"SINGULAR" SOLUTIONS OF THE EQUATIONS FOR PLASMA OSCILLATIONS

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We consider the effect of Coulomb collisions on plasma oscillations with frequencies and decrements that depend on the initial perturbation ("singular" solutions). In total absence of collisions, such oscillations may decay at a slower rate than oscillations with a Landau decrement. It is shown that Coulomb oscillations, even of very low frequency (much lower than the Landau decrement) result in a very rapid decay of the singular solutions.

As is well known, the general solution of the linearized equations for plasma oscillations[1)]

$$\frac{\partial f_k(t,v)}{\partial t} + ikvf_k + \frac{e}{m}E_k\frac{\partial f_0}{\partial v} = 0, \qquad (1)$$

$$-ikE_k(t) = 4\pi eN\int f_k(t,v)\,dv \qquad (2)$$

is of the form[1]

$$E_k(t) = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{d\omega}{2\pi}\frac{r_k(\omega)}{\varepsilon_k(\omega)}e^{-i\omega t}, \qquad (3)$$

$$r_k(\omega) = \frac{4\pi eN}{k}\int_{-\infty}^{\infty}\frac{g_k(v)\,dv}{\omega - kv},$$

$$\varepsilon_k(\omega) = 1 + \frac{\omega_0^2}{k}\int_{-\infty}^{\infty}\frac{dv}{\omega - kv}f_0'(v), \qquad (4)$$

where $f_0(v)$ is the equilibrium distribution function; $E_k(t)$, $f_k(t, v)$ and $g_k(v)$ are the Fourier components of the field, of the distribution function, and of the initial perturbation, with

$$f_k(0, v) = g_k(v); \qquad (5)$$

the functions $\epsilon_k(\omega)$ and $r_k(\omega)$ are analytic in the upper half-plane. It follows from (3) that the asymptotic value of the field is determined, first, by the zeroes of the dielectric constant[2)] $\epsilon_k(\omega)$ and, second, by singularities of the functions $r_k(\omega)$, which in turn are determined by the character of the initial perturbation $g_k(v)$.

If Ω_k is the root of the dispersion equation $\epsilon_k(\omega)$ closest to the real axis, and $g_k(v)$ is an analytic function whose singularities lie above the point Ω_k/k, then the asymptotic form of (3) is $E_k(t) \sim E_0 \exp(-i\Omega_k t)$, and describes plasma oscillations with frequency $\omega_k = \mathrm{Re}\,\Omega_k$ and decrement $\gamma_k = -\mathrm{Im}\,\Omega_k$.

On the other hand if, for example, $g_k(v)$ has a pole at a point u such that $0 \geq \mathrm{Im}\,u > \mathrm{Im}\,\Omega_k/k$, then the field has the asymptotic form $\exp(-ikut)$ and attenuates more slowly than the "plasma" term, which attenuates with a Landau decrement γ_k. A particular case of this type are the solutions of Van Kampen,[2] where $\mathrm{Im}\,u = 0$, so that in general undamped oscillations of frequency ku are obtained. Another example are the solutions considered in [3,4], where the function $g_k(v)$ or its derivatives have finite discontinuities on the real axis, i.e., they are piecewise-smooth functions. If, for example, the n-th derivative of the function $g_k(v)$ has a discontinuity at the point u, and the derivatives of lower order are continuous, then the function $r_k(\omega)$ in (4) has a branch point at $\omega = ku$, leading in the asymptotic expression for the field to a term $\sim t^{-(n+1)}\exp(-ikut)$, which should thus prevail over the "plasma" term if it is sufficiently large.

The solutions that have an asymptotic form determined by the initial perturbation $g_k(v)$ will be called "singular." It is obvious that singular solutions appear when the function $g_k(v)$ or its derivatives vary sufficiently rapidly in some small interval δv of velocity space.

We note that even in the case when $g_k(v)$ is an entire function of v, the term decisive in solution (3) may be the one depending on the initial perturbation. Let, for example,

$$g_k(v) = (1/\sqrt{\pi}\alpha)\exp[-(v-u)^2/\alpha^2].$$

then for sufficiently small ω the solution (3) can be reduced to the form

1)For simplicity we consider throughout electron oscillations without a magnetic field, and the ions are assumed to be infinitely heavy.

2)We are considering only a stable plasma, since all the roots $\epsilon_k(\omega)$ lie in the lower half-plane.

$$E_k(t) \approx \frac{r_k(\Omega_k)}{\partial\varepsilon_k(\Omega_k)/\partial\Omega_k}\exp(-i\Omega_k t) + \frac{4\pi eN}{ik}\frac{\exp(-k^2\alpha^2t^2/4)\exp(-ikut)}{\varepsilon_k(ku)}. \quad (6)$$

For sufficiently small α ($k\alpha/2 < -\mathrm{Im}\,\Omega_k$) the second term in formula (6) can be considerably larger than the first in the entire region of t, where the field is still not small (although formally there exist t so large that the second term becomes considerably smaller than the first, these values of t are no longer of interest, since the field has time to become very small). As $\alpha \to 0$, we obtain from this again the Van Kampen solutions.

2. All this, however, is obtained if we completely neglect the collisions. In an analysis of problems connected with the rate of change of the distribution function, it is necessary to bear in mind that owing to the diffusion character of the Coulomb collisions, the latter smooth out rapidly in the abrupt changes of the distribution function. In this connection, it is of interest to ascertain the influence exerted on the "singular" solutions by the Coulomb collisions (in the case when the frequency of the latter is sufficiently small).

To take the collisions into account, we must add to the right side of (2) the Landau collision integral. The solution of the kinetic equation is in this case a very complicated problem, but the situation is simplified by the fact that we are interested in questions connected with a change in the distribution function in a relatively small interval δv, and with relaxation processes described by the collision integral and having a diffusion behavior in velocity space. This enables us to replace the exact collision integral by the model collision term proposed by Lennard and Bernstein:[5]

$$\mathrm{St}\{F\} = \beta\frac{\partial}{\partial v}\left(vF + \frac{v_T^2}{2}\frac{\partial F}{\partial v}\right), \quad (7)$$

where v is the velocity component in the wave propagation direction, v_T the average thermal velocity of the particles, and β a certain effective collision frequency. Expression (7) retains two most important properties of the Coulomb collision integral; it has a "diffusion" behavior and vanishes if we substitute in it the Maxwellian distribution function

$$f_0(v) = (1/\sqrt{\pi}v_T)\exp(-v^2/v_T^2).$$

Taking (7) into account, the kinetic equation takes the form

$$\frac{\partial f_k(t,v)}{\partial t} + ikvf_k + \frac{eE_k(t)}{m}f_0'(v) = \beta\frac{\partial}{\partial v}\left(vf_k + \frac{v_T^2}{2}\frac{\partial f_k}{\partial v}\right) \quad (8)$$

under the initial condition (5). Applying to (8) the Laplace transformation

$$f_k(\omega, v) = \int_0^\infty f_k(t,v)e^{i\omega t}\,dt, \quad (9)$$

$$f_k(t,v) = \int_{-\infty+i\delta}^{\infty+i\delta} f_k(\omega,v)e^{-i\omega t}\frac{d\omega}{2\pi}, \quad (10)$$

where $f_k(\omega, v)$ is analytic in the upper half-plane of ω, we obtain

$$-i(\omega - kv)f_k(\omega,v) - \beta\frac{\partial}{\partial v}\left[vf_k(\omega,v) + \frac{v_T^2}{2}\frac{\partial f_k(\omega,v)}{\partial v}\right] = -\frac{e}{m}E_k(\omega)f_0'(v) + g_k(v). \quad (11)$$

This equation is easy to solve by taking the Fourier transform in v of both sides. As a result we obtain

$$f_k(\omega,v) = -\frac{e}{m}E_k(\omega)\int_{-\infty}^{\infty}G_k(\omega,v,v')f_0'(v')\,dv' + \int_{-\infty}^{\infty}G_k(\omega,v,v')g_k(v')\,dv', \quad (12)$$

where $G_k(\omega, v, v')$ is the Green's function of the left side of (11), which takes the form

$$G_k(\omega,v,v') = \frac{1}{2\pi k}\int_{-\infty}^{\infty}d\sigma\int_{\sigma}^{k/\beta}d\tau\left(1-\frac{\beta\tau}{k}\right)^{-i\omega/\beta+k^2v_T^2/2\beta-1} \times\left(1-\frac{\beta\sigma}{k}\right)^{i\omega/\beta-k^2v_T^2/2\beta^2}\exp\left[i(\sigma v-\tau v') + \frac{v_T^2\tau^2}{4} \times\frac{v_T^2k\tau}{2\beta} - \frac{v_T^2\sigma^2}{4} - \frac{v_T^2k\sigma}{2\beta}\right]. \quad (13)$$

Going over from σ and τ to new variables of integration s and s':

$$1-\frac{\beta\sigma}{k} = e^{-\beta s}, \qquad 1-\frac{\beta\tau}{k} = e^{-\beta s'}, \quad (14)$$

we obtain

$$G_k(\omega,v,v') = \frac{1}{8\pi}\int_{-\infty}^{\infty}ds\int_{s}^{\infty}ds'\exp\left[-\beta s - \frac{ikv'}{\beta}(1-e^{-\beta s'}) + \frac{ikv}{\beta}(1-e^{-\beta s}) + \left(i\omega - \frac{k^2v_T^2}{2\beta}\right)(s'-s) - \frac{k^2v_T^2}{4\beta^2}(2-e^{-\beta s})^2 + \frac{k^2v_T^2}{4\beta^2}(2-e^{-\beta s'})^2\right]. \quad (15)$$

We now proceed to calculate the field $E_k(t)$. From (2), (9), and (12) we easily obtain

$$E_k(\omega) = \tilde{r}_k(\omega) / \tilde{\varepsilon}_k(\omega), \tag{16}$$

where

$$\tilde{r}_k(\omega) = \frac{4\pi eN}{k} \int_{-\infty}^{\infty} dv\, I_k(\omega, v) g_k(v), \tag{17}$$

$$\tilde{\varepsilon}_k(\omega) = 1 + \frac{\omega_0^2}{k} \int_{-\infty}^{\infty} dv\, I_k(\omega, v) f_0'(v), \tag{18}$$

$$I_k(\omega, v) = -i \int_{-\infty}^{\infty} G_k(\omega, u, v)\, du =$$

$$-i \int_0^{\infty} \exp\left\{ \frac{k^2 v_T^2}{4\beta^2} \left[(1 - e^{-\beta\sigma})^2 + 2(1 - e^{-\beta\sigma}) \right.\right.$$

$$\left.\left. - i \frac{kv}{\beta}(1 - e^{-\beta\sigma}) - \frac{k^2 v_T^2}{2\beta} \sigma \right] + i\omega\sigma \right\} d\sigma. \tag{19}$$

Expressions (16)–(18) are in general similar to formulas (3) and (4), which are valid when the collisions are completely neglected. The only difference is that (17) and (18) contain in place of $(\omega - kv)^{-1}$ the quantity $I_k(\omega, v)$ which goes over into $(\omega - kv)^{-1}$ as $\beta \to 0$.

It is easy to verify that for small β the quantity $\epsilon_k(\omega)$ does not differ qualitatively from $\epsilon_k(\omega)$. Indeed, substituting (19) in (18) and expanding $\tilde{\epsilon}_k(\omega)$ in powers of β we obtain

$$\tilde{\varepsilon}_k(\omega) = \varepsilon_k(\omega)$$

$$- i\beta k\omega_0^2 \int_{-\infty}^{\infty} dv\, f_0(v) \left[\frac{2v}{(\omega - kv)^4} - \frac{2v^2}{v_T^2(\omega - kv)^3} \right] \tag{20}$$

The presence of a small addition $\sim\beta$ is reflected only in a certain shift of the roots of the dielectric constant. Denoting the new root by $\tilde{\Omega}_k$, we obtain $\tilde{\Omega}_k \approx \omega_k - i(\gamma_k + \beta/2)$, where ω_k and γ_k are the frequency and damping decrement of the oscillations in the collisionless theory. Since we have assumed that $\beta \ll \gamma_k$, this correction is generally insignificant and we shall neglect it, putting $\tilde{\epsilon}_k(\omega) \approx \epsilon\ (\omega)$.

Of much greater importance for us is the change in the analytic properties of the numerator in (16) when $\beta \neq 0$, and the associated change in the asymptotic expression for $E_k(t)$, in the case when the initial perturbation changes sufficiently radically in a small region. It is easy to verify, first, that the function $\tilde{r}_k(\omega)$ is an entire function of ω, even if $g_k(v)$ is a piecewise-smooth function. To obtain a detailed idea of the behavior of the field in this case, let us assume that $\epsilon_k^{-1}(\omega)$ can be represented in the form of the following partial-fraction expansion:

$$\varepsilon_k^{-1}(\omega) = 1 + \sum_r \frac{1}{\varepsilon'(\Omega_k^{(r)})(\omega - \Omega_k^{(r)})},$$

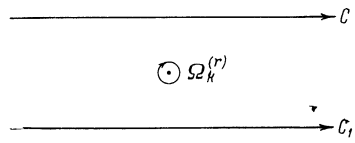
$$\varepsilon'(\omega) = \frac{\partial \varepsilon}{\partial \omega}, \tag{21}$$

where $\Omega_k^{(r)}$ (r = 1, 2, 3, ...) are the zeroes of the dielectric constant, arranged in decreasing order of their imaginary parts. Thus, the value r = 1 corresponds to the root determining the frequency and decrement of the plasma oscillations, that is, $\Omega_k^{(1)} = \Omega_k$. Substituting (21) in (16) and going over to the t-representation, we obtain

$$E_k(t) = \int_C \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{r}_k(\omega)$$

$$+ \sum_r \frac{1}{\varepsilon'(\Omega_k^{(r)})} \int_C \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{r}_k(\omega)}{\omega - \Omega_k^{(r)}}, \tag{22}$$

where C is a contour lying above all the points $\Omega_k^{(r)}$ in the ω plane (see the figure).



We now consider the integral under the summation sign in (22). It can be reduced to an integral along a contour C_1 lying below the point $\Omega_k^{(r)}$, by adding to the latter the corresponding residue relative to the pole at the point $\Omega_k^{(r)}$ (see the figure):

$$\int_C \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{r}_k(\omega)}{\omega - \Omega_k^{(r)}} = -i\tilde{r}_k(\Omega_k^{(r)}) \exp(-i\Omega_k^{(r)} t)$$

$$+ \int_{C_1} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{r}_k(\omega)}{\omega - \Omega_k^{(r)}}. \tag{23}$$

Inasmuch as in the integral along the contour C_1 we have Im $\omega <$ Im $\Omega_k^{(r)}$, we can write

$$(\omega - \Omega_k^{(r)})^{-1} = i \int_{-\infty}^{0} \exp[i\theta(\omega - \Omega_k^{(r)})] d\theta. \tag{24}$$

Substituting now (24) in (23), changing the order of integration, and carrying out elementary calculations, we obtain for the field defined by (22) the following expression:

$$E_k(t) = \tilde{R}_k(t) + \sum_r \frac{\tilde{r}_k(\Omega_k^{(r)})}{i\varepsilon'(\Omega_k^{(r)})} \exp(-i\Omega_k^{(r)} t)$$

$$+ i \sum_r \frac{\exp(-i\Omega_k^{(r)} t)}{\varepsilon'(\Omega_k^{(r)})} \int_t^{\infty} d\tau \exp(i\Omega_k^{(r)} \tau) \tilde{R}_k(\tau), \tag{25}$$

where $\tilde{R}_k(t)$ is the t-representation for the function $\tilde{r}_k(\omega)$:

$$\tilde{R}_k(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{r}_k(\omega) e^{-i\omega t} = \frac{4\pi eN}{k} \int_{-\infty}^{\infty} dv\, I_k(t, v) g_k(v). \tag{26}$$

Here[3)]

$$I_k(t, v) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} I_k(\omega, v) e^{-i\omega t} =$$

$$-i \exp\left\{\frac{k^2 v_T^2}{4\beta^2}\left[(1-e^{-\beta t})^2 + 2(1-e^{-\beta t})\right.\right.$$

$$\left.\left. - i\frac{kv}{\beta}(1-e^{-\beta t}) - \frac{k^2 v_T^2}{2\beta} t\right]\right\}. \tag{27}$$

Thus, the problem reduces to determining the asymptotic expression for $\tilde{R}_k(t)$, and accordingly for $I_k(t, v)$ for sufficiently large t. It follows from (27) that when $t \gg \beta^{-1}$ the quantity $I_k(t, v)$ behaves like $\exp(-k^2 v_T^2 t/\beta)$. This asymptotic expression, however, does not yield useful information on the behavior of $I_k(t, v)$ since, as will be shown later, $I_k(t, v)$ attenuates within a time much shorter than β^{-1}.

When $t \ll \beta^{-1}$ we can expand the expression in the curly brackets of (27) in powers of t. Confining ourselves to the first nonvanishing term, we obtain

$$I_k(t, v) \approx -i \exp(-\beta k^2 v_T^2 t^3/6 - ikvt) \quad (t \ll \beta^{-1}). \tag{28}$$

It follows therefore that $I_k(t, v)$ attenuates within a time

$$\tau \sim (\beta k^2 v_T^2)^{-1/3}. \tag{29}$$

Inasmuch as $\tau\beta \sim (\beta/kv_T)^{2/3} \ll 1$ (for sufficiently small β), the approximation (28) is justified.

Substituting (28) in (26), we obtain

$$\tilde{R}_k(t) \approx \frac{4\pi eN}{ik} \exp\left(-\frac{\beta k^2 v_T^2}{6} t^3\right) \int_{-\infty}^{\infty} dv\, e^{-ikvt} g_k(v), \tag{30}$$

so that $\tilde{R}_k(t)$ attenuates rapidly within the time not longer than the time τ defined by (29), no matter what the distribution function of the initial perturbation $g_k(v)$ may be.

Thus, if the time τ is shorter than the reciprocal Landau decrement γ_k^{-1}, that is, when

$$\beta/\gamma_k > (\gamma_k/kv_T)^2, \tag{31}$$

then the expression for the field (25) for $t > \tau$ takes the form

$$E_k(t) \approx \frac{\tilde{r}_k(\Omega_k)}{i\varepsilon_k'(\Omega_k)} e^{-i\Omega_k t},$$

where $\Omega_k = \Omega_k^{(1)}$ is the root of the dielectric constant with maximum imaginary part, this being valid for any form of $g_k(v)$. Since the right side of (31) is very small compared with unity, the effective collision frequency β, remaining small compared with γ_k, can at the same time be sufficiently large to "suppress" all the singular solutions.

To understand the physical cause of appearance of the rapidly damped factor $\exp[-(t/\tau)^3]$, we consider in greater detail the evolution of the distribution-function term that is determined directly by the initial perturbation $g_k(v)$ (that is, the second term on the right side of (12)). Going over in (15) to the t-representation and calculating the asymptotic value of the Green's function $G_k(t, v, v')$ in analogy with the procedure used for the quantity $I_k(t, v)$,[4)] that is, by expanding the argument of the exponential in powers of βt, we obtain

$$G_k(t, v, v') \approx (2\pi\beta v_T^2 t)^{-1/2} \exp[-(v-v')^2/2\beta v_T^2 t]$$

$$\times \exp(-ikvt - \beta v_T^2 k^2 t^3/24) \exp[ik(v-v')t/2] \tag{32}$$

(it is easy to verify that, in accord with (19), integration of (32) with respect to v leads to (28)).

The product of the first two factors in (32) is the Green's function of the diffusion equation, describing the diffusion spreading of the initial perturbation $g_k(v)$ with a diffusion coefficient

$$D = \beta v_T^2/2. \tag{33}$$

The third factor in (32) has approximately the same time dependence as (28). In order to clarify its physical meaning, we note that if there are no collisions at all ($\beta = 0$) the distribution-function term leading to the singular solution would have the form $g_k(v) \exp(-ikvt)$. This expression is an oscillating function of the velocity v, the oscillation frequency in velocity space being kt and in-

[3)] In order to obtain $I_k(t, v)$, it is sufficient to use the fact that the $I_k(\omega, v)$ are of the form

$$\int_0^{\infty} f(\sigma) e^{i\omega\sigma} d\sigma,$$

where $f(\sigma)$ is the exponential under the integral sign in (19), but without the terms $i\omega\sigma$ in the argument. Therefore $I_k(t, v) = f(t)$.

[4)] If we introduce in (15) a new integration variable σ, namely $s' = \sigma + s$, then the transformation to the t-representation is realized in analogy with the transition from (19) to (27).

creasing in proportion to the time. The diffusion character of the collisions should lead to a rapid damping of these oscillations. The characteristic damping time τ can be obtained from elementary considerations by putting $D \sim (\delta v)^2/\tau$, where D is the diffusion coefficient (33), and δv is the oscillation period at the instant τ, that is, $\delta v \sim (k\tau)^{-1}$. If we determine τ from this, we obtain precisely expression (29), thus explaining the physical meaning of the rapid damping of the singular solutions.

3. In connection with the foregoing, we make one more remark. As shown in the paper of Bernstein, Greene, and Kruskal,[6] the Van Kampen solutions have a simple physical meaning: they are obtained from the stationary nonlinear periodic waves in a plasma in the limiting case of small amplitudes. Namely, if we go in the limit to a small amplitude in the expression for the particle distribution function in a nonlinear periodic wave, then this distribution function, as shown in [6] (see Eq. (21)) will take the form

$$f(t, x, v) = f_0(v) - \mathrm{P}\frac{e(\varphi - \varphi_{min})}{v - u}\frac{df_0(v)}{dv} + C\delta(v - u), \qquad (34)$$

where $u = \omega/k$ is the velocity of the wave, P the principal-value symbol, $\varphi - \varphi_{min}$) is the perturbation of the potential, and C a certain constant connected with the distribution function of the captured particles. There are no limitations here on the velocity of the wave u, and consequently on the frequency for a specified k.

Expression (34) has the same form as Van Kampen's singular solutions. From the results obtained above it follows that when diffusion collisions are taken into account the term with the δ function in the quasistationary state should vanish. But then substitution of (34) in the Poisson equation (2) makes it impossible for the wave velocity u to be arbitrary, since it must satisfy a certain relation in the form, as can be easily seen, $u = \omega_k/k$, where ω_k is determined from the dispersion equation $\mathrm{Re}\,\epsilon(\omega_k) = 0$. Thus, allowance for finite albeit very small diffusion collisions eliminates the arbitrariness and the propagation velocities of the nonlinear periodic waves.

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105