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NONLINEAR EVOLUTION OF DISTURBANCES IN PLASMAS AND OTHER DISPERSIVE MEDIA

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We investigate the nonstationary solutions of the Korteweg–de Vries equation that describes the evolution of nonlinear disturbances in a plasma or other dispersive media. The conditions for the decay of the disturbances into ''solitons'' are found. The physical meaning of the self-similar solution of the Korteweg–de Vries equation is elucidated. Some general asymptotic relations are obtained for the nonstationary solutions.

1. INTRODUCTION. GENERAL RELATIONS

THE quantitative difference between nonlinear plasma dynamics and gas-dynamics consists in the important role of dispersion effects (see, for example, [1]). To explain the character of the resultant phenomena, it is natural to consider first the case when the deviations of the dispersion law from linear are small, so that the dispersion equation can be represented in the form of a series of powers of the wave number:

$$\omega = v_f k(1 \pm \delta^2 k^2 + \ldots), \qquad (1.1)$$

where v_f is the phase velocity of small oscillations at $k \to 0$, and δ is a certain constant characterizing the magnitude of the dispersion effects (dispersion ''length''). If it is possible to confine oneself in (1.1) to the first two terms (δ is small compared with the characteristic wavelengths that are important in this problem), and also to consider perturbations of finite but sufficiently small amplitude, then in many cases the equation for such perturbations (in the first non-vanishing order relative to nonlinear and dispersion effects) reduces to the form

$$u_t + uu_x + \beta u_{xxx} = 0, \qquad (1.2)$$

where u is the magnitude of the perturbation (for example, the velocity or the magnetic field in the plasma), and the parameter β is equal to $\mp v_f \delta^2$. [1)]

Equation (1.2) was first derived by Korteweg and de Vries[2] for surface waves of sufficiently large wavelength (compared with the depth), and sufficiently small but finite amplitude in liquids. A similar equation for plasma waves was first obtained by Gardner and Morikawa[3] (for waves propagating transverse to the magnetic field in a cold plasma) and in [4, 5] (for waves propagating at an angle to the magnetic field).[2)] Of course, the importance of (1.2) is not confined to these cases. As noted in [4], it is valid also for other types of plasma waves of small but finite amplitude, when it is possible to confine oneself to the first two nonvanishing terms in the dispersion equation (1.1), which can be readily obtained from (1.2) after linearization and transition to a reference frame in which the plasma is at rest (see also [6–8]). We note, finally, that Eq. (1.2) is also closely connected with the Fermi-Pasta-Ulam problem dealing with the establishment of stochastic oscillations in a nonlinear string.[9, 10]

It is easy to verify that the solutions of the Korteweg–de Vries equation for $\beta < 0$ can be obtained from the corresponding solutions for the case $\beta > 0$ by making the substitutions

$$u \to -u, \qquad x \to -x, \qquad t \to t. \qquad (1.3)$$

We can therefore confine ourselves to a detailed investigation of (1.2) with $\beta > 0$.

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1)Equation (1.2) is written in a coordinate frame that moves with a velocity $v_f = \lim_{k\to 0} (\omega/k)$ relative to the medium; the quantity u has the dimension of velocity.

2)For waves propagating transversely to the magnetic field in a cold plasma, $\beta = V_A c^2/\omega_{0e}^2$ (when the ions have a sufficiently large Larmor radius, the sign of β reverses[7,8]. For waves propagating at an angle $\pi/2 - \theta$ to the magnetic field, $\beta = V_A c^2\theta^2/\omega_{0i}^2$ (when $1 \gg \theta \gg (m_e/m_i)^{1/2}$. In the case of gravitational-capillary waves on the surface of the liquid, $\beta = (1/2)(gh)^{1/2}(h^2/3 - \alpha/\rho g)$, where α is the surface-tension coefficient, ρ the density, and h the depth of the channel [11], so that, depending on the latter, β can be either positive or negative.

The stationary solutions of (1.2) were already obtained by Korteweg and de Vries,[2] who showed that these solutions are solitary and periodic waves propagating with constant velocity relative to the medium, i.e., they are described by equations of the type $u(x, t) = u(x - Vt)$. For solitary wave, or using the terminology of [10], for a "soliton," $u(x)$ is of the form

$$u(x) = u_0 \operatorname{sch}^2 \{(u_0 / 12\beta)^{1/2} x\}, \qquad (1.4)$$

and the velocity of the wave is determined by its amplitude and is equal to $V = u_0/3$. For periodic waves

$$u(x) = (2a / s^2) \operatorname{dn}^2\{(a / 6\beta)^{1/2}(x / s)\} + \gamma, \quad V = 2a(2 - s^2) / 3s^2 + \gamma. \qquad (1.5)$$

Here dn z is the elliptic Jacobi function with modulus $0 \le s \le 1$, and a and γ are arbitrary constants, with a having the meaning of the wave amplitude. The wavelength λ and the mean value of the amplitude $\bar{u}$ are equal to

$$\lambda = 2(6\beta / a)^{1/2} s K(s^2), \quad \bar{u} = 2aE(s^2) / s^2 K(s^2) + \gamma, \qquad (1.6)$$

where $K(s^2)$ and $E(s^2)$ are complete elliptic integrals with modulus s.

When $s \to 1$ we get

$$K(s^2) \approx \frac{1}{2} \ln\left(\frac{16}{1 - s^2}\right), \quad \lambda \to \infty, \quad \operatorname{dn} z \to \operatorname{sch} z, \qquad (1.7)$$

so that a periodic wave with $s \approx 1$ can be approximately regarded as a sequence of solitons with amplitudes $u_0 = 2a$ (relative to the level $\bar{u} = \gamma$), separated by the logarithmically large distance $\lambda = 2 (6\beta/a)^{1/2} |\ln (1 - s^2)|$. The width of each soliton, according to (1.4), is $(12\beta/u_0)^{1/2}$, and its velocity is $V = (2a/3) + \gamma$ (in the reference frame that moves with velocity $v_f = \lim_{k \to 0} (\omega/k)$ relative to the plasma).

The solutions presented above are stationary. Some nonstationary solutions of the Korteweg–de Vries equation were considered earlier by the authors in [4],3) where a detailed study was made of a self-similar solution of (1.2), and also by Zabusky and Kruskal,[10] where a solution corresponding to the periodic initial condition $u(x, 0) = \cos \pi x$ was obtained by numerical integration. An interesting approximate method for analyzing nonstationary solutions of the Korteweg–de Vries equation, based on the representation of these solutions in the form (1.5) with slowly varying parameters a, s, and γ, was developed by Whitham.[12]

The solution obtained in [10] differs essentially from the self-similar solution investigated in [4]. This difference lies in the fact that the former gives a series of individual solitons, into which the chosen initial perturbation $u(x, 0) = \cos \pi x$ decays, whereas the self-similar solution represents a wave packet, which never decays into solitons. Thus, diffferent initial conditions can correspond to solutions of essentially different types. In this paper we clarify some characteristic features of different types of solutions of the Korteweg–de Vries equation.

2. SIMILARITY PRINCIPLE

Let us formulate first a similarity principle for the Korteweg–de Vries equation. We write the initial condition in the form

$$u(x, 0) = u_0 \varphi(x / l), \qquad (2.1)$$

where u_0 is the amplitude and l the linear dimension of the initial perturbation. By varying the parameters u_0 and l we obtain a family of similar initial conditions characterized by the dimensionless function $\varphi(x/l)$. Introducing new variables

$$\eta = u / u_0, \quad \xi = x / l, \quad \tau = u_0 t / l, \qquad (2.2)$$

we obtain from (1.2) and (2.1)

$$\eta_\tau + \eta\eta_\xi + \sigma^{-2}\eta_{\xi\xi\xi} = 0, \quad \eta(\xi, 0) = \varphi(\xi), \qquad (2.3)$$

where

$$\sigma = l(u_0 / \beta)^{1/2}. \qquad (2.4)$$

It follows from (2.3) that flows corresponding to the same value of the number σ and to the same initial function $\varphi(\xi)$ are similar. For the solitons (1.4) we have

$$\sigma = \sigma_s = \sqrt{12}. \qquad (2.5)$$

The number σ is actually the nonlinearity index of the problem, and its value σ_s for the soliton is characteristic in a definite sense, viz., qualitatively different solutions are obtained for $\sigma \gg \sigma_s$ and $\sigma \ll \sigma_s$ following an initial perturbation $\varphi(\xi)$ of identical form (see below).

3. CERTAIN PECULIARITIES OF THE NONSTATIONARY SOLUTIONS

Let us consider the solutions of (1.2) corresponding to the most typical initial disturbances

3)In [4] the problem was considered not with initial but with boundary conditions (excitation of waves by a source on a plasma boundary); the corresponding equation was

$$u_t + v_f u_x + (u / v_f) u_t + (\beta / v_f^3) u_{ttt} = 0.$$

Introducing new variables $\tau = x/v_f$ and $\xi = v_f t - x$, we arrive at Eq. (1.2) with initial conditions.

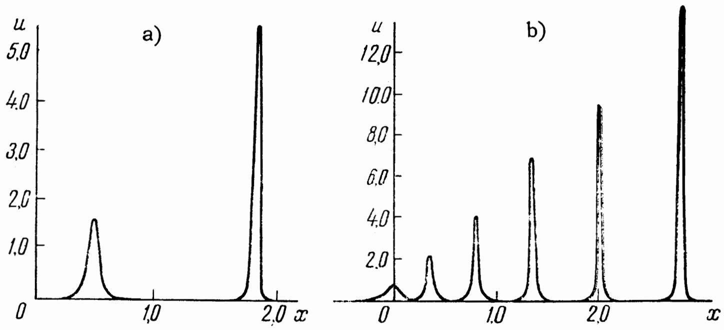

FIG. 1. Profile of perturbation in the case $\beta > 0$ with $\sigma > \sigma_c$: a) $\sigma = 5.9$, b) $\sigma = 16.5$.

that attenuate as $x \to \pm\infty$. We take first an initial perturbation in the form of a single peak, for example

$$u(x, 0) = u_0 \exp(-x^2 / l^2). \qquad (3.1)$$

The character of the solutions will depend on the values of σ for the initial condition (3.1).

As shown by a numerical trial (see Fig. 1), for sufficiently large values of $\sigma \gg \sigma_S$ the initial perturbation (3.1) decays practically completely during the course of the evolution into individual solitons (it will be shown later that a "tail," representing a limited wave packet of small amplitude, is produced in addition to solitons. A solution of similar type (decaying into solitons) was obtained earlier by Zabusky and Kruskal[10] for the periodic initial condition

$$u(x, 0) = \cos \pi x.$$

It follows from a numerical trial that the initial perturbation (3.1) breaks up into two solitons when $4 < \sigma < 7$, into 3 when $7 < \sigma < 11$, into 4 when $\sigma \sim 11$, and into 6 when $\sigma \sim 16$, i.e., with increasing number σ the corresponding perturbation breaks up into a larger number of solitons.

In the opposite limiting case ($\sigma \ll \sigma_S$), "non-soliton" solutions are obtained, corresponding to perturbations which do not decay into solitons. These solutions are rapidly-oscillating wave packets. Qualitatively these solutions are similar to the self-similar solution investigated in [4], although quantitatively they can differ in the law governing the decrease of amplitude in time and in space (it will be shown below that the character of the asymptotic dependence of the wave number on x and t in the rapidly-oscillating part of the packet is common to all solutions of this type, and coincides with that previously obtained for the self-similar solution in [4]).

Finally, we note that at certain initial conditions a mixed type of solution is obtained, containing, besides solitons that move forward, a lagging

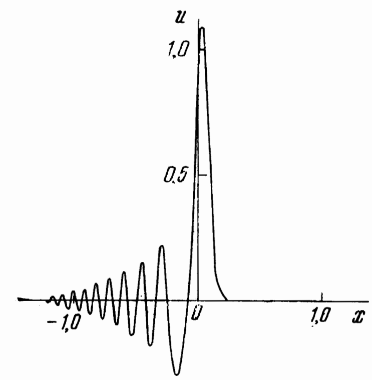

FIG. 2. Profile of perturbation in the case $\beta > 0$ with $\sigma < \sigma_c$ ($\sigma = 1.9$).

"tail" having the same form as a train of fast oscillations (Fig. 2).

Certain qualitative peculiarities of the "pure soliton" solutions can be explained by starting from the conservation laws derived from the Korteweg–de Vries equation. It turns out that in addition to the three known conservation laws given, for example, in [12], many others exist.[4)] These conservation laws can be written in the form

$$\frac{\partial Q_m(x,t)}{\partial t} + \frac{\partial P_m(x,t)}{\partial x} = 0 \qquad (m = 1, 2, \ldots), \quad (3.2)$$

where

$$Q_1 = u, \quad P_1 = {}^1/_2 u^2 + \beta u_{xx},$$

$$Q_2 = {}^1/_2 u^2, \quad P_2 = {}^1/_3 u^3 + \beta (u u_{xx} - {}^1/_2 u_x^2),$$

$$Q_3 = {}^1/_3 u^3 - \beta u_x^2, \quad P_3 = {}^1/_4 u^4 + \beta (u^2 u_{xx} + 2 u_t u_x) + \beta^2 u_{xx}^2. \quad (3.3)$$

4)The conservation laws numbered 4 to 8 were obtained by M. Kruskal, N. Zabusky, and R. Miura (Private communication from Professor M. Kruskal). The number of such conservation laws is apparently infinite.

The first conservation law is Eq. (1.2) itself, written in divergence form. The second law is obtained from (1.2) by multiplying both sides of the equation by u, etc. However, the complexity of the derivations increases rapidly with increasing m.

If the perturbation attenuates at $x \to \pm\infty$, then it follows from (3.2) that

$$S_m = \int_{-\infty}^{\infty} Q_m(x, t)\,dx = \int_{-\infty}^{\infty} Q_m(x, 0)\,dx. \quad (3.4)$$

If N solitons are produced from the first perturbation then, after they have diverged from one another sufficiently far, the conserved quantities S_m will be summations of these values for the individual solitons, i.e., $S_m = \sum_{r=1}^{N} S_m^{(r)}$. Equating this sum to the value of S_m calculated for the initial perturbation (3.1), we obtain the following equations:

$$\sum_{r=1}^{N} \eta_{(r)}{}^{m-1/2} = \frac{\sigma}{\sigma_s} \int_{-\infty}^{\infty} Q_m(\xi, 0)\,d\xi \Big/ \int_{-\infty}^{\infty} q_m(\xi)\,d\xi \quad (m = 1, 2, \ldots), \quad (3.5)$$

where $\eta_r = u_0/u_0^{(r)}$ are the dimensionless amplitudes of the produced solitons, u_0 is the characteristic "amplitude" of the initial perturbation (3.1), the number σ determined from (2.4) pertains to the initial perturbation, and $Q_m(\xi, 0)$ and $q_m(\xi)$ are the densities of the conserved quantities for the initial condition and for the soliton, respectively, expressed in dimensionless variables (where $u_0 = l = 1$).

The system (3.5) makes it possible, in principle, to determine the amplitude of the solitons produced from the specified initial perturbation. Let, for example, this perturbation decay into two solitons (N = 2). In this case, solving the system (3.5), we obtain the following expressions for the amplitudes:

$$(\eta_{1,2})^{1/2} = {}^1\!/_2\{a_1 \pm [(4a_2/3a_1) - a_1^2/3]^{1/2}\}, \quad (3.6)$$

where

$$a_1 = \left(\frac{\sigma}{2\sigma_s}\right) \int_{-\infty}^{\infty} \varphi(\xi)\,d\xi, \quad a_2 = \left(\frac{3\sigma}{4\sigma_s}\right) \int_{-\infty}^{\infty} \varphi^2(\xi)\,d\xi. \quad (3.7)$$

The condition for the roots to be real leads to the inequality $4a_2 > a_1^3$ (the equality sign must be excluded, since this means formation of two solitons of identical amplitude, which cannot diverge because the velocity of the soliton is proportional to its amplitude). Further, we note that in the derivation of (3.6) all the square roots were taken in the arithmetic sense; therefore the right side of (3.6) should be positive, giving the inequality $a_1^3 > a_2$. When $a_1^3 \to a_2$ the amplitude of one of the solitons tends to zero.

Thus, for the initial perturbation (3.1) to decay into two solitons it is necessary that the similarity parameter (2.4) satisfy the conditions

$$\sigma_c < \sigma < 2\sigma_c, \quad (3.8)$$

$$\sigma_c^2 = 6\sigma_s^2 \int_{-\infty}^{\infty} \varphi^2(\xi)\,d\xi \cdot \left[\int_{-\infty}^{\infty} \varphi(\xi)\,d\xi\right]^{-3}. \quad (3.9)$$

The lower limit $\sigma = \sigma_c$ separates the perturbations that decay into solitons from perturbations which evolve principally in a different manner; they either do not decay into solitons at all, or else produce one soliton and a "tail" having an energy comparable with it (Fig. 2).[5] For the initial profile (3.1), $\sigma_c \approx 4$, which coincides in order of magnitude with σ_s (see (2.5)). On the other hand, if we choose as the initial perturbation

$$u(x, 0) = u_0 \operatorname{sch}^2(x/l), \quad (3.10)$$

where u_0 and l are arbitrary constants, then we get $\sigma_c = \sigma_s$. Consequently, the value of the parameter σ for the soliton determines the order of magnitude of the boundary between solutions of qualitatively different types.

When $\sigma > \sigma_c$, the initial perturbation (3.1) always decays into solitons. The crosses in Fig. 3 mark the amplitudes of the solitons derived from (3.1) for different values of the parameter σ. The dependence of the amplitudes of the solitons on σ, obtained from the conservation laws under the assumption that the perturbation decays into two and three solitons, is represented by curves I and II respectively (curve I—two branches of expression (3.6), and curve II was obtained by numerically solving the system (3.5) for N = c). As seen from Fig. 3, the "experimental" values lie well on the corresponding curves obtained under the assumption that the perturbation decays completely into solitons (i.e., that the contribution of the "non-soliton" part of the solution can be neglected).[6] When condition (3.8) is satisfied, the "experimen-

[5]Numerical solutions for times larger than shown in Fig. 2 yield nothing that is essentially new: the "tail," consisting of short-wave oscillations, lengthens and the front maximum, which goes over into a soliton as $t \to \infty$, moves forward.

[6]Apparently the "non-soliton" part of the solution should exist in all those cases, inasmuch, besides those conservation laws from which the amplitudes of the solitons were determined, there exist also others which should also be satisfied.

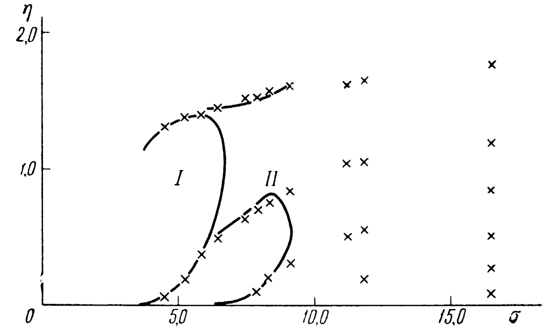

FIG. 3. Dependence of the amplitude of the solitons on the similarity parameter. Crosses—"experimental" values, curve I—decay into two solitons, curve II—decay into three solitons.

tal" values fit well the curve I, with the exception of the direct vicinity of the upper limit $\sigma \sim 2\sigma_c$. Near this limit, the "experimental" points go over already to curve II, i.e., when $\sigma \gtrsim 2\sigma_c \approx 8$ three solitons are produced, which "converge" with curve II, when the amplitudes of two out of the three solitons become sufficiently close to each other.

It is easy to verify that the points at which amplitudes of any two solitons coincide are singular also in the general case—in the sense that the solution giving the final number of solitons becomes unstable at these points. Indeed, if we consider the increments of the amplitudes of the solitons following infinitesimally small variation of the initial condition, then we obtain from the system (3.5)

$$\left(m-\frac{1}{2}\right)\sum_{r=1}^{N}\eta_r^{m-3/2}\,\delta\eta_r = \delta B_m \qquad (m=1,2,\ldots), \quad (3.11)$$

where δB_m are the variations of the right sides of the system (3.5), and $\delta\eta_r$ are the increments of the soliton amplitudes. Relations (3.11) can be regarded as a system of equations for the increments $\delta\eta_r$ for specified η_r and δB_m. If we take the unknowns to be the quantities $\eta_r^{-1/2}\,\delta\eta_r$, then the system of equations will be linear, and its determinant is a Vandermonde determinant made up of the quantities η_r; this determinant, as is well known, vanishes if any two values of η_r coincide. Thus, if $\eta_{r-1} \to \eta_r$ for at least one value of r, then small variations of the initial condition correspond to large changes in the amplitudes of the solitons (i.e., such solutions are unstable against variation of the initial conditions). Near the indicated points, a qualitative change should take place in the character of the solution (a more stable variant is realized).

It must be noted that the solutions which do not give solitons can take place not only when $\sigma < \sigma_c$. For example, if the area of the profile of the initial perturbation is $\int_{-\infty}^{\infty} u(x, 0)dx \le 0$, then its decay into solitons only is impossible, since the area of the profile is conserved, and it must be positive for a soliton. This is correct for the case when the dispersion parameter β is positive. When $\beta < 0$, the area of the profile of the soliton is negative, and therefore, any perturbation for which the profile area is positive cannot decay into solitons. Consequently, for the perturbation to decay into solitons it is necessary that the sign of the area of the profile of this perturbation coincide with the sign of the dispersion parameter β. This condition can be generalized on the basis of the general relations (3.5). Inasmuch as all the square roots in the left sides of (3.5) are taken in the arithmetic sense, these left sides should be positive, i.e., the necessary condition for the decay of the perturbation into solitons is

$$\operatorname{sign}\int_{-\infty}^{\infty} Q_m(x, 0)\,dx = \operatorname{sign}\int_{-\infty}^{\infty} q_m(x)\,dx, \qquad (3.12)$$

where $Q_m(x, 0)$ and $q_m(x)$ are determined by relations (3.3) for the initial perturbation and for the soliton, respectively.

Figure 4 shows by way of illustration the solution of Eq. (1.2) with $\beta < 0$, corresponding to the initial perturbation (3.1) with $u_0 > 0$ ($\sigma = 10$). It is analogous to the solution for the case when $\beta > 0$ and $\sigma < \sigma_c$, the only difference being that the short-wave oscillations go off to the right and not to the left (in accordance with the transformation (1.3)).

4. SELF-SIMILAR SOLUTION OF THE KORTEWEG–De VRIES EQUATION

Using the results presented above, we can obtain additional information with respect to the previously investigated[4] self-similar solution. Let us consider a sequence of initial perturbations, for which the characteristic dimension $l \to 0$, but the product $u_0 l^2$ remains constant. Solutions of (1.2) corresponding to such initial conditions should be similar if β has the same value for them, since σ remains constant. The form of these solutions as $l \to 0$ can be established by starting from the fact that the limiting solution cannot contain constant parameters with dimensions of length of velocity. The only dimensional parameter which can enter in the solution is the dispersion parameter β, with dimensionality cm^3sec^{-1}. Therefore the limiting solution can have only the following form:

$$u(x, t) = \beta^{1/3}t^{-2/3}\psi\{x/(\beta t)^{1/3}\}, \qquad (4.1)$$

where ψ is a dimensionless function. Expression (4.1) coincides with the self-similar solution obtained in [4], where the behavior of the function ψ was also investigated.

It is easy to verify that the initial perturbation which leads to the limiting solution (4.1) is of the form

$$u(x,0) = \beta\sigma^2\delta'(x), \quad (4.2)$$

where σ is the similarity parameter corresponding to the considered sequence of initial perturbations and $\delta'(x)$ is the derivative of the δ function. Indeed, if C is an arbitrary constant, then in the relation

$$C\delta'(x) = \lim_{l\to 0}\{-(Cx/\pi^{1/2}l^3)\exp(-x^2/l^2)\} \quad (4.3)$$

the quantity C/l^2 is the characteristic velocity u_0, and consequently $u_0 l^2 = C = \text{const}$. According to (2.4), $u_0 l^2 = \beta\sigma^2$, from which (4.2) indeed follows. Thus, the physical meaning of the self-similar solution (4.1) is that it describes the evolution of initial perturbations of the type[7)]

$$\frac{\partial}{\partial x}\{(C/\pi^{1/2})\exp(-x^2/l^2)\} \quad \text{for } x \gg l,\ t^{1/3} \gg l\beta^{-1/3}.$$

5. CERTAIN ASYMPTOTIC RELATIONS

It follows from the foregoing that for a rather broad class of initial perturbations, the "evolving" profile will consist of a train of fast oscillations in the left (right) side of the profile when $\beta > 0$ ($\beta < 0$), and a certain sequence of solitons, which goes off to the right (to the left). In certain definite cases, the amplitude of the fast oscillations can be negligibly small compared with the amplitude of the solitons (Fig. 1). For other initial conditions the solution gives a wave packet (Figs. 2 and 4) with fast oscillations, on one hand, and slow ones on the other. It turns out that certain general asymptotic relations can be obtained for the rapidly-oscillating and "soliton" parts of the solution (for the latter, of course, only in the case when the number of solitons is sufficiently large).

To obtain these relations we shall use the procedure developed by Whitham in [12], where he considered solutions characterized by sufficiently slow variation of the parameters of the oscillations at distances that are large compared with their period. In this case, the profile can be represented as a quasi-periodic wave (1.5) with slowly varying parameters (amplitude a, velocity V, etc.). For Eq. (1.2) there are three independent parameters, for which we choose, for example, the amplitude a, the wave number $k = 2\pi/\lambda$, and the mean value $\bar{u}$ (see (15) and (16)). The modulus of the elliptic function s is connected with a and K by the following expression:

$$sK(s^2) = (\pi/k)(a/3\beta)^{1/2}. \quad (5.1)$$

If $s \to 0$, then the elliptic function dn z in (1.5) converges to a trigonometric function, and the amplitude of the oscillations can at the same time remain finite (as $k \to \infty$); such a case is realized in the rapidly-oscillating part of the profile. In the opposite limiting case, when $s \to 1$ at a = const, the wave number $k \sim \{1/\ln(1-s)\} \to 0$ and the wave can be regarded as a sequence of solitons; this case is realized in the "soliton" part of the profile.

Whitham's equations for the "slowly" varying parameters take a form which is convenient for us

$$\frac{\partial}{\partial t}(b_1+b_2) + v_3\frac{\partial}{\partial x}(b_1+b_2) = 0, \ldots, \quad (5.2)$$

where the dots denote cyclic permutation, and $b_i(x, t)$ denotes quantities connected with a, s, and u by the relations

$$a = \frac{1}{2}(b_1 - b_2), \quad s^2 = \frac{b_1-b_2}{b_1-b_3}, \quad \bar{u} = \frac{2aE(s^2)}{s^2K(s^2)} + b_3 \quad (5.3)$$

(for the sake of uniformity we denote here the quantity γ of (1.5) and (1.6) by b_3); the velocities $v_i(x, t)$ are defined

$$v_1 = (V/6) - aK/9(K-E),$$
$$v_2 = (V/6) - a(1-s^2)K/9[E-(1-s^2)K],$$
$$v_3 = (V/6) + a(1-s^2)K/9s^2E, \quad (5.4)$$

$V = (b_1 + b_2 + b_3)/3$ is the velocity of the wave, determined by (1.5). The quantities $v_i(x, t)$ have the meaning of propagation velocities of "Riemann invariants" $b_{i_1} + b_{i_2}$ ($i_1, i_2 \neq i$).

In the general case (5.2)–(5.4) are rather complicated equations, but in the limiting cases $1 - s \ll 1$ and $s \ll 1$ they simplify appreciably. Let us consider these cases in greater detail.

Let $1 - s \ll 1$ (sequence of solitons). Then $b_2 - b_3 \sim 1 - s$, so that $b_2 \approx b_3$. Further, $b_3 = \gamma$ has the meaning of the magnitude of the perturbation between the solitons; therefore $b_3 \approx 0$ (this can also be obtained directly from (5.2)) and the

7)Of course, this pertains to any other sequence of functions that tends to $\delta'(x)$ as $l \to 0$, for example,

$$u(x,0) = Cl^3\frac{\partial}{\partial x}\{(x^2+l^2)^{-1}\}$$

(the solution does not depend on the detailed form of u(x, 0) when x and t are large).

quantity u is logarithmically small ($u \sim 1/\ln(1-s)$).

Thus, $b_1 \approx 2a = u_0$ (u_0 = soliton amplitude), $v_1 \approx 0$, $v_2 \approx v_3 \approx 2u_0$; the second equation of (5.2) becomes degenerate, and the first and third assume an identical form:

$$\frac{\partial u_0}{\partial t} + \frac{1}{3} u_0 \frac{\partial u_0}{\partial x} = 0. \qquad (5.5)$$

The general solution of the equation can be written in the form $3x - u_0(t - t_0) = f(u_0)$, where $f(u_0)$ is an arbitrary function, defined from the conditions at the instant t_0, when the solitons have already been formed. With increasing t, the slope of the profile $u_0(x, t)$ increases and for sufficiently large t the solution becomes multiply valued. In the region where the solution is multiply valued, it becomes physically meaningless, since the averaged equations (5.2)-(5.4) are not valid for large gradients. In the uniqueness region, the solution assumes for sufficiently large $t - t_0$ the asymptotic form

$$u_0(x, t) = 3x / (t - t_0), \qquad (5.6)$$

i.e., at sufficiently large fixed t the peaks of the solitons should lie on a straight line,[8)] the slope of which decreases in inverse proportion to the time.

Relation (5.6) is valid, naturally, when the number of solitons is sufficiently large. In practice its accuracy, as shown by a numerical solution, turns out to be perfectly satisfactory even for a perturbation decay—into six solitons (Fig. 1b). With this, the peaks of the largest and of the smallest solitons deviate somewhat from the straight line, obviously because of the large gradients of the ''average'' amplitude in these regions. It is interesting that relation (5.6) begins to be satisfied with good accuracy even before the solitons traverse a distance which is large compared with the width l of the region of the initial perturbation (the formation of solitons, generally speaking, is very rapid compared with the time necessary to traverse a distance of the order of l).

We can obtain a few other interesting relations characterizing the asymptotic behavior of the quantities k and u for a sequence of solitons. Since these quantities decrease when $s \to 1$ (see (5.1) and (5.3)), it is necessary to take into account in (5.2) small terms, which we have hitherto neglected. We can, however, proceed in simpler fashion by using the ''conservation law'' for the wave number[12]

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(Vk) = 0, \qquad (5.7)$$

which follows from the exact equations (5.2). Confining ourselves in the expression (1.5) for V to terms which do not vanish when $s \to 1$, we obtain $V = u_0/3$. Using formula (5.6) for u_0, we obtain from (5.7) a simple equation, which has a general solution of the type $k = t^{-1} f(x/t)$, where f is an arbitrary function. Just as in the derivation of (5.6), we should stipulate that the asymptotic expression for k be independent of the detailed form of the initial conditions. The wave number k has the dimension of the reciprocal length, and this requirement can be satisfied only when $f(x/t) = Ct/x$, where C is a dimensionless constant. Therefore for sufficiently large x

$$k \approx C / x. \qquad (5.8)$$

Thus, at a fixed point of space the average distance between solitons does not depend on the time. Relation (5.8) can be recast in a more illustrative form by determining the average distance between solitons in the vicinity of a point that moves with some soliton (for example, soliton number r). Denoting its amplitude by $u_0^{(r)}$, we obtain for the soliton coordinate from (5.6) the expression $x = u_0^{(r)} t/3$, from which we get

$$k_r \approx 3C / u_0^{(r)} t, \quad \lambda_r \approx 2\pi u_0^{(r)} t / 3C, \qquad (5.9)$$

where $\lambda_r = 2\pi/k_r$ is the average distance between solitons. It increases in proportion to the time, inasmuch as the solitons move uniformly but with different velocities $V = u_0/3$.

Let us now find $\bar{u}$ for the region under consideration. Taking (5.1) and (5.3) into account and neglecting b_3, we obtain $\pi^2 \bar{u}^{-2} \approx 6\beta u_0 k^2$, whence

$$\bar{u} \approx (18C^2\beta / \pi x t)^{1/2}. \qquad (5.10)$$

Finally, let us consider also the region of fast oscillations in profiles similar to those shown in Figs. 2 and 4. In this region $s^2 \ll 1$. Then it follows from (5.2)–(5.4) that

$$\frac{\partial}{\partial t}(4\bar{u} - 6\beta k^2) + (\bar{u} - 3\beta k^2)\frac{\partial}{\partial x}(4\bar{u} - 6\beta k^2) = 0, \qquad (5.11)$$

and

$$\frac{\partial}{\partial t}\left(\bar{u} + \frac{a^2}{12\beta k^2}\right) + \bar{u}\frac{\partial}{\partial x}\left(\bar{u} + \frac{a^2}{12\beta k^2}\right) = 0, \qquad (5.12)$$
$$a = 3\beta k^2 s^2 / 2$$

When $s^2 \ll 1$ we can put $\bar{u} = 0$ in (5.11), after which we get for sufficiently large x and t the asymptotic relation

8)This was already noted in the analysis of the ''experimental'' data in [10].

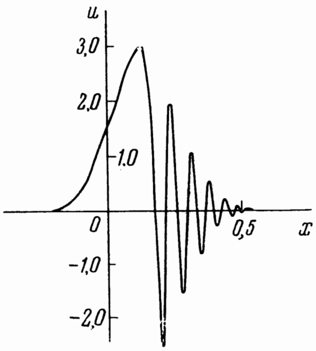

FIG. 4. Profile of perturbation in the case $\beta < 0 (\sigma = 10)$.

$$k^2 \approx -x / 3\beta t. \qquad (5.13)$$

The minus sign denotes here that the region of fast oscillations is situated in the rear ($x < 0$) and front ($x > 0$) parts of the profile, depending on the sign of β (see Figs. 2 and 4 respectively; the origin is chosen at the center of the initial perturbation).

If we recognize now that the complete solution is of the form $u(x, t) = u(x - Vt)$, and that when $s \ll 1$ we can write $u(x) \approx a \sin(kx + \alpha)$ and $V = -\beta k^2$, then

$$u(x, t) \approx a(x, t) \sin [(2/3)(-x^3 / 3\beta t)^{1/2} + \alpha], \qquad (5.14)$$

where $a(x, t)$ is the amplitude determined from the initial conditions. Relation (5.14) has in a definite sense a universal character; it is valid for any rapidly oscillating part of the profile where u is sufficiently small (the amplitude in this case need not necessarily be small). It is therefore natural that the asymptotic expression for the self-similar solution for large x, obtained in [4], has the form (5.14).

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