THE QUASIENERGY OF A QUANTUM-MECHANICAL SYSTEM SUBJECTED TO A PERIODIC ACTION

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Submitted to JETP editor May 24, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1492—1495 (November 1966)

We consider the structure of the solutions of a Schrödinger equation for a quantum-mechanical system which is subjected to the action of a classical force which is a periodic function of time. The concept of quasienergy is introduced for quasistationary states which exhibit periodic time variations. We consider the radiation of a system consisting of single quanta with frequencies which are multiples of the frequency of the external field, or of groups of quanta related to the radiative transitions from one quasistationary state into another, particularly transitions from a lower-lying state into a higher state, at the expense of the energy of the external field.

We consider a quantum-mechanical system subjected to the action of a force which is a periodic function of time. Usually one considers as the zeroth approximation a system in a potential H_0 which does not depend on the time; the states of such a system which correspond to definite values of the energy form a complete orthonormal set. The time-dependent part of the Hamiltonian H_1 is considered as a small perturbation which causes transitions between the eigenstates of H_0 [1)]. However, if H_1 is not small a different approach seems more expedient.

According to general group-theoretical principles, if $H = H_0 + H_1$ is a periodic function of time one should be able to select among the solutions of the Schrödinger equation solutions which go over into themselves after a period T. It is admissible for the solution to be multiplied by a phase factor.

Thus, for the equation

$$i\hbar\partial\psi / \partial t = H\psi, \quad H(t+T) = H(t)$$

we look for solutions such that

$$\psi_\alpha(t+T) = e^{-i\alpha}\psi_\alpha(t).$$

By analogy with the case of a constant H it is natural to designate the quantity

$$E = \hbar\alpha / T, \quad T = 2\pi / \omega.$$

as the quasienergy[2)].

With respect to the operator H(t), states of this kind play the same role as the stationary states do for a constant Hamiltonian. In this case however, the expansion coefficients of an arbitrary function $\psi(x)$ with respect to the states $\psi_\alpha(x, t)$ will depend on the time t, and their absolute values will exhibit a periodic time dependence (with period T). On the contrary, an arbitrary exact solution $\psi(x, t)$ of the Schrödinger equation with the Hamiltonian H(t) will have constant expansion coefficients when expanded with respect to the states $\psi_\alpha(x, t)$.

In the same manner as in a theory with constant H, states belonging to different quasienergies are orthogonal. This can be easily seen if one notes that for two arbitrary solutions $\psi_1(x, t)$ and $\psi_2(x, t)$ of a Schrödinger equation with an arbitrary time-dependent H(t) the hermiticity of the Hamiltonian has as a consequence the fact that the inner product $(\psi_2(t), \psi_1(t))$ does not depend on time. For states ψ_1 and ψ_2 of definite quasienergy E_1 and E_2 one can write the functions $\psi_E(x, t)$ in the form

$$\psi_E(x, t) = e^{-iEt/\hbar}U_E(x, t), \quad U_E(x, t+T) = U_E(x, t).$$

Therefore if $E_2 - E_1 \neq m\hbar\omega$, where m is an integer, the time-independence implies orthogonality for the functions $\psi_{E_1}(t)$ and $\psi_{E_2}(t)$ at any given t:

$$\int dx\, \psi_{E_2}^*(x, t)\psi_{E_1}(x, t) = 0.$$

For solutions with $E_2 = E_1 + m\hbar\omega$ we note that the superposition of two such solutions is again a solution with the same quasienergy (which is defined, as can be easily seen, up to an additive con-

[1)] If the field which produces H_1 is itself considered as a quantized field, the transitions of the system are accompanied by absorption and induced emission of quanta of the field H_1, cf. [1].

[2)] Cf. also the concept of quasi-momentum of a particle in a spatially periodic potential of a crystal lattice.

stant[3] $m\hbar\omega$). Because of this fact one can always orthogonalize the states ψ_{E_1} and ψ_{E_2} for $E_2 = E_1 + m\hbar\omega$, in complete correspondence with the case of a constant H.

The totality of all linearly independent solutions with different quasi-energies forms a complete set of functions (at any given instant of time). It would be desirable to have a formal mathematical proof of this assertion.

An important special case of periodic time dependence of $H_1(t)$ is a purely sine-wave dependence: $H_1(t) = A\cos\omega t$, where A is a Hermitian operator. In this case the system exhibits the group $H(-t) = H(t)$ in addition to the group $H(t) = H(t + T)$. The states with definite quasienergy can also be classified with respect to this additional group.

Of special interest is the situation arising if one takes into account the emission of quanta by the system under consideration. We have in mind quanta of arbitrary direction and frequency; the interaction with the field which gives rise to $H_1(t)$ has already been taken into account. Formally this means that one takes into consideration the electromagnetic field, described by the Hamiltonian H_f of the free field, and the interaction H_i of the free field with the system:

$$H = H_0 + H_1(t) + H_f + H_i.$$

We recall the situation in the absence of H_1: a quantum-mechanical system which interacts with the free field always has a set of levels consisting of a continuum bounded from below. The lowest level E_0 corresponds to the ground state of the system plus the field without free quanta. This state is strictly stationary. Adding quanta of arbitrary direction and energy one can construct states of arbitrary energy $E > E_0$. Owing to the interaction with the field the excited states are not rigorously stationary (due to the possibility of radiation) and their energy is complex.

A complete set of states with real energy consists of the solutions of the scattering problem for quanta scattered by the system. The complete set is determined by the enumeration of the wave-vectors of quanta coming from infinity (solutions of type ψ^+). The distribution of the scattered quanta and the corresponding state of the system are determined by the equation itself. The excitation spectrum of the system plays the role of the poles of the scattering matrix with an especially large scattering cross section and a very large probability of finding the system in an excited state.

[3]We note that if H_1 is adiabatically switched on one can give a well-defined value to the quasienergy of each level.

In the case we consider here of a system subjected to the action of a periodic force $H_1(t)$ the situation is radically different.

The state ψ_d of lowest quasienergy has, in general, a variable dipole moment, and hence the system can radiate from this state. Since $U_\alpha(x, t)$, and consequently the dipole moment, have a period T, it is obvious that the Fourier expansion of the dipole moment contains only frequencies which are multiples of ω. In the language of perturbation theory, we will have scattering of the quanta of the field H_1 as well as a nonlinear doubling, tripling, etc. of the frequency.

In a system subjected to the action of H_1 energy is conserved modulo $\hbar\omega$, i.e., it can change by $m\hbar\omega$, where m is an integer. It is easy to see that the Fourier expansion of the dipole moment for the transition from a state of quasienergy E_1 into a state of quasienergy E_2 contains the frequencies

$$\hbar\omega_{12m} = E_1 - E_2 + m\hbar\omega;$$

$$m \text{ is an integer}, \quad -\infty < m < +\infty.$$

This means, in particular, that there exist positive-frequency Fourier components for the expansion of the dipole moment even for transitions from the lowest quasienergy state (d) into an excited state (2) with $E_d - E_2 < 0$. The system radiates from its ground state, going over into the excited state absorbing energy from the field H_1 in the form of an integral number of quanta $m\hbar\omega$. It is clear that this process is a combined (Raman) scattering of the external field.

Since this is followed by an inverse transition of the excited state into the ground state, one may say that in the stationary problem one deals with the emission of two quanta

$$\hbar\omega' = E_d - E_2 + m'\hbar\omega, \quad \hbar\omega'' = E_2 - E_d + m''\hbar\omega,$$

such that the sum of the energies of the pair is again a multiple of the field quantum.

$$\hbar\omega' + \hbar\omega'' = (m' + m'')\hbar\omega.$$

We consider a state which is stationary in the mean for a system subjected to the action of H_1 and interacts with the field in the absence of quanta coming in from infinity. Such a state is the closest analogue of the ground state of a system in a static field (in the absence of H_1).

In this state in the presence of H_1 there is on the average a stationary flux of quanta emitted by the system. Formally quanta of all frequencies are present but of special importance in the spectrum are: a) monochromatic lines with frequencies $\omega, 2\omega, 3\omega, \ldots$; b) correlated pairs of quanta with

frequencies ω' and ω'', forming a continuous spectrum, but within this continuous spectrum the quasienergy differences of the system (modulo ω) form poles which are responsible for sharp Lorentz maxima in the spectrum; c) the similar emission of three or more quanta. The presence of the additional symmetry ($H(-t) = H(t)$) gives rise to definite selection rules for the radiation.

We have included above the interaction H_1 in the determination of the states with given quasienergy and have considered the interaction with the field as a perturbation. In principle one could have looked for a solution of the complete problem which is stationary in the mean, which becomes unique if one imposes the radiation condition, and contains all the wave functions of the components of radiation enumerated above. This state, for which the existence is intuitively clear[4)], will go over into the ground state of the system characterized by H_0, if H_1 and H_i are adiabatically switched off. If only H_1 is switched off, leaving H_i and H_f on, we obtain the ground state of the system with quantum-electrodynamic corrections (renormalizations) taken into account. Thus, for the example of the hydrogen atom, the Lamb-shift would be taken into account.

Both the problem involving radiation and the radiationless problem are considerably simpler if the system has a finite and small number of levels. The minimal number is obviously two in the case of a particle of spin $\frac{1}{2}$ in a magnetic field, i.e., in the paramagnetic resonance problem.

[4)]The existence of a discrete quasienergy spectrum (in the absence of interaction with radiation) depends on the form of H_0 and H_1. There is no discrete spectrum if, for instance, H_0 is the Hamiltonian of an electron in a Coulomb field and H_1 corresponds to a plane wave: $H_1 = e\mathbf{E}\cdot\mathbf{r}\cos\omega t$ (cf. [2]). There is a discrete spectrum for the same H_1, but H_0 corresponding to an oscillator: $U_0(\mathbf{r}) = k\mathbf{r}^2$.

[1] T. H. Shirley, Phys. Rev. 138, B979 (1965).

[2] L. V. Keldysh, JETP 47, 1945 (1964), Soviet Phys. JETP 20, 1307 (1965).

Translated by Meinhard E. Mayer
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