TUNNELING BETWEEN TWO SUPERCONDUCTORS

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Equations describing the behavior of a system consisting of two superconductors separated by a thin dielectric layer are derived. It is shown that the Josephson current does not depend essentially on the phase difference of the two superconductors, since the parameter it depends on is involved in tunneling processes that occur even between normal metals. Some differential relations are deduced for small barrier voltages and for slowly varying processes. It is found that in tunneling between thin superconducting films the dependence of the tunnel current on the magnetic field strength may strongly differ from the corresponding dependence for bulky samples. This is due to the fact that the currents flowing along the films become comparable with the tunnel current in relatively strong magnetic fields.

JOSEPHSON,[1] using a model approach proposed by Cohen, Falicov, and Phillips,[2] considered the phenomena occurring in a system consisting of two superconductors separated by a thin dielectric layer.

Josephson's results were later duplicated and refined in [3,4]. A nonstationary perturbation theory in the effective interaction between two superconductors was developed in [1,3,4], it being assumed that prior to the application of the interaction the superconductors were described by complex ordering parameters $|\Delta_p| \exp i\varphi_1$ and $|\Delta_q| \exp i\varphi_2$, the flowing Josephson current being connected with the phase difference $\varphi_1 - \varphi_2$. In this article we develop a different approach and show that the Josephson current is expressed in terms of a certain parameter φ which in itself has no physical meaning. However, certain definite operations applied to φ give physically observable quantities. It turns out that the processes occurring upon tunneling between normal metals can also be conveniently expressed in terms of the parameter φ. This gives grounds for assuming that φ has nothing in common with the phase difference between two superconductors.

An examination of the behavior of the Josephson current in the presence of external fields shows that if the thickness of a superconducting film is commensurate with the London depth of penetration, then the dependence of the current on the magnetic field has a more complicated character than is observed in bulky samples.[5-8] The latter circumstance is connected with the fact that for sufficiently strong magnetic fields the currents flowing along superconducting fields reach values comparable with the tunnel current.

1. We start from the presently universally accepted model scheme with a tunnel Hamiltonian. The unperturbed system is characterized by a Hamiltonian

$$H = H_1 + H_2 + T. \qquad (1)$$

Here H_1 and H_2 are the Hamiltonians of the left-hand and right-hand metals, respectively, and the interaction Hamiltonian is

$$T = \sum_{\mathbf{p},\mathbf{q},\sigma} T_{\mathbf{pq}}\, a_{\mathbf{p}\sigma}^{+} b_{\mathbf{q}\sigma} + \text{h.c.}, \qquad (2)$$

where $a^{+}_{p\sigma}$, $a_{p\sigma}$, $b^{+}_{q\sigma}$, and $b_{q\sigma}$ are the operators for the creation and annihilation of particles in states with respective quasimomentum and spin p and σ in the right-hand metal and q and σ in the left-hand metal, while T_{pq} is the matrix element of the effective interaction.

When the system is connected to a closed circuit at the instant t_0, the following operator is added to the Hamiltonian (1):

$$H_i = eV(t)N_1, \quad N_1 = \sum_{\mathbf{p}\sigma} a_{\mathbf{p}\sigma}^{+} a_{\mathbf{p}\sigma}. \qquad (3)$$

The function (V(t) is unknown and will be determined from the relation

$$e\langle \dot{N}_1(V)\rangle = I(V), \qquad (4)$$

where $\langle \ldots \rangle$ is the average over the non-equilibrium ensemble described by the Hamiltonian $H + H_i$; I(V) depends on the concrete choice of the external circuit. In the simplest case I(V)

$= (E - V)/R$; here E and R are the emf and the internal resistance of the source. In the general case I(V) is a certain differential equation.

We shall not develop a nonequilibrium perturbation theory in T, because the perturbation leading to the appearance of the current is in our case not T but H_i.[1)] However, we cannot expand in powers of H_i, since this operator is not small. We shall show that the tunnel current flowing in the system described by the Hamiltonian

$$H_{\rm I} = H_1 + H_2 + \theta(t - t_0) eV(t) N_1 + \sum_{\mathbf{p},\mathbf{q},\sigma} (T_{\mathbf{pq}} a_{\mathbf{p}\sigma}^+ b_{\mathbf{q}\sigma} + \text{э.с.}), \tag{5}$$

is equal to the current of the system whose Hamiltonian is

$$H_{\rm II} = H_1 + H_2 + \sum_{\mathbf{p},\mathbf{q},\sigma} (T_{\mathbf{pq}} \exp[i\theta(t - t_0)\varphi(t)] a_{\mathbf{p}\sigma}^+ b_{\mathbf{q}\sigma} + \text{h.c.}). \tag{6}$$

Here H_1 and H_2 are the Hamiltonians of the left and right metals, $\theta(t - t_0)$ is the usual step function, and $\varphi(6)$ satisfies the relation

$$\partial\varphi / \partial t = eV(t), \quad \varphi(t_0) = 0. \tag{7}$$

Using the definition (4), we obtain

$$I_{\rm I} = 2e \operatorname{Im} \sum_{\mathbf{p},\mathbf{q},\sigma} T_{\mathbf{pq}} \operatorname{Sp}(\rho_{\rm I} a_{\mathbf{p}\sigma}^+ b_{\mathbf{q}\sigma}), \tag{8}$$

$$I_{\rm II} = 2e \operatorname{Im} \sum_{\mathbf{p},\mathbf{q},\sigma} T_{\mathbf{pq}} \operatorname{Sp}(\rho_{\rm II} a_{\mathbf{p}\sigma}^+ b_{\mathbf{q}\sigma}) \exp[i\theta(t - t_0)\varphi(t)], \tag{9}$$

where $\rho_{\rm I}$ and $\rho_{\rm II}$ satisfy the equations of motion

$$i\frac{\partial}{\partial t}\rho_{\rm I} = [H_{\rm I}\rho_{\rm I}]_-, \quad i\frac{\partial\rho_{\rm II}}{\partial t} = [H_{\rm II}\rho_{\rm II}]_- \tag{10}$$

and the initial condition

$$\rho_{\rm I}(t_0) = \rho_{\rm II}(t_0) = \rho_0 = \frac{\exp[-(H - \mu N)/\Theta]}{\operatorname{Sp}\exp[-(H - \mu N)/\Theta]}, \quad \Theta = kT.$$

It is then easy to find the solutions of (10):

$$\rho_{\rm I} = \theta(t - t_0)\exp\left[-i\int_{t_0}^{t} dt' H(t')\right] S(t, t_0)\rho_0 S^{-1}(t, t_0) \times \exp\left[i\int_{t_0}^{t} dt' H(t')\right] + \theta(t_0 - t)\rho_0; \tag{11}$$

$$\rho_{\rm II} = \theta(t - t_0)\exp[-i(H_1 + H_2)(t - t_0)] S(t, t_0)\rho_0 S^{-1}(t, t_0) \times \exp[i(H_1 + H_2)(t - t_0)] + \theta(t_0 - t)\rho_0. \tag{12}$$

Here

$$H(t) = H_1 + H_2 + \theta(t - t_0) eV(t) N_1,$$

and $S(t, t_0)$ satisfies the equation (see, e.g., [9]):

$$i\frac{\partial}{\partial t} S(t, t_0) = T(t, t_0) S(t, t_0),$$

where

$$T(t, t_0) = \exp[i(H_1 + H_2)(t - t_0)] \times \sum_{\mathbf{pq}\sigma} (T_{\mathbf{pq}} \exp[i\theta(t - t_0)\varphi(t)] \times a_{\mathbf{p}\sigma}^+ b_{\mathbf{q}} + \text{h.c.}) \exp[-i(H_1 + H_2)(t - t_0)].$$

Substituting (11) and (12) in (8) and (9) we verify that $I_{\rm I} = I_{\rm II}$.

2. Assuming that the tunnel probability is small, let us calculate the current flowing through a barrier separating the superconductors. Using (6), (9), and (12) we obtain in first nonvanishing order of perturbation theory

$$I = 2e \operatorname{Re}\int_{t_0}^{t} dt' \sum_{\substack{\mathbf{pq}\sigma \\ \mathbf{p}',\mathbf{q}',\sigma'}} T_{\mathbf{pq}} e^{i\varphi(t)} \times \{T_{\mathbf{p}'\mathbf{q}'} e^{i\varphi(t')} \langle [a_{\mathbf{p}'\sigma'}^{+}(t') B_{\mathbf{q}'\sigma'}(t') \,|\, a_{\mathbf{p}\sigma}^+(t) b_{\mathbf{q}\sigma}(t)]_-\rangle_0 + \overset{*}{T}_{\mathbf{p}'\mathbf{q}'} e^{-i\varphi(t')} \langle [b_{\mathbf{q}'\sigma'}^{+}(t') a_{\mathbf{p}'\sigma'}(t') \,|\, a_{\mathbf{p}\sigma}^+(t) b_{\mathbf{q}\sigma}(t)]_-\rangle_0\}. \tag{13}$$

Here $\langle \ldots \rangle_0$ stands for averaging over the equilibrium ensemble, when $T_{pq} = 0$, and

$$a_{\mathbf{p}\sigma}(t) = \exp[i(H_1 - \mu N_1)t] a_{\mathbf{p}\sigma} \exp[-i(H_1 - \mu N_1)t],$$
$$b_{\mathbf{q}\sigma}(t) = \exp[i(H_2 - \mu N_2)t] b_{\mathbf{q}\sigma} \exp[-i(H_2 - \mu N_2)t].$$

After calculating (13), we write the result in the form

$$I = \int_{t_0}^{t} dt' \{K_s(t - t')\sin[\varphi(t) + \varphi(t')] + K_n(t - t')\sin[\varphi(t) - \varphi(t')]\}. \tag{14}$$

Here[2)]*

$$K_s(t - t') = \frac{1}{\pi e R_N}\int_{-\mu}^{\infty}\int_{-\mu}^{\infty} d\xi_1\, d\xi_2 \frac{\Delta(\xi_1)\Delta(\xi_2)}{E_1 E_2} \times \operatorname{th}\frac{E_2}{2\Theta}\sin E_1(t - t')\cos E_2(t - t'),$$

$$K_n(t - t') = \frac{1}{\pi e R_N}\int_{-\mu}^{\infty}\int_{-\mu}^{\infty} d\xi_1\, d\xi_2 \nu(\xi_1)\nu(\xi_2) \times \left(\operatorname{th}\frac{E_1}{2\Theta} - \frac{\xi_1\xi_2}{E_1 E_2}\operatorname{th}\frac{E_2}{2\Theta}\right)\sin E_1(t - t')\cos E_2(t - t'),$$

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1)It was assumed in [1-4] that the transparency of the barrier is turned on adiabatically by the time the instant t is reached, and that V = const. It will be shown below that in superconductors we cannot assume that V is constant (at least for $eV \ll \Delta_1 + \Delta_2$), apart from the trivial case V = 0.

2)In calculating (14) we took account of the fact that prior to connecting the contact in the circuit the requirement that the thermodynamic potential be a minimum yields $\varphi_1 - \varphi_2 = 0$. For the sake of simplicity we are confining ourselves also to the case of identical metals, $\Delta_1 = \Delta_2 = \Delta$.

*th ≡ tanh.

where R_N is the resistance of the normal junction at $\Theta \to 0$ and $V \to 0$. ξ is the electron energy, reckoned from the Fermi surface, $\Delta(\xi)$ is the gap in the elementary-excitation spectrum, $\nu(\xi)$ is the relative density of the state ($\nu(0) = 1$), and $E = \sqrt{\xi^2 + \Delta^2}$ is the energy of the elementary excitation.

When account is taken of the conditions in the external circuit, Eq. (14) is a rather complicated integro-differential equation for $\varphi(t)$. This relation can be simplified somewhat by recognizing that in practically all cases of experimental interest the junction has a rather large intrinsic capacitance C,[6, 8, 10] such that $\tau^{-1} = (RC)^{-1} \ll 2\Delta$. The quantum-mechanical fluctuation transients will then become smoothed out by the long time constant τ, i.e., we can in practice let $t_0 \to -\infty$ in (14). Then we obtain for the simplest circuit the equation

$$\frac{C}{e}\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{eR}\frac{\partial\varphi}{\partial t}+\int_{-\infty}^{t} dt' K_s(t-t')\sin[\varphi(t)+\varphi(t')]$$
$$+\int_{-\infty}^{t} d\tau' K_n(t-t')\sin[\varphi(t)-\varphi(t')]-I_{sc}=0, \qquad (15)$$

where I_{SC} is the short-circuit current of the source.

Relation (15) can in general be used even for the analysis of very fast processes with frequencies commensurate with and larger than 2Δ. Only the steady-state solution will be meaningful here. To analyze the establishment of these processes it is essential to make use of (14).

We, however, will consider here slowly varying processes, such that $\Omega \ll 2\Delta$, where Ω is the characteristic frequency. If the inequality $\Omega \ll 2\Delta$ is satisfied, then Eq. (15) can be greatly simplified in two extreme limiting cases. It would be natural to call the first the Josephson case. This is the case of sufficiently low temperatures $\Theta \ll \Delta$ and small voltages $eV_m < \Omega$, $e\overline{V} \ll 2\Delta$ ($eV_m \sim \max \partial\varphi/\partial t - \min \partial\varphi/\partial t$; $\overline{V}$—average voltage on the barrier). In this case the term of (15) containing K_n gives an exponentially small contribution, $\sim \exp(-\Delta/\Theta)$, and the term with K_s can be transformed, with accuracy to terms $\sim e\overline{V}/2\Delta$ and $eV_m/2\Delta$, into $I_0 \sin 2\varphi$, so that we obtain

$$\frac{C}{e}\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{eR}\frac{\partial\varphi}{\partial t}+I_0\sin 2\varphi-I_{sc}=0. \qquad (16)$$

Here $I_0 = \Delta\pi/2eR_N$ (see [1, 3]).

We shall call the second case "normal." This is the case of large average voltages $e\overline{V} \gg 2\Delta$. The term containing K_S can be neglected here with accuracy $\sim 2\Delta/e\overline{V}$, and the expression for K_n is also greatly specified at the stipulated accuracy.

The equation takes the form

$$\frac{C}{e}\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{eR}\frac{\partial\varphi}{\partial t}+\int_{-\infty}^{\infty} dt'\widetilde{K}_n(t-t')\sin[\varphi(t)-\varphi(t')]-I_{sc}=0. \qquad (17)$$

Here

$$\widetilde{K}_n(t-t')=\frac{1}{\pi eR_N}\int_{-\mu}^{\infty}\int_{-\mu}^{\infty} d\xi_1\, d\xi_2\,\nu(\xi_1)\nu(\xi_2)$$
$$\times[n(\xi_2)-n(\xi_1)]\sin(\xi_1-\xi_2)(t-t').$$

It is interesting to note that if we were to consider tunneling in a normal metal, then we would arrive at Eq. (17). Thus, for large average barrier voltages a superconducting tunnel contact behaves just like a normal one. We note that in the case of a quadratic electron dispersion law Eq. (17) has a stationary solution $\partial\varphi/\partial t$ = const. In the general case a stationary solution of (17) for the voltage on the barrier will be oscillatory with a large dc component. In the case when (17) admits of a stationary solution, we obtain the usual expression for the normal tunnel current.[11, 12]

It is typical that the general equation,(15), just as (16), has no stationary solution other than the trivial $\partial\varphi/\partial t = 0$. For purpose of illustration let us see how the trivial solution of (16) is established. This solution describes the establishment of direct current in the circuit at zero voltage on the barrier ("Josephson" current). For simplicity we consider the case $I_{SC} \ll I_0$. Then we can linearize (16) and the solution is

$$\varphi=\frac{I_{sc}}{2I}\left[1-\exp\left(-\frac{t-t_0}{2\tau}\right)\left(\operatorname{ch}\alpha\frac{t-t_0}{2\tau}+\frac{1}{\alpha}\operatorname{sh}\alpha\frac{t-t_0}{2\tau}\right)\right], \qquad (18)^*$$

where $\alpha = \sqrt{1 - 8e\tau RI_0} \sim 1$. Thus, $\varphi \approx I_{SC}/2I_0$ for $t - t_0 \gg \tau$.

It is easy to see that for $I_{SC} \ll I_0$ there exists a stationary solution $\varphi = \frac{1}{2}\sin^{-1}(I_{SC}/I_0)$. A similar solution is possessed also by the general equation (15) for the case $I_{SC} \le \bar{I}_0$.

$$\bar{I}_0=\int_0^{\infty} dtK_s(t)=\frac{\pi\Delta}{2e}\operatorname{th}\frac{\Delta}{2\Theta}$$

(see [3]).

If the foregoing inequalities are not satisfied, then (15) and (16) have no stationary solutions. However, even if the inequalities are satisfied, then the stationary solution does not necessarily exist, since an equation of general form can have several times of steady-state solutions, depending on the concrete initial conditions (see also [13]).

*ch ≡ cosh, sh ≡ sinh.

The latter circumstance can lead to hysteresis loops on the experimental average current vs. average voltage curves.

3. Let us consider now the electrodynamic behavior of the tunnel contact for the "Josephson" case. We shall assume that φ can depend on two coordinates in the plane of the junction, but the distances over which it changes significantly are sufficiently large and therefore we can assume the current flowing through a unit junction area to be proportional to sin 2φ. It is customary to express it in the form (see, e.g., [1, 3, 7]):

$$I = I_0 \sin \varphi. \tag{19}$$

But then we must write in lieu of (8)

$$\partial \varphi / \partial t = 2eV(t). \tag{20}$$

We use the induction law

$$\oint \mathbf{E}d\mathbf{l} + \frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{H}d\mathbf{S} = 0$$

for a loop passing through the barrier perpendicular to the planes of both superconductors and closed by segments $\Delta\mathbf{r}$ lying on the surfaces of the superconductors. Using also the well known relations for London superconductors (we consider here for simplicity only London superconductors), we obtain with account of (2)

$$\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial \mathbf{r}} = \frac{2e}{c}\frac{\partial}{\partial t}\left(\frac{8\pi}{c}\lambda_L^2 \mathbf{j} + d[\mathbf{nH}_1]\right). \tag{21*}$$

Here λ_L is the London depth of penetration, d the width of the barrier, $\mathbf{j}$ the current on the surface of the superconductor, $\mathbf{n}$ a unit vector normal to the barrier surface, and $\mathbf{H}_1$ the magnetic field inside the barrier.

Relations (21) can be integrated with respect to time, and the integration constant can be determined from the homogeneity condition in the absence of currents along the surfaces of the superconductors, i.e.,

$$\frac{\partial \varphi}{\partial \mathbf{r}} = \frac{2e}{c}\left(\frac{8\pi}{c}\lambda_L^2 \mathbf{j} + d[\mathbf{nH}_1]\right). \tag{22}$$

Recognizing that usually the field along the barrier varies essentially over distances $l \gg \lambda_L$, and assuming a superconductor film thickness equal to a, we obtain

$$\frac{\partial \varphi}{\partial \mathbf{r}} = \frac{2e}{c}\left\{[\mathbf{nH}_1(\mathbf{r},t)]\left(2\lambda_L \operatorname{cth}\frac{a}{\lambda_L} + d\right) - [\mathbf{nH}_2(\mathbf{r},t)]\,2\lambda_L \operatorname{sh}^{-1}\frac{a}{\lambda_L}\right\}. \tag{23}$$

*$[\mathbf{nH}_1] \equiv \mathbf{n} \times \mathbf{H}_1$.

Here $\mathbf{H}_2$ is the field on the outer sides of the superconductors. Using Maxwell's equations for the region of the barrier, and Eqs. (19), (20), and (23), we can eliminate the field $\mathbf{H}_1$ and write the equation for φ in the form

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\varphi - \frac{1}{\bar{c}^2}\left(\operatorname{cth}\frac{a}{\lambda_L} + \frac{d}{2\lambda_L}\right)\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\lambda_j^2}\left(\operatorname{cth}\frac{a}{\lambda_L} + \frac{d}{2\lambda_L}\right)\sin\varphi - \frac{4e\lambda_L}{c}(\operatorname{rot}\mathbf{H}_2)_z \operatorname{sh}^{-1}\frac{a}{\lambda_L}, \tag{24}$$

where $\lambda_j^2 = c/16\pi e\lambda_L I_0$ is the square of the "Josephson" depth of penetration,[5, 7] and $\bar{c} = c\sqrt{d/2\epsilon\lambda_L}$ is the velocity of propagation of the electromagnetic waves in the insulator between two bulky superconductors.[14] To determine φ it is necessary to add to (24) boundary conditions, which can be determined if one knows the concrete configuration of the junction, the specified external constant magnetic field, and the parameters of the external circuit. It is obvious that in the case when the magnetic field can be neglected (a possible case, as is clear from (24), if the transverse dimensions of the junction are considerably smaller than λ_j), the boundary conditions will yield relations of the type (3) and (16).

4. In this article we obtained the most general relations obtainable for tunneling between two metals. Furthermore, it turns out that the generalization can be made only by foregoing the treatment of φ as a phase difference between two superconductors. We see from (17) that φ participates actively in processes occurring during tunneling even in normal metals and semiconductors. The obtained relation (24), which φ satisfies for tunneling in superconductors when $a \gg \lambda_L$, goes over into the expression first given by Eck et al.[6] Equation (24) is in qualitatively good agreement with the latest experimental data on the Josephson effect,[15] where an anomalous behavior of the Josephson current was observed, not described by the relations of [5-7]. The latter circumstance is connected with the fact that, as can be seen from (24), the thickness of the superconducting films in these experiments was of the order of the depth of penetration of the magnetic field. Unfortunately, a quantitative comparison with the experimental data of [15] is impossible, since we do not know the circuit parameters and the junction configurations, and without them we cannot write down correctly the boundary conditions corresponding to the given experiment.

*cth ≡ coth, rot ≡ curl.

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