THE QUASILINEAR THEORY OF ION-ACOUSTIC WAVES

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A one-dimensional theory of ion-acoustic waves in a plasma is developed in the quasilinear approximation by taking collisions into account. An explicit expression is found for the spectral density of ion-acoustic waves as a function of time for a current plasma which is unstable in the linear approximation, and which is located in an external electric field. It is shown that in a stationary state, the wave intensity is determined by the magnitude of collision damping and Landau ion damping. The case of the propagation in a plasma of intense ion-acoustic wave is also considered. It is shown that the character of its damping also depends significantly on the collision frequency.

As is well known, linear theory enables us to describe the propagation of waves in a plasma only in the case in which their intensity is sufficiently small so that the effect of the waves on the distribution function of the particles can be neglected. If the intensity of noise (waves) in a plasma appreciably exceeds the thermal noise, then such a neglect is generally invalid and it is necessary to also take into account the scattering of particles by the plasma noise, that is, the inverse effect of the plasma waves on the distribution function of the particles. Account of this effect of the waves on the distribution function in lowest order in nonlinearity is the content of the so-called quasilinear approximation,[1] the use of which is essential in the case of systems that are unstable in the linear approximation.

A sufficiently detailed analysis of the one-dimensional model of the beam instability of Langmuir waves was worked out by Vedenov[1] within the framework of the quasilinear theory; the character of their damping in a plasma with account of electron-ion collisions was also investigated. The present research is devoted to a similar analysis of the development of ion-acoustic instability in a plasma located in an external electric field, and to the investigation of the character of the damping of ion-acoustic noise in a current-free collision plasma.[1)] Here, as in [1], we restrict ourselves to the study of a one-dimensional model, which allows us to obtain an analytic solution of the problem very easily.

Thus we consider a nonisothermal electron-ion plasma located in an external electric field **E**. We assume for concreteness that the velocity distribution of the ions is Maxwellian with a temperature equal to T_i, and the electron distribution is close to Maxwellian[2)] and is characterized by a temperature $T_e \gg T_i$. We denote by $f(v,t)$ the one-dimensional electron distribution in the plasma, normalized to unity, and by $w(s,t)$ the one dimensional energy spectral density of ion-acoustic noise, so normalized that the total noise energy density is

$$\mathscr{E}(t) = \int ds\, w(s,\ t),$$

where $s = \omega(k)/k$ is the phase velocity of ion-acoustic waves. Then the initial set of nonlinear equations which determines the behavior of the function $f(v,t)$ and $w(s,t)$ will have the form (see, for example, [6])[3)]

$$\frac{\partial f}{\partial t} + \Gamma \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} D(v) \frac{\partial f}{\partial v} + \mathrm{St}(f), \tag{1}$$

$$\frac{\partial w}{\partial t} = [\gamma_s - \gamma]\, w, \tag{2}$$

1)Without account of collisions, the quasilinear theory of a plasma located in a strong electric field was developed in the researches of Field and Fried[2] and Korablev and Rudakov.[3]

2)If runaway electrons are neglected,[4,5] the electron distribution function can be regarded as Maxwellian in first approximation because, as we shall see below, the presence of noise leads only to a small change of it in a small region of velocities of the order of the phase velocities of ion sound.

3)The equations of the three-dimensional problem can also be reduced to such a form in the case of a strongly magnetized plasma if **H** || E and $eH/Mc > \sqrt{m/M}\,\omega_0$. Here k and v are the components of the corresponding vectors parallel to the direction of the external fields E and **H**.

where the first component on the right side of Eq. (1) is proportional to the function

$$D(v)=\frac{\pi\delta^{-2}}{nm}\omega_0\frac{v\,[s_m^2-v^2]^{1/2}}{v_{Te}}w(v), \quad (3)$$

and takes into account the scattering of the plasma electrons by the ion-acoustic noise and the second, which has the form

$$\mathrm{St}(f)=\nu_e\frac{\partial}{\partial v}\left[vf+v_{Te}^2\frac{\partial f}{\partial v}\right], \quad (4)$$

takes the electron collisions into account. The quantity γ_0 in Eq. (2) is the growth increment (damping decrement) of linear theory, and is equal to

$$\gamma_s=\pi\delta^{-2}\omega_0\frac{\sqrt{s_m^2-s^2}}{v_{Te}}s^2\frac{\partial f}{\partial s}, \quad (5)$$

while the component $\gamma = \gamma_i + \gamma_{coll}$ takes into account the Landau damping of ion sound by ions, and by ion-ion collisions, [7] respectively:

$$\gamma_i=\sqrt{\frac{\pi}{2}}\delta\omega_0\alpha^3\frac{s^3}{s_m^3}\frac{\sqrt{s_m^2-s^2}}{s_m}\exp\left(-\frac{\alpha^2s^2}{2s_m^2}\right),$$
$$\gamma_{coll}=\nu_i\frac{v_{Ti}^2}{s^2}\approx\nu_e\frac{s_m^2}{s^2}\left(\frac{mT_e}{MT_i}\right)^{1/2}. \quad (6)$$

In Eqs. (3)—(6), m—mass, n—density, ω_0—Langmuir frequency of the plasma electrons, z and M—the charge number and the mass number of the ions, $v_{Te}^2 = T_e/m$ and $v_{Ti}^2 = T_i/M$—the squares of the thermal velocities of the electrons and ions, ν_e and ν_i—the frequencies of the electron and ion collisions, $s_m = \sqrt{zT_e/M}$—the maximum possible velocity of sound waves and, finally, $\Gamma = eE/m$, $\delta^2 = zm/M$, $\alpha^2 = zT_e/T_i$.

We denote by $\gamma_s^0 = \sqrt{\pi/2}\,\delta^{-2}v_{Te}^{-3}ks^3(u-s)$ the increment (5), found according to the linear theory ($u = \Gamma/\nu_e$). Analysis of Eqs. (1) and (2) shows that the process of development of the instability (or absorption of an intense wave in the case of the absence of an external field) can develop in two stages. During the course of the first, comparatively brief stage, which lasts a time of the order of several reciprocals of the increment γ_s^0, the noise intensity grows very rapidly, almost exponentially, and an explicit dependence of the distribution function on time (that is, the component $\partial f/\partial t$) is very significant. Collisions play practically no role here.

Next, a second, longer stage of development of a certain stationary (or, in the absence of an external field **E**, quasistationary) distribution sets in when the explicit dependence of the distribution function on time (that is, the component $\partial f/\partial t$) becomes insignificant, and the fundamental role is played by the collision of electrons with ions and their scattering by waves. Just this second stage will be the basis for the present investigation.

Simple estimates show that if the intensity of the noise is sufficiently large, so that[4)]

$$\gamma_s^0\ll\nu_e[1+w/w_0]\approx\nu_e[1+\mathscr{E}/\mathscr{E}_0], \quad (7)$$

where

$$w_0(s)=\frac{\nu_e}{\pi\omega_0}\delta^2nT_e\frac{v_{Te}}{s\sqrt{s_m^2-s^2}},\mathscr{E}_0=\int w_0ds\approx\frac{\nu_e}{\omega_0}\delta nT_e, \quad (8)$$

then the component $\partial f/\partial t$ in Eq. (1) can be neglected in comparison with the other terms. In this case, Eq. (7) is easily integrated, and we find

$$f(v,t)=A\exp\left[\frac{1}{v_{Te}^2}\int_u^v\frac{u-v}{[1+w/w_0]}dv\right],$$
$$A\approx\frac{1}{\sqrt{2\pi}\,v_{Te}},\quad u=\frac{\Gamma}{\nu_e}. \quad (9)$$

Equation (2) takes the form

$$\frac{\partial w}{\partial t}=\gamma w\frac{w_{coll}-w}{w_0+w}, \quad (10)$$

where w_0 was given by Eq. (8) and

$$w_{coll}(s)=w_0\left[\frac{\gamma_s^0}{\gamma}-1\right]$$
$$=\delta\frac{nT_e}{s_m}\left\{\frac{\delta^2}{\sqrt{2\pi}}\frac{\nu_e}{\gamma}\frac{s(u-s)}{s_m^2}-\frac{1}{\pi}\frac{\nu_e}{\omega_0}\frac{s_m^2}{s\sqrt{s_m^2-s^2}}\right\}. \quad (11)$$

Equation (10) is easily integrated and for $w \gg w_0$, we find

$$w(s,t)=w_{coll}(s)\left[1-e^{-\gamma(s)t}\right]+w_{init}e^{-\gamma(s)t}, \quad (12)$$

where by $w_{init}(s)$ is meant the initial value of the noise.

It follows from (12) that account of absorption γ leads to a limitation of the amplitude of ion-acoustic noises and to the establishment of a certain stationary state characterized by the spectral distribution (11). The characteristic time for establishing the stationary state is determined by the quantity γ, that is, by the collision damping and Landau ion damping, which also determine the corresponding upper and lower (in s) boundaries of the spectrum. Here the total noise energy in the stationary state for $u \gtrsim s_m$ is equal to

$$\mathscr{E}_{coll}=\int ds\,w_{coll}(s)\approx nT_e\delta^3\frac{u}{s_m}\frac{\nu_e}{\gamma(s_m)}. \quad (13)$$

[4)]This condition, as is easy to see, is essentially equivalent to the requirement that the intensity of the ion-acoustic noise in the plasma greatly exceeds the level of thermal noise.

Since the quantity s is proportional to the collision frequency ν_e over a wide range of values of s,[5)] it follows from (11) and (13) that the noise intensity of the stationary state depends only on the ratio $u = eE/m\nu_e$. In other words, in the case of a sufficiently weak electric field $E \ll mv_{Te}\nu_e/e$, when one can neglect the runaway electrons, the maximum noise energy in the stationary state is practically independent of the collision frequency, and is equal to[6)]

$$\mathscr{E}_{\text{coll}}^{max} \approx nT_e\delta^2 \frac{\nu_e}{\gamma(s_m)} \approx nT_e\left(\frac{mT_i}{MT_e}\right)^{1/2}. \quad (14)$$

We now turn to the investigation of the character of the damping of intense ion-acoustic waves in a plasma without an external electric field, and in one excited by any sort of "external" source. In this case, setting $u = 0$ in Eqs. (9) and (10), we find

$$f(v,t) = \frac{1}{\sqrt{2\pi}\, v_{Te}} \exp\left\{-\frac{1}{v_{Te}^2}\int_0^v \frac{v\,dv}{[1 + w/w_0]}\right\}, \quad (15)$$

$$\frac{\partial w}{\partial t} = \gamma_s^0 \frac{w_0 w}{w_0 + w} - \gamma w \approx -a - \gamma w,$$

$$a = -\gamma_s^0 w_0 = \delta^3 \nu_e \frac{nT_e}{\sqrt{2\pi}} \frac{s^2}{s_m^2}. \quad (16)$$

Similar to the above, these equations are valid only for sufficiently large initial noise intensities, when after a time $\sim 1/|\gamma_s^0|$ Eq. (15) will be valid for the distribution function (or, for example, for the boundary condition when an "outside" source of noise acts for a sufficiently long time).

Denoting by w_{in} the value of the spectral density of the noise at the moment $t = 0$, and taking it into account that in the case of interest to us, $w \gg w_0$, we find from (16)

$$w(s,t) = w_{\text{init}}e^{-\gamma t} - a\gamma^{-1}[1 - e^{-\gamma t}]. \quad (17)$$

Thus we see that for sufficiently high intensity the noise is damped more slowly than follows from the linear theory (under the condition of course that $\gamma_s^0 \gg \gamma$). Here, if $w_{\text{init}} \gg a/\gamma$, then the damping has an exponential character. In the inverse case, when $w_{\text{init}} \ll a/\gamma$, the intensity of the noise falls off according to a linear law similar to what was obtained in reference [1] for the Langmuir waves.

APPENDIX

In the considerations given above, we have completely neglected the nonlinear interactions of the waves with one another (nonlinear damping or s-s scattering) inasmuch as such interaction is absent in the one-dimensional model. However, taking it into account that the results obtained in this approximation can usually be of value even in the more general case, it is of interest to obtain the condition under which such a neglect is appropriate.[7)] This condition evidently is identical with the condition of smallness of the "nonlinear" increment γ_n in comparison with γ_s. Using the expression for the nonlinear increments [6,8,9] and assuming for the estimate $k = \omega_0/v_{Te}$, $s = s_m$, and taking (13) into account, we find

$$\gamma_n/\gamma_s = (2\pi)^2\theta_0^2 \frac{\omega_0\nu_e}{\gamma^2}\delta^4\left(\frac{T_i}{T_e}\right)\frac{u}{s_m}, \quad (A.1)$$

where θ_0 is the maximal angle between the wave vector $\mathbf{k}$ and the vector of the electric field $\mathbf{E}$ for which the vibrations are still unstable. Then, taking it into account that $\gamma \approx \delta(T_e/T_I)^{1/2}\nu_e$, we find that the nonlinear interaction of the waves is unimportant if the frequency of the electronic collisions

$$\nu_e \gg (2\pi)^2\theta_0^2\omega_0\frac{m}{M}\left(\frac{T_i}{T_e}\right)^2\frac{u}{s_m}. \quad (A.2)$$

On the other hand, by virtue of the inequality $\gamma \ll \gamma_s^0$, we should have the condition

$$\nu_e \ll \frac{u}{s_m}\left(\frac{mT_i}{MT_e}\right)^{1/2}\omega_0. \quad (A.3)$$

It is seen from a comparison of (A.2) and (A.3) that there is a sufficiently broad region of values of the parameters when the account of the nonlinear interaction is unimportant. In the case of sufficiently small values of the collision frequency, when condition (A.2) is violated, the nonlinear damping must be considered and it can have a significant effect on the character of the stationary spectrum (if such is present).

In this connection, we want to make a remark in connection with the solution found in [9] (see also [10]) for the spectral density of ion-acoustic noise,

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5)For a sufficiently strong nonisothermal process, the Landau ion damping γ_i is important only in the region of small phase velocities $s \sim s_{min} \approx 2v_{Ti}\ln(a^3/\delta)$.

6)We emphasize that the theory developed here is valid only for sufficiently weak fields $E \ll mv_{Te}\nu e/e$ when the number of runaway electrons is exponentially small. In the opposite case, all the electrons with $v > s_m$ enter into a regime of continuous acceleration: this follows from the fact that for $v > s_m$ we have $w(v) = 0$.

7)We note that Petviashvili, [9] in a paper devoted to the study of the stationary spectrum of ion-acoustic waves, uses for the increment γ_0 an expression obtained in the case of the one-dimensional model.

which takes into account the nonlinear damping; in our view, this is in error.

In order to show this, we shall start out from a somewhat more general equation having the form[8)]

$$k\lambda + \gamma_s(k, x) W(k, x)$$

$$+ ak^2 W(k, x) \frac{\partial}{\partial k} k^3 \int_{-1}^{1} dx' \varkappa(x, x') W(k, x') = 0. \quad \text{(A.4)}$$

Here λ and a are certain constants whose specific values are of no importance for us, $\gamma_S(k, x)$ is the increment of the quasilinear theory, $W(k, x)$ is the spectral energy density of the ion-acoustic waves, x is the cosine of the angle between the wave vector **k** in the direction of the external electric field, while the kernel is

$$\varkappa(x, x') = 1 + {}^5/_7 P_2(x) P_2(x') - {}^{12}/_7 P_4(x) P_4(x'), \quad \text{(A.5)}$$

where $P_n(x)$ is the Legendre polynomial of order n.

For a value of the parameter $\lambda = 0$, we obtain the equation used as the starting point in the research of [9]. This equation was solved by averaging over x, which, in our opinion, is incorrect, because of the degeneracy of the kernel $\kappa(x, x')$. Actually, taking Eq. (A.5) into account, it is not difficult to establish the fact that the solution of Eq. (A.4) for the kernel $\kappa(x, x')$ should have the form

$$W(k, x) = \frac{\lambda k}{A + BP_2(x) + CP_4(x) - \gamma_s(k, x)}, \quad \text{(A.6)}$$

where the quantities A, B, and C are functions only of k and not x.

It is then seen that as $\lambda \to 0$ the solution $W(k, x)$ also tends to zero. An exception exists only in the very special case in which the increment is

$$\gamma_s(k, x) = A + BP_2(x) + CP_4(x),$$

that is, in particular, it is an even function of x.

Consequently, the solution obtained in [9] is generally incorrect because the equation used there for $\lambda = 0$ is the same as Eq. (A.4) which has only a trivial solution in this case.[9)]

[1] A. A. Vedenov, Voprosy teorii plazmy (Problems of Plasma Theory), Atomizdat, 1963, v. 3, p. 203.

[2] E. F. Field and B. D. Fried, Phys. Fluids **7**, 1937 (1964).

[3] L. V. Korablev and L. I. Rudakov, JETP **50**, 220 (1966), Soviet Phys. JETP **23**, 145 (1966).

[4] H. Dreicer, Proc. Second Int. Conf. Peaceful Uses of Atomic Energy, Geneva, 1958.

[5] L. M. Kovrizhnykh, JETP **37**, 1394 (1959), Soviet Phys. JETP **10**, 989 (1960).

[6] L. M. Kovrizhnykh, JETP **48**, 1114 (1965), Soviet Phys. JETP **21**, 744 (1965); Rapport EUR-CEA-FC-258, Fontenay-aux-Roses, France, 1964.

[7] E. E. Lovetskiĭ and A. A. Rukhadze, JETP **41**, 1845 (1961), Soviet Phys. JETP **14**, 1312 (1962).

[8] L. M. Gorbunov and V. P. Silin, JETP **47**, 200 (1964), Soviet Phys. JETP **20**, 135 (1965).

[9] V. I. Petviashvili, DAN 153, 1295 (1963), Soviet Phys. Doklady **8**, 1218 (1964).

[10] B. B. Kadomtsev, Voprosy teorii plazmy (Problems of Plasma Theory), Atomizdat, 1964, v. 4, p. 258.

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[8)]We note that in the case of axial symmetry, the equation for W reduces to such a form if one also takes into account the processes of spontaneous emission (see for example, [6]).

[9)]It is possible that account of absorption in the walls and the dependence of W on the spatial coordinates will also lead to a solution of the type obtained in [9]. However, this is nowhere evident and requires additional investigation.