF 37, 1967 (in press).

[5] G. P. Tolstov, Ryady Fur'e (Fourier Series), Gostekhizdat, 1951.

[6] I. S. Gradshtein and I. M. Ryzhik. Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), 4th ed., Fizmatgiz, 1962.

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