ELECTRON SCATTERING BY AN IMPURITY WITH A SPIN

S. V. MALEEV

A. F. Ioffe Physico-technical Institute, Academy of Sciences, U.S.S.R.

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The problem of scattering of an electron in a metal by an impurity with a spin is solved by means of the analyticity and unitarity relations. An expression for the scattering amplitude is found which is valid throughout the whole region near the Fermi surface. A formal expansion of the solution in powers of the interaction is identical, up to quadratic terms, with the respective terms of the perturbation-theory series. The behavior of the amplitude as a function of the energy depends on the sign of the exchange part of the interaction. If the sign is negative then the cross section is a maximum at $E = E_F$ and equals $4\pi k_F^{-2}$; however, at $E \approx E_F$ the behavior of the scattering phase shifts has nothing in common with the resonance described by the Breit-Wigner formula. If the exchange interaction sign is positive then the cross section is a minimum at $E = E_F$.

KONDO[1] called attention to the fact that perturbation theory is not applicable to the amplitude of the scattering of an electron in a metal by an impurity possessing a spin. This problem must therefore be solved by other methods. There are already many papers devoted to this question (see, for example, the articles by Abrikosov[2], Donach[3], Nagaoka[4], and Suhl[5-7]. The problem, however, cannot yet be regarded as solved. In particular, the results obtained by different authors are not the same. In addition, the scattering amplitude obtained in a number of papers[3,4,7] has incorrect analytic properties (complex poles on the physical sheet), as noted by the authors themselves.

We solve the problem in this paper at zero temperature, using equations obtained on the basis of the analyticity and unitarity properties. Such an approach to the problem was already used earlier by Suhl[6,7]. However, the expressions obtained in the present paper for the scattering amplitude are internally consistent at all energies, unlike Suhl's results[7]. In particular, there are no complex poles on the physical sheet.

We begin with consideration of the model problem in which an ideal Fermi gas at $T = 0$ interacts with a pointlike impurity. At the end of the article we shall discuss briefly the question of applicability of the obtained results to a real system of interacting electrons (Fermi liquid).

We must clarify first the analytic properties of the scattering amplitude. To do so we must derive for the scattering amplitude formulas analogous to the formulas of Lehmann, Zimmerman, and Symanzik in field theory[8] (see also[6]).

When $t \to \pm\infty$, the electron creation operators can be represented in the form

$$a^+_{k\alpha} = V^{-1/2}\int d\mathbf{x}\, e^{i\mathbf{kx}-iE_k t}\, u_\alpha(\nu)\,\psi_\nu^+(\mathbf{x}), \qquad (1)$$

where V is the volume of the system. A similar formula holds obviously also for the annihilation operators. With the aid of these formulas we obtain for the S-matrix elements corresponding to electron scattering

$$\begin{aligned}\langle M'\mathbf{k}'\alpha'|S|\mathbf{k}\alpha M\rangle &= \langle M'|a_{k'\alpha'}Sa^+_{k\alpha}|M\rangle\\ &= V^{-1/2}\int d\mathbf{x}\, e^{i\mathbf{kx}-iE_k t}u_\alpha(\nu)\langle M'|a_{k'\alpha'}S\psi_\nu^+(\mathbf{x})\,|M\rangle|_{t=-\infty}\\ &= -V^{-1/2}\int d^4x\,\frac{\partial}{\partial t}\,e^{i\mathbf{kx}-iE_k t}u_\alpha(\nu)\langle M'|a_{k'\alpha'}T(S\psi_\nu^+(x))\,|M\rangle\\ &+\langle M'|a_{\alpha' k'}a^+_{k\alpha}S|M\rangle = \delta_{\alpha'\alpha}\delta_{M'M}\delta_{\mathbf{k}'\mathbf{k}}\\ &-\frac{i}{V}\int d^4x\, e^{ikx}\left(i\frac{\partial}{\partial t}+H_0\right)_x\langle M'|a_{k'\alpha'}T(S\psi_\alpha^+(x))\,|M\rangle.\end{aligned} \qquad (2)$$

Here α and M are the projections of the electron spin and of the impurity spin before the scattering, while α' and M' are the same quantities after scattering; $H_0 = -\nabla^2/2m$ (we shall henceforth put $m = 1/2$), and $k, k' > k_F$, where k_F is the Fermi momentum. A similar procedure is applied also to the operator $a_{k'\alpha'}$; as a result we get:

$$\begin{aligned}\langle M'\mathbf{k}'\alpha'|S|M\mathbf{k}\alpha\rangle &= \delta_{\mathbf{k}'\mathbf{k}}\delta_{\alpha'\alpha}\delta_{M'M}\\ &+V^{-1}\int d^4x d^4y e^{ikx-ik'y}\left(i\frac{\partial}{\partial x_0}+H_0\right)_x\left(i\frac{\partial}{\partial y_0}-H_0\right)_y\\ &\times\langle M'|T(S\psi_{\alpha'}(y)\psi_\alpha^+(x))\,|M\rangle.\end{aligned} \qquad (3)$$

A similar formula can be obtained also for the

matrix element of hole scattering.

We shall assume that the impurity is located at the origin. Outside the action range of the impurity potential, the operators ψ and ψ^+ satisfy the free equations of motion in the Heisenberg representation. Therefore, taking into account the fact that the radius of the impurity is equal to zero, we can neglect the dependence on the vectors $\mathbf{k}$ and $\mathbf{k}'$ in the expressions for the electron and hole scattering matrix elements. As a result, these formulas can be written in the form

$$\langle M'\mathbf{k}'\alpha'|S|M\mathbf{k}\alpha\rangle = \delta_{\alpha'\alpha}\delta_{M'M}\delta_{\mathbf{k}'\mathbf{k}} + 2\pi i\delta(E_k - E_{k'})\frac{4\pi}{V}T^{M'M}_{\alpha'\alpha}(E_k), \quad E_k > E_F; \tag{4a}$$

$$\langle M'\mathbf{k}'\alpha'|S|M\mathbf{k}\alpha\rangle = -\delta_{\alpha'\alpha}\delta_{M'M}\delta_{\mathbf{k}'\mathbf{k}} + 2\pi i\delta(E_{k'} - E_k)\frac{4\pi}{V}T^{M'}_{\alpha\alpha'}(E_k), \quad E_k < E_F; \tag{4b}$$

$$T^{M'M}_{\alpha'\alpha}(E) = \frac{i}{4\pi}\int d\mathbf{x}\,dy\,dt\,e^{iEt}\left(i\frac{\partial}{\partial t} - H_0\right)_x\left(i\frac{\partial}{\partial t} - H_0\right)_y \times\langle M'|T\psi_{\alpha'}(\mathbf{y}, t)\psi_\alpha(\mathbf{x}, 0)|M\rangle. \tag{4c}$$

In the expression for T we have gone over in the usual fashion from the operators in the interaction representation to the Heisenberg operators. The factor $4\pi/V$ is introduced in order that the matrix T represent the usually normalized scattering amplitude; the correctness of such a normalization can be readily verified by considering the first perturbation-theory approximation for the S matrix. We note that in formula (4b) the sequence of the initial and final electron spin indices is different than in (4a). Therefore when $E < E_F$ the scattering is described not by the matrix T(E), but by the matrix $T^T(E)$, where the transposition sign pertains only to the spin indices of the electron.

In order to clarify the analytic properties of the matrix T as a function of the energy, it is necessary to obtain for it a representation similar to the Lehmann representation for the single-particle Green's function (see, for example, the book by Abrikosov, Gor'kov, and Dzyaloshinskiĭ[9]). Such a representation is of the form[1)]

$$T^{M'M}_{\alpha'\alpha}(E) = \frac{1}{\pi}\int_0^{E_F} dE'\frac{\rho^{M'M}_{\alpha'\alpha}(E')}{E' - E + i\delta} + \frac{1}{\pi}\int_{E_F}^{\infty} dE'\frac{\rho^{M'M}_{\alpha'\alpha}(E')}{E' - E - i\delta},$$

$$\rho^{M'M}_{\alpha'\alpha}(E')\,dE' = \begin{cases} \frac{1}{4}\sum_n \langle M'|j_{\alpha'}(E')|n\rangle\langle n|j_\alpha^+(E')|M\rangle, & E > E_F \\ \frac{1}{4}\sum_n \langle M'|j_\alpha^+(E')|n\rangle\langle n|j_{\alpha'}(E')M\rangle, & E < E_F \end{cases} \tag{5}$$

Here

$$j_\alpha(E') = \int d\mathbf{x}\,(E' - H_0)\psi_\alpha(\mathbf{x}), \quad E' < E_n < E' + dE'.$$

Expression (5) differs from that obtained by Suhl[5,6] in that it has no constant term, which in our normalization is equal to the Born scattering amplitude. The reason for this difference is as follows. In the derivation of (5) we have interchanged the order of integration with respect to the coordinates and the summation over the intermediate states. Since this sum converges very slowly (like $\int dk\, k^{-1}\exp[ikx]$), such a permutation is not valid. However, the difference between the two expressions—with the correct and with the rearranged sequence of integration and summation—is equal to a constant, which cancels out the constant occurring when the derivatives with respect to time are introduced under the sign of the T-product. The correctness of this statement can be readily verified by considering scattering in the absence of the Fermi sphere. Inasmuch as we are dealing with large intermediate-state energies, the presence of the Fermi sphere cannot change the result. We note also that the amplitude (5) decreases when $E \to \infty$, which agrees with the results of the usual theory of scattering by a pointlike center—scattering of neutrons by protons in the zero-radius approximation (see, for example, the book by Akhiezer and Pomeranchuk[10]).

We write the scattering amplitude in the form $T = A + B(\mathbf{S}\cdot\boldsymbol{\sigma})$. Then there exist for the functions A and B representations which are obtained from (5) by replacing the matrix ρ by the functions $\rho_A = (1/2)\mathrm{Sp}\,\rho$ and $\rho_B = [2S(S+1)]^{-1}\mathrm{Sp}(\mathbf{S}\cdot\boldsymbol{\sigma}\rho)$. Using the definition of ρ, we can readily verify that the function ρ_A is not negative, and the function ρ_B is real. By virtue of the foregoing, the functions A and B are analytic in the plane of the complex variable E with a cut along the real axis from zero to infinity, their values on the lower edge of the cut being the complex conjugates of the values of the upper edge. These functions assume physical values on the upper edge of the cut if $E > E_F$, and on the lower one if $E < E_F$.

We now proceed to the unitarity condition $S + S = 1$. Let $E > E_F$; then by virtue of (4)

[1)] Generally speaking, (5) can have poles when $E < 0$. However, the presence of such poles should not affect the phenomena that occur near the Fermi surface.

$$i(T^+ - T) = T^+T.$$

In the right side of this equation is the sum over the intermediate states, which contain, besides one electron, an arbitrary number of electron-hole pairs; all the particles in the intermediate states are real, i.e., their energies are connected with the momentum in the usual fashion: $E = p^2$. If we assume now that the matrix elements of the T-matrix for transitions from the single-particle state to the many-particle states are finite[2)], then the contribution of the many-particle states to the unitarity condition vanishes when the energy of the scattered particle approaches the Fermi energy, owing to the vanishing of the statistical weights of the intermediate states (for more details see the paper by Migdal[11]).

We are interested in phenomena that occur near the Fermi surface. It is therefore reasonable to neglect the contribution of the many-particle states to the unitarity condition. Then for $E > E_F$, using (4), we obtain

$$\begin{aligned} \operatorname{Im} A &= k(|A|^2 + S(S+1)|B|^2), \\ \operatorname{Im} B &= k(A^*B + AB^* - |B|^2), \end{aligned} \tag{6}$$

where $k = \sqrt{E}$; in the derivation of these equations we used the fact that $(\mathbf{S}\cdot\boldsymbol{\sigma})^2 = S(S+1) - (\mathbf{S}\cdot\boldsymbol{\sigma})$.

As already mentioned, when $E < E_F$ the scattering amplitude is the matrix $T^T = A + B(\mathbf{S}\cdot\boldsymbol{\sigma}^T)$, and therefore the unitarity relations become

$$\begin{aligned} \operatorname{Im} A &= -k(|A|^2 + S(S+1)|B|^2), \\ \operatorname{Im} B &= -k(A^*B + AB^* + |B|^2). \end{aligned} \tag{7}$$

In the derivation of (7) we used the equality $(\mathbf{S}\cdot\boldsymbol{\sigma}^T)^2 = S(S+1) + (\mathbf{S}\cdot\boldsymbol{\sigma}^T)$, which can be readily obtained by recognizing that $-\boldsymbol{\sigma}^T$ is obtained from with the aid of the unitary transformation $-\boldsymbol{\sigma}^T = \sigma_y\boldsymbol{\sigma}\sigma_y$. Expressions (7) have been written out for the lower edge of the cut ($E = E - i\delta$). On the other hand, $A^*, B^*(E - i\delta) = A, B(E + i\delta)$, so that we get on the upper edge

$$\begin{aligned} \operatorname{Im} A &= k(|A|^2 + S(S+1)|B|^2), \\ \operatorname{Im} B &= k(A^*B + AB^* - |B|^2\varepsilon(E - E_F)). \end{aligned} \tag{8}$$

Unless otherwise stipulated, we shall henceforth assume that $E = E + i\delta$.

We introduce in lieu of A and B the quantities $\alpha_\pm$, which for $E > E_F$ are the scattering amplitudes in states with total angular momentum $J = S \pm 1/2$:

$$\alpha_+ = A + BS, \quad \alpha_- = A - B(S+1). \tag{9}$$

The unitarity relations for these quantities are

$$\begin{aligned} \operatorname{Im}\alpha_+ &= k\Big\{ |\alpha_+|^2 + \frac{2S}{(2S+1)^2}[|\alpha_+|^2 + |\alpha_-|^2 \\ &\quad - \alpha_+^*\alpha_- - \alpha_+\alpha_-^*]\theta(E_F - E)\Big\} \\ \operatorname{Im}\alpha_- &= k\Big\{ |\alpha_-|^2 - \frac{2(S+1)}{(2S+1)^2}[|\alpha_+|^2 + |\alpha_-|^2 \\ &\quad - \alpha_+^*\alpha_- - \alpha_+\alpha_-^*]\theta(E_F - E)\Big\}. \end{aligned} \tag{10}$$

We shall use a device proposed by Froissart[12]; we seek $\alpha_\pm$ in the form

$$\alpha_\pm = (2ik)^{-1}(\eta_\pm e^{2i\delta_\pm} - 1). \tag{11}$$

We then get from (10)

$$\begin{aligned} \eta_\pm^2 &= 1 \mp \frac{2(S + 1/2 \mp 1/2)}{(2S+1)^2}[\eta_+^2 + \eta_-^2 \\ &\quad - 2\eta_+\eta_-\cos 2(\delta_+ - \delta_-)]\theta(E_F - E). \end{aligned} \tag{12}$$

As expected, when $E > E_F$ we have $\eta_\pm^2 = 1$. When $E < E_F$ it follows from (12), in particular, that

$$(S+1)\eta_+^2 + S\eta_-^2 = 2S + 1. \tag{13}$$

With the aid of the functions $\eta_\pm$ we can write the amplitudes $\alpha_\pm$ in the form

$$\begin{aligned} \alpha_\pm &= \frac{1}{2ik}\{\exp[2i(\nu_\pm + \varphi_\pm)] - 1\}, \\ \varphi_\pm &= -\frac{k}{4\pi}\int_0^{E_F} dE' \frac{\ln\eta_\pm^2}{k'(E' - E)} = \varphi_\pm' - \frac{i}{2}\ln\eta_\pm, \\ &\qquad \nu_\pm + \varphi'_\pm = \delta_\pm. \end{aligned} \tag{14}$$

Let us set up the combinations[12]

$$\alpha_\pm' = \frac{1}{2ik}[(2ik\alpha_\pm + 1)e^{-2i\varphi_\pm} - 1] = \frac{1}{2ik}(e^{2i\nu_\pm} - 1). \tag{15}$$

By virtue of the fact that $\nu_\pm$ is real, we have for $\alpha_\pm'$ the equalities $\operatorname{Im}\alpha_\pm' = k|\alpha_\pm'|^2$. When $E > 0$, the $\alpha_\pm'$ have no singularities other than holes and the cut along the real axis.

Functions possessing such properties were investigated in detail by Gribov, Zel'dovich, and

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2)It is essential in what follows that ImB takes on different forms when $E > E_F$ and when $E < E_F$. As follows from the results (32) and (35) below, as $E \to E_F$ we get $B \sim (\ln|E - E_F|)^{-1}$, i.e., this difference vanishes when $E \approx E_F$. The contribution of the many-particle intermediate states also leads to different forms of ImA and ImB when $E > E_F$ and $E < E_F$. It is therefore legitimate to neglect this contribution if it approaches zero as $E \to E_F$ more rapidly than $\ln^{-2}|E - E_F|$. In the case when the matrix elements of the transition from the single-particle state to the many-particle states are bounded, this is certainly the case[11]. A situation in which the contribution from the many-particle states can not be neglected appears little likely to us.

Perelomov[13]. Their most general form is

$$\alpha_\pm' = -\left(-\frac{1}{a_\pm} + ik + c_\pm E + \sum_m \frac{R_m^{(\pm)}}{E_m^{(\pm)} - E}\right)^{-1}, \quad (16)$$

where $a_\pm$ and $E_m^{(\pm)}$ are real, and $c_\pm$ and $R_m^{(\pm)}$ are positive. We now recall that when $E_F = 0$ the amplitudes for the scattering by a pointlike potential take the form[10]

$$\alpha_\pm^{(0)} = a_\pm^{(0)} (1 - ika_\pm^{(0)})^{-1}. \quad (17)$$

When $E \gg E_F$, Eq. (16) should go over into (17), since the phases $\varphi_\pm'$ decrease and therefore $c_\pm = 0$. We cannot forbid the appearance of poles; this, however, seems extremely unlikely to us, and we therefore assume that

$$\alpha_\pm' = a_\pm(1 - ika_\pm)^{-1}, \qquad \nu_\pm = \operatorname{arctg} ka_\pm. \quad (18)$$

It is obvious here that if $k_F|a_\pm^{(0)}| \ll 1$ then $a_\pm \approx a_\pm^{(0)}$.

This question is discussed in detail later.

We set up, following Suhl[7], the function

$$u = \frac{1 + 2ikA}{B} = \frac{2S + 1 + 2ik[(S+1)\alpha_+ + S\alpha_-]}{\alpha_+ - \alpha_-}. \quad (19)$$

By virtue of the unitarity conditions (8) and (10), we get the equality

$$u - u^* = 2ik\varepsilon(E - E_F), \quad (20)$$

which enables us to represent the function u in the form

$$u(E) = P(E) + ik - \frac{4k^2}{\pi}\int_0^{k_F} \frac{dk'}{k'^2 - E}$$

$$= P(E) + ik\varepsilon(E - E_F) + I_0,$$

$$I_0 = -\frac{2k}{\pi}\ln\left|\frac{k_F - k}{k_F + k}\right|. \quad (21)$$

Here P(E) is a rational function of E, satisfying the only condition $P^*(E) - P(E) = 0$ on the entire real axis.

With the aid of (14) and (19) we readily obtain the equality

$$\exp 2i(\varphi_- - \varphi_+) = \exp 2i(\nu_+ - \nu_-)\frac{u - 2ik(S+1)}{u + 2ikS}, \quad (22)$$

from which it follows that

$$\frac{\eta_-^2}{\eta_+^2} = \left|\frac{u - 2ik(S+1)}{u + 2ikS}\right|^2. \quad (23)$$

It is easy to verify, using the definition (21) of u, that when $E > E_F$ we get $\eta_-^2 = \eta_+^2$, as should be the case. When $E < E_F$, taking (13) into account, we obtain

$$\eta_\pm^2 = \frac{(P + I_0)^2 + (2S + 1 \mp 2)^2 E}{(P + I_0)^2 + (2S+1)^2 E}. \quad (24)$$

Let us determine now the function P(E). When $E_F = 0$, taking (17) and (19) into account, we get

$$P_0(E) = (2S+1)\frac{1 + a_+^{(0)}a_-^{(0)}E}{a_+^{(0)} - a_-^{(0)}}. \quad (25)$$

Let us assume that $P(E) = P_1 + P_2E$ also when $E_F \neq 0$. Then the constants P_1 and P_2 can be determined with the aid of (18) and (22). This is done in the appendix, where it is shown that if $k_F|a_\pm| \ll 1$, we get

$$P(E) \approx (2S+1)\frac{1 + a_+a_-E}{a_+ - a_-} - \frac{4k_F}{\pi}. \quad (26)$$

As a result we obtain a solution of the problem in the form

$$\alpha_\pm = \frac{1}{2ik}\left[\frac{1 + ika_\pm}{1 - ika_\pm}e^{2i\varphi_\pm} - 1\right], \quad (27)$$

$$\varphi_\pm = -\frac{k}{4\pi}\int_0^{E_F}\frac{dE' \ln \eta_\pm^2(E')}{k'(E' - E - i\delta)},$$

$$\eta_\pm^2 = \left[\left(1 - \frac{4k_Fb}{\pi} + a_+a_-E - \frac{2kb}{\pi}\ln\frac{k_F - k}{k_F + k}\right)^2 + (2S + 1 \mp 2)^2 b^2 E\right]\left[\left(1 - \frac{4k_Fb}{\pi} + a_+a_-E - \frac{2kb}{\pi}\ln\frac{k_F - k}{k_F + k}\right)^2 + (2S+1)b^2E\right]^{-1}.$$

Here $b = (2S+1)^{-1}(a_+ - a_-)$. The functions $\varphi_\pm$ are regular in the entire complex E plane, with a cut along the real axis from zero to E_F. In particular, they are bounded for all complex E, and therefore the scattering amplitudes cannot have any singular points, including poles, for complex E on the physical sheet.

The logarithm which enters in $\eta_\pm^2$ becomes infinite when $E = E_F$, and therefore the amplitudes $\alpha_\pm$ cannot be expanded in powers of b. However, making such an expansion formally, we obtain for $E > E_F$

$$\alpha_\pm = a_\pm + ika_\pm^2 \pm \frac{2b^2}{\pi}\left(S + \frac{1}{2} \mp \frac{1}{2}\right)\int_0^{E_F}\frac{dE'k'}{E' - E}. \quad (28)$$

These expressions coincide exactly with the results obtained in second order of perturbation theory, if $a_\pm$ and b are replaced by $a_\pm^{(0)}$ and $b^{(0)}$, and if the interaction potential is chosen in the form

$$V(\mathbf{r}) = -4\pi(a_+^{(0)}P_+ + a_-^{(0)}P_-)\delta(\mathbf{r})\frac{\partial}{\partial r}r,$$

$$P_+ = (2S+1)^{-1}(S + 1 + (\mathbf{S}\sigma)),$$

$$P_- = (2S+1)^{-1}(S - (\mathbf{S}\sigma)), \quad (29)$$

where the factor $(\partial/\partial r)r$ has been added to elimin-

ate the divergences at large momenta in the intermediate states. In this form, the potential takes correct account of the boundary conditions at zero for a zero effective radius of the forces[14]. When $E_F = 0$, the expressions obtained with its aid for the amplitudes coincide with (17). From a comparison with perturbation theory it is natural to conclude that the amplitudes $a_\pm$ coincide with $a_\pm^{(0)}$ at least accurate to terms of order $(k_F a_\pm)^2$.

We now proceed to a more detailed analysis of our solution. We are interested in the energy region near the Fermi surface. We note first that according to (27) $\eta_-^2(S) = \eta_+^{-2}(S+1)$ and $\varphi_-(S) = -\varphi_+(S+1)$. It is therefore sufficient to investigate only one of the phases, say φ_+; the final expressions will be written out throughout for both phases. It is necessary to consider separately the two cases $b < 0$ and $b > 0$.[3)] The greatest interest attaches to the first case, with which we begin.

The logarithm under the integral sign in the expression for φ_+ cannot be expanded in powers of b and of a_+a_- only in a narrow region of energies

$$|E' - E_F| \sim E_F \exp(-\pi/2k_F|b|);$$

outside this region, the integrand is small, and the corresponding contribution to φ_+ can be neglected. As a result we obtain for $\varphi_+' = \mathrm{Re}\,\varphi_+$ the approximate expression

$$\varphi_+'(E) = -\frac{1}{4\pi}\int_0^{E_F}\frac{dE'}{E'-E} \times \ln\frac{\left(1+g\ln\frac{E_F-E'}{E_F}\right)^2+(2S-1)^2\left(\frac{\pi g}{2}\right)^2}{\left(1+g\ln\frac{E_F-E'}{E_F}\right)^2+(2S+1)^2\left(\frac{\pi g}{2}\right)^2},$$

$$g = \frac{2k_F|b|}{\pi}\left[1-\frac{4k_F|b|}{\pi}(1-\ln 2)\right]^{-1}. \qquad (30)$$

In the derivation of (30) we have allowed an error of the order of $E_F b^2$, as can be readily verified by considering the difference between the exact expression for φ_+' and the approximate one. As will be shown later, when $E_F \gg |E - E_F| \gg E_F \exp(-1/g)$, Eq. (30) contains a term proportional to $E_F b^2 \ln|E - E_F|$, which is not contained in the aforementioned difference, since the integrand in this difference is finite when $E = E_F$. Therefore terms of the type $E_F b^2 \ln|E_F - E|$ must be taken into account in the calculation of (30).

[3)] By virtue of (29), the sign of b coincides with the sign of the exchange integral defined in the customary manner.

By making a change of variable, Eq. (30) can be transformed to

$$\varphi_+'(E) = \frac{1}{4\pi g}\int_0^\infty \frac{dy}{\varepsilon e^{y/g}+1}\ln\frac{(1-y)^2+(1/2\,\pi g)^2(2S-1)^2}{(1-y)^2+(1/2\,\pi g)^2(2S+1)^2}, \qquad (31)$$

where $\epsilon = (E - E_F)E_F^{-1}$. When $\epsilon = 0$, this integral can be readily calculated and is equal to $-\pi/2 - Sbk_F$, i.e., the phases $\varphi_\pm'$ are close to $\mp\pi/2$. It can be shown that when $|\epsilon| \ll \exp(-1/g)$

$$\varphi_+' \approx -\pi/2 - Sbk_F - \pi(\ln|\varepsilon|)^{-1},$$
$$\varphi_-' \approx \pi/2 + (S+1)bk_F + \pi(\ln|\varepsilon|)^{-1}. \qquad (32)$$

The large value of the phase at small ϵ is connected with the fact that when $y \approx 1$ the logarithm under the integral sign in (31) has a peak with a width of the order of $k_F|b|$. It is also obvious that if the factor $(\epsilon\exp[+y/g] + 1)^{-1}$ is small in the region of this peak, then the phase is also small. If $|\epsilon| \gg \exp(-1/g)$, then the region of the peak makes an exponentially small contribution to the integral, and the logarithm can be expanded in powers of g^2 and y; as a result we get for $|\epsilon| \gg \exp(-1/g)$

$$\varphi_+' \approx \frac{2E_F b^2 S}{\pi}\ln|\varepsilon|; \quad \varphi_-' \approx -2E_F b^2(S+1)\frac{1}{\pi}\ln|\varepsilon|. \qquad (33)$$

The scattering amplitude calculated with the aid of (33) coincides with that obtained by perturbation theory.

The behavior of the phases $\varphi_\pm'$ is shown qualitatively in Fig. 1. The phases $\varphi_\pm'$, which are small when $|\epsilon| \gg \exp(-1/g)$, increase with decreasing $|\epsilon|$ and reach extremal values close to $\mp\pi/2$ when $\epsilon = 0$. With this, in the entire range of variation of ϵ we have $\varphi_+' < 0$ and $\varphi_-' < 0$. This behavior of the phases has nothing in common with their behavior at resonance, which is then described by the Breit-Wigner formula.

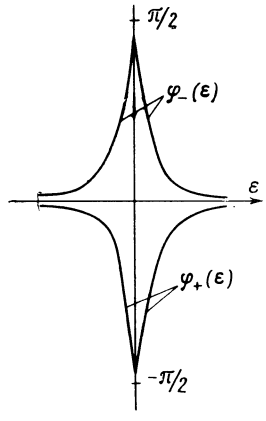

FIG. 1.

At the same time, when $\epsilon = 0$ the scattering amplitudes $\alpha_\pm$ turn out to be close to their maximum values, equal to i/k_F, just as in ordinary resonance. The scattering cross section turns out in this case to be equal simply to $4\pi k_F^{-2}$, whereas when $|E_F - E| \gg E_F \exp(-1/g)$ we have

$$\sigma \approx 4\pi \frac{(S+1)a_+^2 + Sa_-^2}{2S+1} = 4\pi[a^2 + S(S+1)b^2],$$

$$a = (2S+1)^{-1}[a_+(S+1) + a_-S],$$

$$b = (2S+1)^{-1}(a_+ - a_-); \qquad (34)$$

a and b coincide, with accuracy to terms of order $(k_F a_\pm)^2$, respectively with the spin-independent and spin-dependent parts of the Born scattering amplitude.

We now proceed to the case $b > 0$. For the phase φ_+' we obtain now a formula similar to (31), but in which the sign in front of y under the logarithm sign is reversed. As a result, the logarithm can be expanded in powers of $E_F f^2$ for all values of y, and simple calculations yield the following formulas for the phases at small values of ϵ:

$$\varphi_+' \approx -Sbk_F - S\pi(2\ln|\varepsilon|)^{-1},$$

$$\varphi_-' \approx (S+1)bk_F + (S+1)\pi(2\ln|\varepsilon|)^{-1}, \qquad (35)$$

and the expressions for the phases coincide with (33) for large ϵ ($|\epsilon| \gg \exp(-1/g)$).

In the case just considered, the phases are small compared with $\pi/2$, but are nevertheless anomalously large (their order of magnitude at $\epsilon \approx 0$ is bk_F and not $(bk_F)^2$). However, the cross section for $\epsilon = 0$ is no longer a maximum, as in the preceding case, but a minimum; namely, for $\epsilon = 0$ we have

$$\sigma \approx 4\pi\left[\frac{(S+1)a_+^2 + Sa_-^2}{2S+1} - S(S+1)b^2\right] = 4\pi a^2,$$

whereas for large ϵ it is determined, as before, by (34).

Inasmuch as we are dealing with s-scattering, the cross sections calculated by us coincide with the "transport" cross sections, knowledge of which is essential in the calculation of the resistance. However, our formulas can be used only at low temperatures ($T \ll E_F \exp(-1/g)$). It is clear here that when $b < 0$ the contribution to the resistance from the scattering by impurities is maximal, and when $b > 0$ it is minimal.

The results obtained are applicable practically in their entirety to real systems of interacting electrons (Fermi liquid). Indeed, the following three circumstances, which also take place in Fermi liquids, were essential to us: 1) single-particle spectrum, 2) analytic properties of the scattering amplitude, 3) different forms of the unitarity conditions for $E > E_F$ and $E < E_F$. It is merely necessary to remember here that, owing to the interaction between the electrons, the quantities $a_\pm$ and b are not related with $a_\pm^{(0)}$ and $b^{(0)}$ as simply as in our case. Moreover, in the unitarity conditions it is necessary now to replace k by ka', where a' is a renormalization constant in the pole term of the single-electron Green's function[9]. However, by introducing the renormalized scattering amplitude $T' = a'^{-1}T$, the unitarity relations are reduced to the form (8).

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APPENDIX

The purpose of this appendix is to determine the constants P_1 and P_2 in the expression for $P = P_1 + P_2 E$ with the aid of Eqs. (18) and (22).

We note first that (22) can be rewritten in the form

$$\exp\left\{2i\left[\nu_- - \nu_+ - \frac{k}{4\pi}\int_0^{E_F} \frac{dE' \ln|z|^2}{k'(E'-E-i\delta)}\right]\right\} = z(E+i\delta), \qquad (A.1)$$

$$z = \frac{u - 2ik(S+1)}{u + 2ikS}$$

$$= \left[P - ik(2S+1) - \frac{4k^2}{\pi}\int_0^{k_F} \frac{dk'}{k'^2 - k^2 - i\delta}\right]$$

$$\times\left[P + ik(2S+1) - \frac{4k^2}{\pi}\int_0^{k_F} \frac{dk'}{k'^2 - k^2 - i\delta}\right]^{-1} = \frac{\xi_+}{\xi_-}. \qquad (A.2)$$

The function z is an analytic function of the complex variable E with a cut along the real axis from zero to infinity. Its values on the different edges of the cut are complex conjugates, and furthermore, when $E > E_F$ its modulus is unity. Because of the latter circumstance, we have $\ln|z|^2 = 0$ when $E > E_F$, and the integral in the argument of the exponential in (A.1) can be written in the form of an integral along the contour C_1 (see Fig. 2).

As a result we get

$$-\frac{2ik}{4\pi}\int_0^{E_F} \frac{dE' \ln|z|^2}{k'(E'-E-i\delta)} = \frac{k}{2\pi i}\int_{C_1} \frac{dE' \ln z}{k'(E'-E-i\delta)}$$

$$= \ln z(E+i\delta) + \frac{k}{2\pi i}\int_{C_2} dE' \frac{\ln z}{k'(E'-E)}. \qquad (A.3)$$

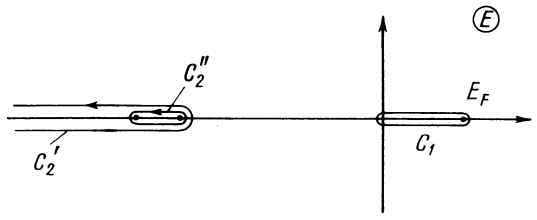

FIG. 2.

The contour C_2 circles in a positive direction around all the singular points of the function ln z, except those lying on the positive part of the real axis. Expression (A.3) together with (A.1) and (18) leads to the equation

$$2i(\operatorname{arctg} ka_+ - \operatorname{arctg} ka_-) - \frac{k}{2\pi i}\int_{C_2}\frac{dE' \ln z}{k'(E'-E)} = 2\pi i n, \tag{A.4}$$

where n is an arbitrary integer.

The singular points ln z of interest to us obviously coincide with the zeroes of the functions lying on the physical sheet:

$$\zeta_\pm = P_1 + P_2 E \mp (2S+1)\frac{E}{\pi}\int_0^\infty \frac{dE'}{k'(E'-E)} - \frac{4E}{\pi}\int_0^{k_F}\frac{dk'}{k'^2-k^2}$$

$$= P_1 + P_2 E \pm (2S+1)\sqrt{-E} + \frac{4\sqrt{-E}}{\pi}\operatorname{arctg}\frac{k_F}{\sqrt{-E}}. \tag{A.5}$$

It is natural to assume, inasmuch as $k_F|a_\pm| \ll 1$, that

$$P_1 \approx 1/b, \qquad P_2 \approx a_+a_-/b,$$

and if $\sqrt{|E|}\,|a_\pm| \ll 1$ the last term in (A.5) can be neglected in first approximation; we obtain therefore the following values for the roots of the functions $\zeta_\pm$:

$$(\sqrt{-E})_+ \approx \begin{cases} a_-^{-1} \\ -a_+^{-1}, \end{cases} \qquad (\sqrt{-E})_- \approx \begin{cases} -a_-^{-1} \\ a_+^{-1}. \end{cases} \tag{A.6}$$

Further, since we have $-\sqrt{E} > 0$ on the physical sheet when $E < 0$, we are interested only in the positive roots of (A.6). For all possible signs of a_+ and a_-, there are two such roots. Using the known theorem concerning the number of roots, we can easily show that the functions $\zeta_\pm$ have no other roots on the physical sheet. By altering slightly the values of the constants P_1 and P_2, we can make the positive roots of (A.6) to coincide exactly with the quantities $|a_+|^{-1}$ and $|a_-|^{-1}$. With this, in first order in k_F, only P_1 changes. ($P_1 \approx 1/b - 4k_F/\pi$; this value of P_1 was used in (26).)

Thus, the contour C_2 should encompass two singular points corresponding to positive roots (A.6) lying on the real axis when $E < 0$. If both roots pertain to one of the functions $\zeta_\pm$, then the contour C_2 should have the form C_2' (see Fig. 2), and if it pertains to the functions ζ_+ and ζ_-, the proper contour is C_2''. The contour integral obtained in this fashion can be readily evaluated by parts. Thus, for example, if $a_+ < 0$, $a_- < 0$, and $a_+ < a_-$, then

$$\frac{k}{2\pi i}\int_{C_2''}\frac{dE' \ln z}{k'(E'-E)} = \frac{\sqrt{E}}{2\pi i}\int_{C_2''} dE' \frac{\ln \zeta_+(E') - \ln \zeta_-(E')}{k'(E'-E)}$$

$$= -2i\frac{\sqrt{E}}{2\pi i\sqrt{E}}\operatorname{arctg}\sqrt{\frac{-E'}{E}}\ln\frac{\zeta_+(E')}{\zeta_-(E')}\Big|_{\text{contour } C_2''}$$

$$+\frac{1}{\pi}\int_{C_2''} dE' \operatorname{arctg}\sqrt{\frac{-E'}{E}}\left[\frac{\zeta_+'(E')}{\zeta_+(E')} - \frac{\zeta_-'(E')}{\zeta_-(E')}\right]$$

$$= 2i\left(\operatorname{arctg}\frac{1}{k|a_-|} - \operatorname{arctg}\frac{1}{k|a_+|}\right). \tag{A.7}$$

It is easy to verify that substitution of (A.7) in (A.4) yields zero. It is easy to verify in similar fashion that Eq. (A.4) is satisfied also for all other possible signs of $a_\pm$.

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