ELECTROMAGNETIC RADIATION IN AN ABSORBING MEDIUM

A. I. ALEKSEEV and V. V. YAKIMETS

Moscow Engineering Physics Institute

Submitted to JETP editor June 14, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1569–1574 (November, 1966)

The influence of the absorbing properties of a medium on radiation processes is determined by the methods of both classical and quantum electrodynamics. Dipole, magnetic dipole and quadrupole radiation from classical and quantum objects is considered. Correction factors taking absorption into account are found for the usual radiation formulas for a transparent medium. The possible effect of neighboring resonance atoms on the radiation from an impurity atom is studied in a uniaxial crystal.

ELECTROMAGNETIC processes in a medium involving a long wavelength field depend strongly on the properties of the medium. Numerous investigations (cf., for example, the review articles[1,2]) which usually consider three types of radiation: Cerenkov, transition and diffraction radiation, have been devoted to this problem. The types of radiation mentioned above are characterized by the fact that they disappear in the absence of a medium. We shall consider radiation from a charged system which occurs in a vacuum, too. The presence of the surrounding medium in this case either alters the intensity of the source, or leads to the radiation of specific waves which are absent in a vacuum.[3] Below we investigate the effect of absorption by the medium on the radiation from classical and quantum objects. The emission from a classical oscillator in a nonequilibrium medium with a negative absorption has been studied in [4]. In contrast to the cited work we consider absorbing media which are in thermodynamic equilibrium. It has turned out that taking absorption into account leads to additional factors in the formulas for the long wavelength radiation in a vacuum. These factors can appreciably alter the intensity of the radiation for example in a conducting medium due to the large value of the conductivity.

In studying longitudinal waves the effect of the absorption of a medium is more pronounced. The results obtained have been applied to the problem of the radiation by an impurity atom in ruby. Because of the large scatter in the levels of neighboring impurity atoms the effect due to the absorption by the surrounding medium is in this case insignificant, and an individual excited atom radiates as in a pure corundum crystal. The dielectric constant of ruby, found for a weak field with account taken of the scatter in the levels of impurity atoms, is utilized to determine the cross section for the absorption of a resonance quantum. There is satisfactory agreement with experiment.

1. We determine the electromagnetic radiation from a classical charged system of finite size placed into an unbounded absorbing medium with a given dielectric permittivity $\epsilon_{\alpha\beta}(\omega, \mathbf{k})$. The radiation is due to the external currents

$$\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}, \omega) e^{-i\omega t} + \mathbf{j}^*(\mathbf{x}, \omega) e^{i\omega t}, \tag{1}$$

which excite in the medium an electromagnetic field

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, \omega) e^{-i\omega t} + \mathbf{E}^*(\mathbf{x}, \omega) e^{i\omega t}. \tag{2}$$

Since in an absorbing medium there is no flux of the Poynting vector over an infinitely distant surface the energy radiated by the external currents is completely absorbed by the medium. The energy $d\mathscr{E}/dt$ accumulated per unit time in a homogeneous medium on being averaged over a period $2\pi/\omega$ is equal to[5]

$$\frac{d\mathscr{E}}{dt} = \frac{i\omega}{4\pi} \int [\varepsilon_{\alpha\beta}^*(\omega, \mathbf{k}) - \varepsilon_{\beta\alpha}(\omega, \mathbf{k})] E_\alpha(\mathbf{k}, \omega) E_\beta^*(\mathbf{k}, \omega) d\mathbf{k}, \tag{3}$$

where

$$\mathbf{E}(\mathbf{k}, \omega) = \int \mathbf{E}(\mathbf{x}, \omega) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x}.$$

Formula (3) is also valid for a nonmagnetic anisotropic medium. With the aid of Maxwell's equations we express the electric field in (3) in terms of the sources (1). We then obtain

$$\frac{d\mathscr{E}}{dt} = -\frac{\omega}{4\pi^3} \int j_\alpha^*(\mathbf{k}, \omega) j_\beta(\mathbf{k}, \omega) \operatorname{Im} L_{\alpha\beta}(\mathbf{k}, \omega) d\mathbf{k}, \tag{4}$$

where the function $L_{\alpha\beta}(\mathbf{k}, \omega)$ is determined by the equation

$$[\omega^2\varepsilon_{\alpha\beta'}(\omega, \mathbf{k}) - c^2k^2\delta_{\alpha\beta'} + c^2k_\alpha k_{\beta'}]L_{\beta'\beta}(\mathbf{k}, \omega) = 4\pi\delta_{\alpha\beta}. \quad (5)$$

In a transparent medium the electromagnetic field in the wave zone falls off inversely proportional to the distance from the charged system and, therefore, the integrals (3) and (4) lose their meaning. Formally this is expressed by the fact that the function $L_{\alpha\beta}(\mathbf{k}, \omega)$ acquires poles on the real axis of integration in the complex k-plane (cf. later). We shall treat a transparent medium as the limiting case of an absorbing medium for which the imaginary part of the dielectric permittivity is negligibly small. Such a condition leads to the correct method of going around the poles in the integral (4) so that the final expression for the radiated energy remains valid also in transparent media. With respect to the latter, formula (4) is also of interest from the methodological point of view since it enables one to determine the multipole radiation avoiding the necessity of calculating the field intensity at large distances.

2. We consider the important case of an isotropic medium without spatial dispersion:

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega). \quad (6)$$

We rewrite the integral (4) in the following manner:

$$\frac{d\mathscr{E}}{dt} = \frac{\varepsilon''}{2\pi^2\omega}\int d\mathbf{x}_1 d\mathbf{x}_2 \int_{-\infty}^{\infty} dk \int d\Omega_n j_\alpha^*(\mathbf{x}_1, \omega) j_\beta(\mathbf{x}_2, \omega)$$
$$\times(\delta_{\alpha\beta} - n_\alpha n_\beta)\frac{k^2 \exp\{ik'\mathbf{n}(\mathbf{x}_1 - \mathbf{x}_2) + k''\mathbf{n}(\mathbf{x}_1 + \mathbf{x}_2)\}}{(k^2c^2/\omega^2 - \varepsilon')^2 + \varepsilon''^2}, \quad (7)$$

$$\mathbf{k} = k\mathbf{n}, \qquad k = k' + ik'' \quad (8)$$

We displace the contour of integration with respect to the variable k into the upper half-plane of the complex variable $k = k' + ik''$, if $\mathbf{n}(\mathbf{x}_1 + \mathbf{x}_2) < 0$, and into the lower half-plane if $\mathbf{n}(\mathbf{x}_1 + \mathbf{x}_2) > 0$. We assume that the following inequalities are satisfied

$$a \ll c/\varkappa_1\omega, \qquad a \ll c/\varkappa_2\omega, \quad (9)$$

where a is the maximum linear dimension of the charged system while κ_1 and κ_2 are respectively the index of refraction and the coefficient of absorption for the medium:

$$\varkappa_1 = \left(\frac{|\varepsilon| + \varepsilon'}{2}\right)^{1/2}, \quad \varkappa_2 = \left(\frac{|\varepsilon| - \varepsilon'}{2}\right)^{1/2}. \quad (10)$$

In this case the exponent in (7) can be expanded in a series after integration over the variable k. Finally we obtain

$$\frac{d\mathscr{E}}{dt} = \frac{d\mathscr{E}_d^0}{dt}\frac{\varepsilon''}{2\varkappa_2} + \frac{d\mathscr{E}_\mu^0}{dt}\frac{|\varepsilon|\varepsilon''}{2\varkappa_2} + \frac{d\mathscr{E}_Q^0}{dt}\frac{|\varepsilon|\varepsilon''}{2\varkappa_2} + \ldots, \quad (11)$$

where $d\mathscr{E}_d^0/dt$, $d\mathscr{E}_\mu^0/dt$ and $d\mathscr{E}_Q^0/dt$ is the average energy radiated in vacuo per unit time respectively by an electric dipole, by a magnetic dipole moment and by a quadrupole. Each successive term in the multiple expansion contains the factor $|\epsilon|$ raised to a power greater by unity than in the preceding term. Thus, the formulas for the long wavelength radiation in an absorbing medium differ from the vacuum formulas by an additional factor. In the limit $\epsilon'' \to 0$ the usual law for radiation in a transparent medium follows from (11).

3. In accordance with the correspondence principle analogous formulas hold for the emission of photons by a quantum object. Indeed, use of the method proposed previously in [6] leads to the following result for the probability W of the transition per unit time from an excited state 2 to the original state 1 of an atom in an absorbing medium:

$$W = -\frac{n_{\omega_{21}} + 1}{4\pi^3\hbar^2c^2}\int M_\alpha^*(\mathbf{k})M_\beta(\mathbf{k})\,\mathrm{Im}\,D_{\alpha\beta}{}^R(\mathbf{k}, \omega_{21})\,d\mathbf{k}, \quad (12)$$
$$n_{\omega_{21}} = [\exp(\hbar\omega_{21}/\varkappa T) - 1]^{-1}, \quad (13)$$

where ω_{21} is the frequency of the atomic transition, κ is the Boltzmann constant, while $\mathbf{M}(\mathbf{k})$ is the matrix element for the transition

$$\mathbf{M}(\mathbf{k}) = -\frac{e}{m}\int \psi_1^*(x)e^{-i\mathbf{kx}}\{\mathbf{p} + i[\mathbf{k}, \mathbf{s}]\}\psi_2(x)\,dx. \quad (14)$$

Here e, m, $\mathbf{s}$, and $\mathbf{p}$ are respectively the charge, the mass, the spin and the momentum operator of the emitting electron, while $\psi_1(x)$ and $\psi_2(x)$ are its wave functions. The Fourier component $D^R_{\alpha\beta}(\mathbf{k}, \omega)$ of the retarded temporal Green's function taking the temperature T into account is written in a gauge for which the scalar potential vanishes. Therefore, the vector indices α and β everywhere take on three values: 1, 2, and 3. The function $D^R_{\alpha\beta}(\mathbf{k}, \omega)$ satisfies Eq. (5) with the right hand side equal to $4\pi\hbar c^2\delta_{\alpha\beta}$ and with a dielectric permittivity which depends on the temperature.[7] The factor $(n_{\omega_{21}} + 1)$ in (12) takes into account the effect of the fluctuating electromagnetic field on the process of emission by an atom. In the absence of absorption it describes the induced and the spontaneous radiation.

Calculations using formula (12) can be carried out comparatively easily for radiation in a medium with dielectric permittivity (6) at zero temperature. In analogy to (11) the probabilities for dipole W_d, magnetic dipole W_μ and quadrupole W_Q radiation can be written in the form

$$W_d = W_d{}^0\frac{\varepsilon''}{2\varkappa_2}, \quad W_\mu = W_\mu{}^0\frac{|\varepsilon|\varepsilon''}{2\varkappa_2}, \quad W_Q = W_Q{}^0\frac{|\varepsilon|\varepsilon''}{2\varkappa_2}, \quad (15)$$

where the index zero denotes the corresponding probabilities of radiation in vacuo, while the functions $\epsilon'(\omega)$ and $\epsilon''(\omega)$ are taken at the point $\omega = \omega_{21}$. We emphasize that formulas (11) and (15) are valid for either sign of the real part of the dielectric permittivity. For example, for $\epsilon' < 0$ radiation is possible only in an absorbing medium.

Similarly to (11) each of the probabilities (15) for the emission by an atom in an absorbing medium differs from the vacuum probability by a correction factor. These factors also appear in calculations involving other processes occurring in vacuo accompanied by the emission of a quantum. For example, in an absorbing medium (6) the differential cross section $d\sigma$ for the bremsstrahlung involving a soft quantum of frequency ω by a nonrelativistic electron in the field of a nucleus of charge $Z|e|$ assumes the form

$$d\sigma = \left(\frac{Ze^2}{2mv_1^2}\right)^2 \frac{d\Omega_2}{\sin^4(\theta_2/2)} \frac{e^2[\mathbf{k}(\mathbf{v}_2 - \mathbf{v}_1)]^2\, d\Omega}{(2\pi)^2 c^3 k\hbar\omega} \frac{\varepsilon''(\omega)}{2\varkappa_2(\omega)} d\omega, \tag{16}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are respectively the velocities of the electron before and after scattering, θ_2 is the angle of scattering of the electron, while $d\Omega$ is the solid angle which contains the propagation vector $\mathbf{k}$ of the emitted quantum. Relation (16) can also be obtained from the general formula for bremsstrahlung in a medium given in [8].

4. We apply the results obtained above to a resonant medium consisting of identical two-level atoms scattered throughout a uniaxial crystal. The presence of the two-level atoms alters the dielectric permittivity of the crystal. In order to determine the latter we shall utilize the method developed in [9]. The Hamiltonian H of a system of N two-level atoms situated in a crystal of volume V can be written in the following manner:

$$H = \sum_r \frac{\hbar\omega_r}{2} N_-{}^r + \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda}^+ c_{\mathbf{k}\lambda} - \frac{1}{c} \sum_{r,\mathbf{k}\lambda} \mathbf{A}_\lambda(\mathbf{k})\, \mathbf{j}_{r\lambda}(-\mathbf{k}), \tag{17}$$

$$\mathbf{A}_\lambda(\mathbf{k}) = (2\pi\hbar c^2/\omega_{\mathbf{k}\lambda}\varepsilon_{\alpha\beta} l_\alpha^{\mathbf{k}\lambda} l_\beta^{\mathbf{k}\lambda})^{1/2} (c_{\mathbf{k}}{}^\lambda \mathbf{l}^{\mathbf{k}\lambda} + \overset{+}{c}_{-\mathbf{k}\lambda} \mathbf{l}^{\mathbf{k}\lambda}), \tag{18}$$

$$\mathbf{j}_{r\lambda}(\mathbf{k}) = \sum_j{}' i\omega_r V^{-1/2} \mathbf{l}^{\mathbf{k}\lambda} (\sigma_+{}^j d_\alpha{}^j + \sigma_-{}^j d_\alpha{}^{*j}) l_\alpha{}^{\mathbf{k}\lambda} e^{-i\mathbf{k}\mathbf{x}_j}, \tag{19}$$

$$N_-{}^r = \sum_j{}' \sigma_3{}^j, \quad \sigma_\pm = \frac{\sigma_1 \pm i\sigma_2}{2}, \tag{20}$$

where the first and second terms in (17) are the energy operators respectively for the impurity atoms and for the radiation field in a crystal without impurity atoms, while the last term is the operator for their interaction energy. σ_1, σ_2 and σ_3 are the Pauli matrixes. $\mathbf{x}_j$, $\mathbf{d}^j$ and $\hbar\omega_j$ are respectively the coordinate of the center of gravity, the dipole moment for the transition and the difference between the energies of the upper and the lower levels of the j-th impurity atom. The prime on the summation sign with respect to the index j denotes summation over all the impurity atoms in the crystal the difference between whose energy levels is equal to $\hbar\omega_r$. It is assumed that their number is sufficiently great. Physical quantities which refer to the above group of impurity atoms are denoted by the index r. Further, $c_{\mathbf{k}\lambda}$ and $c_{\mathbf{k}\lambda}^+$ are the annihilation and creation operators for a macroscopic quantum of propagation vector $\mathbf{k}$ and polarization $\mathbf{l}^{\mathbf{k}\lambda}$, which is determined by the solution of the equation

$$[c^2(k_\alpha k_\beta - k^2\delta_{\alpha\beta}) + \omega_{\mathbf{k}\lambda}{}^2 \varepsilon_{\alpha\beta}] l_\beta{}^{\mathbf{k}\lambda} = 0. \tag{21}$$

The connection between $\omega_{\mathbf{k}\lambda}$ and $\mathbf{k}$ is obtained from the Fresnel equation for the ordinary ($\lambda = 1$) and the extraordinary ($\lambda = 2$) waves. We assume the tensor for the dielectric permittivity $\epsilon_{\alpha\beta}$ of a crystal without impurity atoms to be real and constant.

In a weak electromagnetic field when the radiation width of the excited level is neglected the Fourier components of the operators for the vector potential (18) and the polarization current (19) satisfy the equations

$$\ddot{\mathbf{A}}_\lambda(\mathbf{k}) + \omega_{\mathbf{k}\lambda}{}^2 \mathbf{A}_\lambda(\mathbf{k}) = \frac{4\pi c}{\varepsilon_{\alpha\beta} l_\alpha{}^{\mathbf{k}\lambda} l_\beta{}^{\mathbf{k}\lambda}} \sum_r \mathbf{j}_{r\lambda}(\mathbf{k}), \tag{22}$$

$$\ddot{\mathbf{j}}_{r\lambda}(\mathbf{k}) + \omega_r{}^2 \mathbf{j}_{r\lambda}(\mathbf{k}) = \frac{2\omega_r{}^3 \mathbf{l}^{\mathbf{k}\lambda}}{\hbar c V} \sum_j{}' (\mathbf{d}^j \mathbf{l}^{\mathbf{k}\lambda}) (\mathbf{d}^{*j}(\mathbf{A}_1(\mathbf{k}) + \mathbf{A}_2(\mathbf{k})). \tag{23}$$

Due to the thermal vibrations of the crystalline lattice similar levels of impurity atoms are scattered around their mean positions.[10] The number of impurity atoms at a given instant of time t having an energy difference between the upper and the lower levels lying in the interval between $\hbar\omega$ and $\hbar(\omega + d\omega)$ is equal to

$$\sum_r N^r = N f(\omega)\, d\omega \equiv \frac{N}{2\pi} \frac{\delta}{(\omega - \omega_0)^2 + \delta^2/4} d\omega, \quad \omega \leqslant \omega_r \leqslant \omega + d\omega, \tag{24}$$

where $\hbar\delta$ is the width of the scatter of similar levels near a resonance value $\hbar\omega_0$, while the distribution function $f(\omega)$ has for the sake of simplicity been taken in dispersion form. The dielectric permittivity of such a medium near resonance can be written in accordance with (22)–(24) in the following manner ($\delta \ll \omega_0$):

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta} \left(1 + \frac{8\pi^2\rho\lambda^3}{\omega_0 - \omega - i\delta/2} \frac{dW_{\mathbf{k}\lambda}}{d\Omega}\right) \tag{25}$$

Here $\lambdabar = 1/k$, ρ is the average concentration of impurity atoms in the crystal, while

$$dW_{\mathbf{k}\lambda} = |\mathbf{d}|^2 k^3 d\Omega \,/\, 6\pi\hbar\varepsilon_{\alpha\beta} l_\alpha^{\mathbf{k}\lambda} l_\beta^{\mathbf{k}\lambda}$$

is the probability of emission of a quantum of propagation vector $\mathbf{k}$ lying in the solid angle $d\Omega$ and with polarization $\mathbf{l}^{\mathbf{k}\lambda}$. In (25) averaging has been performed over the directions of the dipole moment for the transition. The numerical value of $\epsilon_{\alpha\beta} l_\alpha^{\mathbf{k}\lambda} l_\beta^{\mathbf{k}\lambda}$ is given as a function of the angles in [3]. Taking into account the intrinsic width $\hbar\Gamma$ of an excited level of the atom leads to replacement in (25) of the quantity δ by the sum $\delta + \Gamma$. However, usually $\delta \gg \Gamma$ and it is superfluous to take into account the width $\hbar\Gamma$.

If we substitute into (25) the experimental values of δ, ρ and $dW_{\mathbf{k}\lambda}/d\Omega$ for impurity atoms in ruby then the imaginary part of the dielectric permittivity (25) due to the large scatter of levels will be a small quantity in spite of the strong inequality $\rho\lambdabar^3 \gg 1$. Therefore, the radiation from an individual impurity atom in ruby in a weak electromagnetic field in accordance with (12) coincides with the radiation in a corundum crystal without neighboring impurity atoms. The latter assertion is particularly simply proven for the case of emission of the ordinary wave for which $\epsilon_{\alpha\beta} l_\alpha^{\mathbf{k}\lambda} l_\beta^{\mathbf{k}\lambda} = \epsilon_\perp$ and (15) is valid.

Formula (25) enables us to establish a relation between the effective cross section σ for the absorption of a quantum and the width of scatter of the levels. For example, for a beam of quanta of the ordinary wave distributed over the frequencies ω in accordance with the distribution function $f(\omega)$ from (24) the average cross section for the absorption of a quantum is equal to

$$\sigma = 2\pi\lambdabar_0^2 W_1 / \varepsilon_\perp\delta, \qquad W_1 = 2\omega_0^3 |\mathbf{d}|^2 \sqrt{\varepsilon_\perp} / 3\hbar c^3, \qquad (26)$$

where $\lambdabar_0 = c/\omega_0$, W_1 is the probability of emission of an ordinary quantum by an impurity atom in a pure crystal the dielectric permittivity of which in a direction perpendicular to the optic axis is equal to $\epsilon_\perp$. For ruby we have $\delta^{-1} \cong 10^{-12}$ sec, $\epsilon_\perp \cong 3$, $2W_1 = 3 \times 10^2$ sec^{-1}, $2\pi\lambdabar_0 = 7 \times 10^{-5}$ cm.[11] Therefore, the cross section (26) for the absorption of a quantum in ruby is equal to $\sigma \cong 3 \times 10^{-20}$ cm^2. The experimental value of the cross section according to the measurements of Maiman et al.[12] is equal to $\sigma \cong 2.5 \times 10^{-20}$ cm^2 which agrees with the value calculated from (26).

The authors are grateful to V. M. Galitskiĭ for discussions.

1 B. M. Bolotovskiĭ, UFN **62**, 201 (1957); **75**, 295 (1961), Soviet Phys. Uspekhi **4**, 781 (1962).

2 B. M. Bolotovskiĭ and G. V. Voskresenskiĭ, UFN **88**, 209 (1966), Soviet Phys. Uspekhi **9**, 73 (1966).

3 A. I. Alekseev and Yu. P. Nikitin, JETP **48**, 1669 (1965) and **50**, 915 (1966); Soviet Phys. JETP **21**, 1121 (1965) and **23**, 608 (1966).

4 V. L. Ginzburg and V. Ya. Éidman, JETP **43**, 1865 (1962), Soviet Phys. JETP **16**, 1316 (1963).

5 V. P. Silin and A. A. Rukhadze, Elektromagnitnye svoĭstva plasmy i plasmopodobnykh sred (Electromagnetic properties of plasma and of plasma-like media), Atomizdat, 1961.

6 A. I. Alekseev, in: Primenenie metodov kvantovoĭ teorii polya k zadacham mnogikh tel (Application of the Methods of Quantum Field Theory to Many Body Problems), Gosatomizdat, 1963.

7 A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1963.

8 V. M. Galitskiĭ and V. V. Yakimets, JETP **46**, 1066 (1964), Soviet Phys. JETP **19**, 723 (1964).

9 A. I. Alekseev and V. M. Galitskiĭ, JETP **47**, 1893 (1964), Soviet Phys. JETP **20**, 1273 (1965).

10 A. Kiel, Phys. Rev. **126**, 1292 (1962).

11 N. G. Basov, R. V. Abartsumyan, V. S. Zuev, P. G. Kryukov and V. S. Lekhotov, DAN SSSR **165**, 58 (1965), Soviet Phys. Doklady **10**, 1039 (1966).

12 T. H. Maiman, R. H. Hoskins, I. J. D'Haenens, C. K. Asawa and V. Evtuhov, Phys. Rev. **123**, 1151 (1961).

Translated by G. Volkoff
189