ELECTRICAL CONDUCTIVITY OF ONE-DIMENSIONAL SYSTEMS

Yu. A. BYCHKOV and A. M. DYKHNE

Institute of Theoretical Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor July 12, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1914–1917 (December, 1966)

We consider in an exact treatment the problem of calculating the electrical conductivity (as function of frequency) in a one-dimensional system of randomly distributed δ-function-like scattering. The averaging over an ensemble characterized by a distribution function for the distances between neighboring scatterers is done explicitly. We obtain an integral equation such that the electrical conductivity can be expressed in terms of its solution through quadrature. The equation obtained makes it possible to average also other quantities, in particular the Green function.

IT is well known that the evaluation of kinetic coefficients in the case when the wavelength of the particles is comparable or large compared with the mean free path encounters appreciable difficulties. It turns out that in a one-dimensional model which has already been used by the authors in an earlier paper [1] one can find an exact solution of the problem of finding the electrical conductivity. In the model considered, the particles do not interact with one another but are scattered by impurities, the potential of which is taken in the form of δ-functions. Halperin [2] has considered an especially particular case of this model (the so-called "white noise" model) by a different method.

We shall start from the well-known expression for the real part of the conductivity:

$$\mathrm{Re}\,\sigma(\omega) = -\frac{\pi e^2}{\omega m^2}\int dE \sum_{n,m} |p_{nm}|^2 \delta(E_n - E') \times \delta(E_m - E)[f(E') - f(E)]. \quad (1)$$

Here p_{nm} is a matrix element of the momentum operator taken between exact eigenstates for the case where there is no electric field, $f(E)$ the equilibrium energy distribution function, and $E' = E + \omega$. We note that we can find the chemical potential μ by using the results of [1] (we put $\hbar = 1$).

As a result of rather elementary transformations Eq. (1) can be rewritten as follows:

$$\mathrm{Re}\,\sigma(\omega) = -\frac{\pi e^2}{m^2\omega}\int dE \Phi(E,E')[f(E') - f(E)],$$

$$\Phi(E,E') = \frac{ik^2}{(2\pi)^2\omega^2}\int_{-\infty}^{\infty} dp\,\delta(E_p - E)[A_{pp}(E') - A_{pp}^*(E')]$$

$$+ \frac{ik'^2}{(2\pi)^2\omega^2}\int_{-\infty}^{\infty} dp\,\delta(E_p - E')[A_{pp}(E) - A_{pp}^*(E)]$$

$$+ \frac{1}{(2\pi)}\int\int_{-\infty}^{\infty} p_1 p_2\, dp_1\, dp_2 [G_{p_1}^{(0)}(E) G_{p_2}^{(0)}(E) A_{p_1p_2}(E) - G_{p_1}^{(0)*}(E) G_{p_2}^{(0)*}(E) A_{p_2p_1}(E)][G_{p_1}^{(0)}(E') G_{p_2}^{(0)}(E') A_{p_2p_1}(E') - G_{p_1}^{(0)*}(E') G_{p_2}^{(0)*}(E') A_{p_1p_2}^*(E')], \quad (2)$$

where

$$k^2 = 2mE, \quad k'^2 = 2mE', \quad E_p = p^2/2m,$$

$G_p^{(0)}(E) = (E_p - E - i\delta)^{-1}$ is the free Green function of a particle, and the exact Green function at frequency E is equal to

$$G_{p_1p_2}(E) = \delta_{p_1p_2} G_{p_1}^{(0)}(E) - G_{p_1}^{(0)}(E) A_{p_1p_2}(E) G_{p_2}^{(0)}(E). \quad (3)$$

Equations (2) and (3) are valid in the general case (their form is independent of the dimensionality of space and of the form of the potential). In the one-dimensional case and for δ-function-like potentials we can obtain recurrence relations for the scattering amplitudes $A_{p_1p_2}(E)$; we shall now derive those. To do this we number the impurities in order of increasing coordinates ($x_n > x_{n-1}$) and we denote by $G_n(xx')$ the particle Green function when the n impurities on its left are absent (while the total number of impurities is finite and equal to N and the length of the system equal to L). We choose, as in [1], the potential in the form $U\Sigma_i \delta(x - x_i)$, where x_i is the coordinate of the i-th impurity. We then have clearly

$$G_n(xx') = G_{n+1}(xx') - \frac{UG_{n+1}(xx_{n+1})(x_{n+1}x')}{1 + UG_{n+1}(x_{n+1}x_{n+1})}. \quad (4)$$

Using the results of [1] one shows easily that in the model considered

$$A_{p_1p_2}(E)=\sum_{n,m} e^{-ip_1x_n}\left[\frac{U}{1+U\hat{G}^{(0)}}\right]_{nm} e^{ip_2x_m}. \quad (5)$$

We bear in mind that $\hat{G}^{(0)}$ is a matrix with elements

$$G^{(0)}_{nm}=G^{(0)}(x_nx_m)=\frac{im}{k}\exp\{ik|x_n-x_m|\}.$$

Using Eqs. (3) to (5) we get the following set of recurrence relations:

$$A^{(n)}_{p_1p_2}(E)=A^{(n+1)}_{p_1p_2}(E)+U\left[e^{-ip_1x_{n+1}}-\frac{im}{k}A^{(n+1)}_{pk}e^{-ikx_{n+1}}\right]$$
$$\times\left[e^{ip_2x_{n+1}}-\frac{im}{k}A^{(n+1)}_{-p_2k}e^{-ikx_{n+1}}\right]$$
$$\times\left[1+\frac{imU}{k}-U\left(\frac{m}{k}\right)^2 e^{2ikx_{n+1}}A^{(n+1)}_{k,-k}(E)\right]^{-1}. \quad (6)$$

We introduce the notation

$$y_n(p,E)=1-\frac{im}{k}e^{i(p-k)x_n}A^{(n)}_{pk}(E),$$
$$V_n(E)=1+\frac{imU}{k}-U\left(\frac{m}{k}\right)^2 e^{2ikx_n}A^{(n)}_{k,-k}(E). \quad (7)$$

Using (6) and (7) we then get

$$A^{(n)}_{p_1p_2}(E)=A^{(n+1)}_{p_1p_2}(E)$$
$$+e^{-i(p_1-p_2)x_{n+1}}\frac{Uy_{n+1}(p_1,E)y_{n+1}(-p_2,E)}{V_{n+1}(E)}. \quad (8)$$

Solving Eq. (8), bearing in mind that $A^{(N)}_{p_1p_2}\equiv 0$, we get

$$A_{p_1p_2}(E)=U\sum_i e^{-i(p_1-p_2)x_i}\frac{y_i(p_1,E)y_i(-p_2,E)}{V_i(E)}. \quad (9)$$

From (8) and (7) we easily see that

$$y_n(p,E)=1-e^{i(k-p)r_{n+1}}+e^{i(k-p)r_{n+1}}\frac{y_{n+1}(p,E)}{V_{n+1}(E)},$$
$$V_n(E)=1+\frac{ik_0}{k}+\left(1-\frac{ik_0}{k}\right)e^{2ikr_{n+1}}-\frac{e^{2ikr_n}}{V_{n+1}(E)} \quad (10)$$

where $K_0=mU$ and $r_{n+1}=x_{n+1}-x_n$.

It is clear from (10) that the quantity V_n is the same as the ratio of determinants introduced in [1].

We now shall average Eq. (2) for the conductivity over the positions of the impurities. The recurrence relations which we obtained enable us to do this easily. To do this we introduce the function

$$F(y,y',V,x_i;p_1,p_2,E)$$
$$=\langle\delta(y-y_i(p_1,E))\delta(y'-y_i(-p_2,E))\delta(V-V_i(E))\rangle.$$

It is clear that after averaging this quantity depends on an index which is a function of the coordinate x_i. If we use Eq. (10) we get the following equation for the distribution function F

$$F(y,y',V,x_i;p_1,p_2,E)=\int_0^\infty\frac{\alpha^3f(r)\,dr}{\beta\gamma(\lambda-V)^4}F$$
$$\times\left(\frac{y-1+\beta}{\beta(\lambda-V)},\frac{y'-1+\gamma}{\gamma(\lambda-V)},\frac{\alpha}{\lambda-V},x_i+r;\ p_1,p_2,E\right), \quad (11)$$

where f(r) is the distribution function of the distances between neighboring impurities,

$$\alpha=e^{2ikr},\quad \beta=e^{i(k-p_1)r},\quad \gamma=e^{i(k+p_2)r},$$
$$\lambda=1+\frac{ik_0}{k}+\left(1-\frac{ik_0}{k}\right)\alpha.$$

It is clear that

$$\langle A_{pp}\rangle=UN\int\int\int F(y,y',V)\frac{yy'}{V}\,dy\,dy'\,dV.$$

Averaging of terms such as

$$\left\langle e^{-i(p_1-p_2)(x_i-x_j)}\frac{y_i(p_1,E)y_i(-p_2,E)y_j(p_2,E')y_j(-p_1,E')}{V_i(E)V_j(E')}\right\rangle.$$

is somewhat more complicated. Quantities referring to different indices can be averaged independently. We then get (we do not write down equations for i = j as they are cumbersome):

$$\langle A_{p_1p_2}(E)A_{p_2p_1}(E)\rangle$$
$$=2U^2N\int\int F(y_1,y_2,V;\ p_1,p_2,E)F(y_1',y_2',V';\ p_2,p_1,E')$$
$$\times\frac{y_1y_1'y_2y_2'}{VV'}\,dy_1\,dy_1'\,dy_2\,dy_2'\,dV\,dV'\,\mathrm{Re}\,\psi(p_1,p_2),$$

where

$$\psi(p_1,p_2)=\frac{a}{1-a},\qquad a=\int_0^\infty e^{i(p_1-p_2)r}f(r)\,dr.$$

The relations found here make it in principle possible to solve the problem of evaluating the conductivity in the one-dimensional model. It is clear that, using F(y, y', V), we can also find the particle Green function.

[1] Yu. A. Bychkov and A. M. Dykhne, JETP Pis'ma 3, 313 (1966), Soviet Phys. JETP Letters 3, 202 (1966).

[2] B. I. Halperin, Phys. Rev. 139, A104 (1965).

Translated by D. ter Haar
231