ELECTRODYNAMICS OF THE JOSEPHSON EFFECT

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The magnitude and location of the resonance maxima on the plot of the dc component of the Josephson tunneling current is calculated. The theory is based on an account of the electromagnetic field of the Josephson current and of the conditions of inclusion of the tunnel junction in the external electric circuit.

JOSEPHSON[1] has shown that the current through a tunnel junction in a system of coupled superconductors will be nonstationary at nonzero voltage on the junction. This result was subsequently duplicated by Ambegaokar and Baratoff.[2] A more accurate derivation, which takes into account the possibility of the time variation of the bias, is given in [3].

The physical interpretation proposed in [3] for the effect is based essentially on a consideration of the entire electric circuit in which the tunnel junction is connected. The point is that inclusion in the circuit "prepares" states with definite phase difference φ between the superconductors. The condition for the connection into the circuit is expressed mathematically in the form of a differential equation with respect to the unknown phase difference (henceforth simply phase). From this point of view, the tunnel junction is a nonlinear element (since the current through the junction is equal to $j_S \sin\varphi$). The occurrence of nonstationary solutions in a certain range of the circuit parameters is essentially connected with the nonlinearity of the system as a whole. In [3] there was considered, for purposes of illustration, an idealized case of the simplest type of circuit, consisting of a voltage source of internal resistance R. The real situation, however, is much more complicated.

The tunnel junction is a thin film (thickness d ~ 10 Å) of dielectric, with transverse dimensions $l \sim 1$ mm, separating two superconductors. The flow of nonstationary current through the junction induces in it an electromagnetic field which can propagate in the junction practically without damping, but with a reduced velocity (under ordinary conditions, by a factor of approximately 30).[4]

Since the transverse dimensions of the junction are comparable with the wavelength, the field inside it should be described by Maxwell's equations. In the presence of a magnetic field (the current's own field or an external field) the phase φ becomes a function not only of the time, but also of the coordinates, Maxwell's equations lead, as shown in [5], to a nonlinear partial differential equation with respect to the phase φ.

As noted in [6-8], resonance can occur in the tunnel junction if the Josephson frequencies, which are determined by the applied emf, coincide with the natural frequencies of the electromagnetic field inside the junction. This leads to the appearance of maxima on the voltage-current curve of the component of the Josephson current. A theory for the voltage-current characteristic was constructed on this basis in several recent papers.[7-9] However, as will be shown below, that theory is not fully satisfactory. For this reason we shall review it here anew.

According to Josephson[1] (for a microscopic derivation see [3]), the voltage V on the barrier is connected with the phase by

$$V = \frac{\hbar}{2e}\frac{d\varphi}{dt}. \tag{1}$$

In the presence of an alternating electromagnetic field, this relation must be generalized, for in this case the potential difference loses its direct meaning and should be replaced by the gauge-invariant concept of the voltage $\int_2^1 \mathbf{E}\, d\mathbf{l}$, which depends both on the choice of the initial and final points 1 and 2, and on the path of integration. Let the junction plane be the XOY coordinate plane. We must consider the voltage between points lying on a line perpendicular to the surface separating the superconductors. The integration path is then naturally taken to be the straight line joining points 1 and 2. The direction of the external magnetic field is taken to be the OY axis.

It is easy to show, in analogy with [4], that the dependence of the voltage on the distance of the chosen points 1 and 2 to the interface between the superconductors is quite weak. Thus, if 1′ and 2′ are points located on the interfaces between the dielectric and the first and second superconductors, respectively, and 1 and 2 are points inside each of the superconductors ($|z_{12}| > \lambda_L$), then the corresponding voltages differ by a quantity on the order of $(\lambda_L/\lambda)^2$, where λ_L is the London depth of penetration and λ is the wavelength, i.e., a quantity negligibly small at the wavelengths customarily used in experiment. It will be convenient below to take points 1 and 2 inside the superconductors.

Let us consider Maxwell's equation

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{1}{c}\frac{\partial H_y}{\partial t} = 0 \tag{2}$$

and let us integrate it along a straight line parallel to the z axis, from the point 1 to the point 2. Using the definition of the voltage as well as formula (1), and recognizing that by virtue of the Meissner effect the fields decrease exponentially along the London penetration depth, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x} - \frac{4e\lambda_L}{\hbar c} H_y\right) = 0. \tag{3}$$

Integrating this relation with respect to time and assuming the integration constant equal to zero, in inasmuch as the phase is constant in the absence of the field, we have finally

$$\frac{\partial \varphi}{\partial x} = \frac{4e\lambda_L}{\hbar c} H_y. \tag{4}$$

In perfect analogy, we obtain

$$\frac{\partial \varphi}{\partial y} = -\frac{4e\lambda_L}{\hbar c} H_x. \tag{5}$$

We note that in (4) and (5) the values of the fields are taken at a certain point z inside the dielectric layer. Let us consider now at the same point Maxwell's second equation

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{\varepsilon}{c}\frac{\partial E_z}{\partial t} = \frac{4\pi}{c} j_z. \tag{6}$$

In view of the slow variation of E_z in the dielectric layer we have

$$E_z \cong \frac{1}{d}\int_{1'}^{2'} E_z dz \cong \frac{1}{d}\int_{1}^{2} E_z dz = \frac{V}{d} = \frac{\hbar}{2ed}\frac{\partial \varphi}{\partial t}. \tag{7}$$

Substituting relations (4), (5), and (7) in (6), we obtain

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \frac{1}{\bar{c}^{2}}\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\lambda_j^2}\sin\varphi, \tag{8}$$

where

$$\bar{c}^2 = c^2 d / 2\varepsilon\lambda_L, \qquad \lambda_j^2 = \hbar c^2 / 16\pi e \lambda_L j_s. \tag{9}$$

In the derivation of (8) we have assumed that the current density is $j_z = j_s \sin\varphi$. This is approximately valid at temperatures much lower than critical, and at voltages smaller than the threshold $V < (\Delta_1 + \Delta_2)/e$, where Δ_1 and Δ_2 are the gaps in the energy spectra of the first and second superconductors. Under these conditions the quasiparticle current is small, and the tunnel current reduces merely to the Josephson current. Equation (8) was obtained first in [5].

Since in practice there are losses due to the excitation of the normal component in the superconductors by the alternating field, to the weak quasiparticle current, and possibly also to other mechanisms, Eq. (8) should in principle be supplemented by the term $-(1/\bar{c}^2\tau)\,\partial\varphi/\partial t$, where τ is a quantity which takes effective account of the damping. In view of the presence of damping, the problem of solving Eq. (8) must naturally be formulated without initial conditions.

For simplicity we consider henceforth a "linear" junction, for which we can neglect the dependence of φ on y.

The experimentally measured quantity is the dc component of the total current through the junction

$$J = j_s \int_0^l dx \int_0^T \frac{dt}{T} \sin\varphi(x,t) = j_s \lambda_j^2 \left(\overline{\varphi_x'(l,t)} - \overline{\varphi_x'(0,t)}\right). \tag{10}$$

The bar denotes averaging with respect to time.

We now proceed to formulate the boundary conditions. It is physically clear that they should be determined by the manner in which the tunnel junction is connected in the electric circuit. It must be noted here that under the experimentally realized conditions, the ac component is rapidly damped outside the junction. This circumstance can be taken into account effectively by loading the output with a complex impedance ρ. We therefore have in terms of Fourier components

$$V(0,\omega) + \rho_1 i(0,\omega) = 0,$$
$$V(l,\omega) - \rho_2 i(l,\omega) = 0, \qquad \omega \neq 0. \tag{11}$$

For the dc component the relations are

$$R\left(\overline{i(0)} - \overline{i(l)}\right) + \overline{V(0)} = \mathscr{E}, \qquad \overline{V(0)} = \overline{V}(l). \tag{12}$$

To formulate the boundary conditions in terms of the phase, it is necessary to have, besides (1), a relation between the phase and the current. This relation can be readily obtained from the London equation, and is of the form

$$i = -\frac{\hbar c^2}{16\pi e\lambda_L}\frac{\partial \varphi}{\partial x}. \tag{13}$$

Using (1) and (13), we rewrite the boundary conditions in the form

$$\varphi_x'(0,\omega)=iq_1\varphi(0,\omega),\quad \varphi_x'(l,\omega)=-iq_2\varphi(l,\omega),\quad \omega\neq 0,$$
$$q=\frac{8\pi\lambda_L}{c^2\rho}\omega; \tag{14}$$

$$\overline{\varphi_x'(l)}-\overline{\varphi_x'(0)}+\frac{8\pi\lambda_L}{Rc^2}\overline{\varphi_t'(0)}=\frac{16\pi e\lambda_L}{\hbar c^2}\frac{\mathscr{E}}{R},$$
$$\varphi_t'(0)=\varphi_t'(l) \tag{15}$$

To solve (8) we use perturbation theory, assuming the nonlinear term to be small. In the zeroth order we obtain

$$\varphi_0=\omega t+kx.$$

It follows from (15) that $\omega = 2e\mathscr{E}/\hbar$. Inasmuch as in the zeroth approximation the only currents in the superconductors are the eddy currents induced by the external magnetic field H_e, we find from (4) that $k = H_e 4e\lambda_L/\hbar c$.

The first-approximation correction to the phase is determined from the equation

$$\frac{\partial^2\varphi_1}{\partial x^2}-\frac{1}{\bar{c}^2}\frac{\partial^2\varphi_1}{\partial t^2}=\frac{1}{\lambda_j^2}\sin(\omega t+kx). \tag{16}$$

Inasmuch as $\overline{\varphi}_1$ vanishes, $\varphi_1(x, y)$ satisfies the boundary conditions (14) and the equation

$$\frac{d^2\varphi_1(x,\omega)}{dx^2}+\varkappa^2\varphi_1(x,\omega)=\frac{1}{2i\lambda_j^2}e^{ikx},\qquad \varkappa=\frac{\omega}{\bar{c}}, \tag{17}$$

and the dc component of the total current is obtained from the formula

$$J=j_s\int_0^l \operatorname{Re}\varphi_1(x,\omega)e^{-ikx}\,dx. \tag{18}$$

We present the solution of the problem for two limiting cases: symmetrical, $\rho_1 = \rho_2 = \infty$, and assymetrical $\rho_1 = \rho$, $\rho_2 = \infty$. For simplicity, we neglect the imaginary part of ρ in the final formulas.

In the symmetrical case

$$J=\frac{j_s}{2\lambda_j^2}\frac{q}{\varkappa^2\sin^2(\varkappa l/2)+q^2\cos^2(\varkappa l/2)}\times\left\{\left(\frac{\sin[(\varkappa-k)l/2]}{\varkappa-k}\right)^2+\left(\frac{\sin[(\varkappa+k)l/2]}{\varkappa+k}\right)^2\right\}. \tag{19}$$

In the asymmetrical case

$$J=\frac{j_s}{2\lambda_j^2}\frac{q}{\varkappa^2\sin^2\varkappa l+q^2\cos^2\varkappa l}\left\{\left(\frac{\sin[(\varkappa-k)l/2]}{\varkappa-k}\right)^2+\left(\frac{\sin[(\varkappa+k)l/2]}{\varkappa+k}\right)^2+2\cos\varkappa l\frac{\sin[(\varkappa-k)l/2]\sin[(\varkappa+k)l/2]}{\varkappa^2-k^2}\right\} \tag{20}$$

Assuming that $q \ll \kappa$, corresponding to small distortion of the boundary conditions for an open cavity,[8] we obtain from (19) and (20) the locations of the resonant maxima $\kappa l = 2\pi n$ in the symmetrical case and $\kappa l = \pi n$ in the asymmetrical one. Thus, in the asymmetrical case the theory leads to a series of equidistant resonance maxima with distances between them equal (in terms of voltage) to $\Delta\mathscr{E}_n = \pi\hbar\bar{c}/2el$.

In the symmetrical case there are only "even" maxima. The largest height of the n-th maximum is reached in both cases in a magnetic field $H_n = \mathscr{E}_n \times (\epsilon/2\lambda_L d)^{1/2}$. The height of the n-th maximum is given in the symmetrical case by the formula

$$J_{max}=\frac{J_0}{2\lambda_j^2}\frac{l}{q}\frac{1}{2\pi^2}\frac{n^2+x^2}{(n^2-x^2)^2}\sin^2 2\pi x, \tag{21}$$

where Φ/Φ_0, Φ is the magnetic flux through the junction, $\Phi_0 = \pi\hbar c/e$ the flux quantum, and $J_0 = j_s l$.

In the asymmetrical case

$$J_{max}=\frac{J_0}{2\lambda_j^2}\frac{l}{q}\frac{x^2}{\pi^2}\frac{1}{[(n/2)^2-x^2]^2}\begin{cases}\sin^2\pi x, & n=2p\\ \cos^2\pi x, & n=2p+1\end{cases} \tag{22}$$

($p = 0, \pm 1, \pm 2 \ldots$).

The results of the theory explain the experimental data obtained in [10].

In the case of small junctions, for which $\kappa l \ll 1$, the expression for J reduces in both cases to the following:

$$J=\frac{1}{2}\frac{J_0^2}{\mathscr{E}/\tilde{\rho}}\frac{1}{1+(\omega\tilde{\rho}C)^2}\left(\frac{\sin\pi x}{\pi x}\right)^2, \tag{23}$$

where, using the expression for λ_j^2, c^2, and q, we have introduced a junction capacitance C and an ohmic load $\tilde{\rho}$, equal to ρ for the asymmetrical case and $\rho/2$ for the symmetrical one.

Since the exact solution of the nonlinear equation (8) cannot be obtained in the general case, it is of interest to note that for nearly pointlike junctions, we can obtain an exact solution by neglecting the junction capacitance and disregarding the magnetic field. The average current is in this case

$$J=\frac{\mathscr{E}}{R}\left(1-\sqrt{1-(J_0R/\mathscr{E})^2}\right), \tag{24}$$

and the ac frequency is $\omega\sqrt{1-(J_0R/\mathscr{E})^2}$. Neglecting the self-action, $J_0 \ll \mathscr{E}/R$, (24) agrees with (23), and the frequency equals the Josephson frequency. Formula (24) pertains to the case when $\mathscr{E}/R > J_0$; in the opposite case, the stationary mode corresponds to the Josephson direct current.

An exact solution can be obtained also for the Josephson direct current in a magnetic field, taking into account the effects of "magnetic self-action." Equation (8) for a linear junction becomes[11]

$$\frac{d^2\varphi}{dx^2}=\frac{1}{\lambda_j^2}\sin\varphi, \qquad (25)$$

and the boundary conditions are written in the form

$$\varphi'(l)-\varphi'(0)=\frac{16\pi e\lambda_L}{\hbar c^2}\frac{\mathscr{E}}{R}, \qquad (26)$$

$$\varphi'(l)+\varphi'(0)=\frac{8\pi e\lambda_L}{\hbar c}H_e. \qquad (27)$$

The solution of (25) is expressed in terms of elliptic functions. Two possibilities must be distinguished: 1) the external field predominates over the self field, and the total magnetic field, and consequently the current along the x axis, does not reverse sign, 2) the opposite case. The density of the Josephson current in the first and second cases respectively is given by the formula

$$j_z=2j_s\,\mathrm{cn}\frac{x_0-x}{k\lambda_j}\,\mathrm{sn}\frac{x_0-x}{k\lambda_j}, \qquad (28)$$

$$j_z=2j_sk\,\mathrm{sn}\frac{x_0-x}{\lambda_j}\,\mathrm{dn}\frac{x_0-x}{\lambda_j}, \qquad (29)$$

and the current along the x axis is respectively

$$i_x=-\frac{2\pi\hbar}{e\lambda_L\lambda_j}\frac{1}{k}\,\mathrm{dn}\frac{x_0-x}{k\lambda_j}, \qquad (30)$$

$$i_x=-\frac{2\pi\hbar}{e\lambda_L\lambda_j}k\,\mathrm{cn}\frac{x_0-x}{\lambda_j}. \qquad (31)$$

Here sn(z, k), cn(z, k) and dn(z, k) are elliptic functions with modulus k. The constants x_0 and k are fixed by the boundary conditions.

If the self field can be neglected compared with the external field, then formula (28) leads to the well known result

$$J_{max}(H)=J_{max}(0)\left|\frac{\sin\pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}\right|. \qquad (32)$$

To the contrary, for a vanishing external field we obtain from (29)

$$J_{max}=4j_s\lambda_j\max_{(k)}k\sqrt{1-k^2}\frac{\mathrm{sn}(l/2\lambda_j)}{\mathrm{dn}(l/2\lambda_j)}. \qquad (33)$$

For small junctions ($l \ll \lambda_j$) (33) reduces to the equality $J_{max}=j_s l$. In the opposite case ($l \gg \lambda_j$) we obtain the effect of self-limiting of the current

$$J_{max}\approx 2j_s\lambda_j. \qquad (34)$$

The self-limiting effect was first predicted by Ferrel and Prange.[11] However, they considered the exceptional case when k = 1, at which the equation can be integrated in terms of elementary functions. Physically, on the other hand, there is nothing to single out this case at all. The general physical picture differs from that considered in [11] in that the magnetic field does not decrease with increasing depth in the junction, but oscillates. The cause of self-limitation lies in the appearance of a layered current structure, as seen from formulas (29) and (31).

An examination of the limiting cases that admit of exact calculation shows that the method of linearizing Eq. (8) is valid if 1) $J_0 \ll \mathscr{E}/\rho$ and 2) $l \ll \lambda_j$ or $J_0 \ll J_m$, where J_m is the current induced by the magnetic field.

To conclude the article, we make several remarks concerning papers [7-9] in which the solution of Eq. (8), supplemented by a term that takes damping into account, is sought (in our notation) in the form[1)]

$$\varphi=\omega t+kx+\Phi(x,t), \qquad (35)$$

where the small addition Φ(x, t) satisfies the boundary conditions $\Phi'_x(0,t)=\Phi'_x(l,t)=0$ (resonator with open ends[8]). Relation (10), which with allowance for damping takes the form

$$\overline{\varphi_x'(l,t)}-\overline{\varphi_x'(0,t)}=\frac{1}{\lambda_j^2}\int_0^l\overline{\sin\varphi}\,dx+\frac{1}{\bar{c}^2\tau}\int_0^l\frac{\varphi(T)-\varphi(0)}{T}\,dx, \qquad (36)$$

reduces in first order in the current (and second order in φ) to the equality

$$\overline{\varphi_x^{(2)\prime}(l,t)}-\overline{\varphi_x^{(2)\prime}(0,t)}=\frac{1}{\lambda_j^2}\int_0^l\overline{\varphi^{(1)}(x,t)\cos\varphi_0(x,t)}\,dx+\frac{1}{\bar{c}^2\tau}\int_0^l\frac{\varphi^{(2)}(T)-\varphi^{(2)}(0)}{T}\,dx. \qquad (37)$$

It is clear therefore that, by virtue of the boundary conditions, the dc component of the total current (Josephson and ohmic) vanishes and the average current in the circuit is equal to $\omega l/\bar{c}^2\tau$ — the zeroth-approximation term—which naturally has no maxima. The authors of [7-9], however, identify the observed current with the first term in the right side of (37), obviously without any justification. We emphasize that, as follows from our work, the existence of a dc component of the tunnel current is due physically just to the deviation of the boundary conditions from those for an open cavity.

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1)Inasmuch as now ωt + kx is not a solution of the zeroth-approximation equation, it would be necessary to take φ_0 in the form $\omega t+kx+\omega x^2/2\bar{c}^2\tau$.

[1] B. D. Josephson, Phys. Lett., **1**, 251 (1962); Revs. Modern Phys. **36**, 221 (1964).
[2] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. **10**, 486 (1963).
[3] A. V. Svidzinskiĭ and V. A. Slyusarev, JETP **51**, 177 (1966), this issue p. 118 (1966).
[4] J. Swihart, J. Appl. Phys. **32**, 471 (1961).
[5] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Phys. Rev. Lett. **13**, 15 (1964).
[6] I. M. Dmitrenko, I. K. Yanson, and V. M. Svistunov, JETP Letters **2**, 17 (1965), transl. p. 10.
[7] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Preprint, (1964).
[8] D. Coon and M. Fiske, Phys. Rev. **138**, A744 (1965).
[9] I. O. Kulik, JETP Letters **2**, 134 (1965), transl. p. 84.
[10] I. K. Yanson, V. M. Svistunov, and I. M. Dmitrenko, JETP **48**, 976 (1965), Soviet Phys. JETP **21**, 650 (1965).
[11] R. Ferrel and R. Prange, Phys. Rev. Lett. **10**, 479 (1963).

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