NONLINEAR THEORY OF THERMOMAGNETIC WAVES

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The nature of thermomagnetic waves and their amplification in the case of instability are described qualitatively. Two possible experiments in which these waves can be detected are considered. In one of them the thermomagnetic waves are traveling waves and in the other standing waves. An exact solution of the nonlinear equation for the stationary state is given for the first case; the conditions for its realization are investigated and are found to be identical with the condition for softness of the excitation regime. The kinetics of development of instability, the conditions for soft and hard excitation, and the stationary state for a small excess of the temperature gradient relative to its critical value are studied for the second case. Finally, the conditions for the realization of the two experiments are compared.

1. THERMOMAGNETIC WAVES

THE instability of a collision-free plasma with a temperature gradient ∇T has been studied by a number of authors (see the review [1]). The instability in the presence of an external current and ∇T perpendicular to the magnetic field was considered by Kadomtsev.[2] In [3] it was shown by the authors that waves of a special type, called thermomagnetic (TM-waves), can exist in semimetals and metals in the presence of ∇T and in the absence of an external current. In order to understand their origin, let us consider a conductor in which there is a temperature gradient and a compensating thermoelectric field

$$\mathbf{E}_0 = \alpha \nabla T, \tag{1.1}$$

where α is the thermal emf. The electric field is expressed in terms of the current $\mathbf{j}$, the magnetic field $\mathbf{H}$ and ∇T by the formula

$$\mathbf{E}_0 = \sigma_0^{-1}\mathbf{j} + \alpha\nabla T + \alpha_1[\nabla T\mathbf{H}], \tag{1.2*}$$

where σ_0 is the electrical conductivity in the absence of a magnetic field. The coefficient for the corresponding field term is denoted by α_1. An external field is absent but there can be a magnetic field of the wave. For simplicity, let us consider a wave propagating along ∇T, when temperature and density oscillations are absent. The equations satisfied by the TM-waves are:

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}. \tag{1.3†}$$

*$[\Delta \mathbf{TH}] \equiv \Delta \times \mathbf{TH}$.
† rot ≡ curl.

By linearizing the equations, we find the following dispersion equation from the condition of their compatibility:

$$\omega = -c\alpha_1(\mathbf{k}\nabla T) - i\nu_m k^2, \tag{1.4}$$

where $\nu_m = c^2/4\pi\sigma_0$ is the magnetic viscosity and $\mathbf{k}$ is the wave vector.

Thus, we obtain a new branch of waves in the presence of ∇T. We can understand their nature in the following way. Suppose that a current fluctuation in the shape of a ring arose at the instant of time $t = 0$ in the plane perpendicular to ∇T. The magnetic lines of this current ring run parallel to ∇T and then diverge. At points where the magnetic field $\mathbf{H}'$ of the fluctuations is not parallel to $\mathbf{E}_0$, a drift of the electrons takes place as a consequence of the Hall effect; a similar drift occurs under the action of $\mathbf{H}'$ and ∇T. As a result of the total drift, new current rings are formed on the two sides of the original ring; on one side, the current in the new ring is in the same direction as in the old, on the other side, it is in the opposite direction. The drift of the electrons in the magnetic field of these rings forms new current rings, and so on. This is the TM-wave. Its phase velocity $-c\alpha_1|\nabla T| \times \cos(\mathbf{k}, \hat{\nabla} T)$ has only one sign, in contrast with sound or Alfvén waves; therefore the TM-waves can be propagated in only one direction. The TM-waves are weakly damped if

$$ck / 4\pi\sigma_0 |\alpha_1 \nabla T| \ll 1. \tag{1.5}$$

This criterion can be satisfied, for example, in such a metal as copper, and in such a semimetal as bismuth, at $T = 30°K$ and below.

Let us consider the TM-waves in the case in which there is an external magnetic field $\mathbf{H}_0$ in the conductor in addition to ∇T. For an arbitrary value of $\Omega\tau$ (Ω is the Larmor frequency and τ the relaxation time of the current carriers) the electric field $\mathbf{E}$ is expressed in terms of $\mathbf{j}$, $\mathbf{H}$ and ∇T by the formula

$$\mathbf{E} = \eta\mathbf{j} + \eta_1[\mathbf{jH}] + \eta_2\mathbf{H}(\mathbf{jH}) + \alpha\nabla T + \alpha_1[\nabla T, \mathbf{H}] + \alpha_2\mathbf{H}(\mathbf{H}\nabla T), \tag{1.6}$$

where η is the resistivity, η_i and α_i are coefficients.

A more favorable case for growth is that in which $\mathbf{k} \parallel \mathbf{H} \parallel \nabla T$.[3] We shall consider this case separately. The dispersion equation for $\mathbf{k} \parallel \mathbf{H} \parallel \nabla T \parallel x$ in a magnetic field of arbitrary intensity has the form

$$\omega + ku_\pm + \nu_m k^2(i \pm \eta_1 H_0/\eta) = 0. \tag{1.7}$$

Here $u_\pm = u_1 \mp iu_2 = c\alpha_1\partial T/\partial x$ is the phase velocity of the TM-wave in the absence of a magnetic field, $u_2 = c\alpha_2(H_0\nabla T)$ is some characteristic velocity whose existence is connected with the instability, as we shall soon see; the ratio $u_2/u_1 \sim \Omega\tau$. The signs $\pm$ in (1.7) correspond to two circularly polarized (helicoidal) waves with right and left polarizations. The x axis is directed so that $u_1 > 0$.

It is seen from (1.7) that the imaginary part of the frequency, $\operatorname{Im}\omega$, can be positive; in this case the TM-waves are converted from damped to growing. The instability region lies between $k = 0$ and $k = |u_2|/\nu_m$. Of course, the wave vector $\mathbf{k}$ cannot actually be equal to zero; for this representation of the TM-waves to be applicable, the wavelength should be much less than the distance over which the temperature, together with all the kinetic coefficients, changes appreciably. If we denote this characteristic length of the inhomogeneity by $L = T/|\nabla T|$, then the condition just mentioned can be written in the form $kL \gg 1$. Here the frequencies of the amplified waves have an upper limit $\omega \sim u_1u_2/\nu_m$ for $\Omega\tau \lesssim 1$.

The amplification of the TM-waves, which is associated with the term $\alpha_2\mathbf{H}(\mathbf{H}\cdot\nabla T)$ in the electric field, can be understood in the following way. For $\mathbf{H}_0 \parallel \nabla T$, its contribution to the oscillating field will be $\mathbf{E}' = \alpha_2\mathbf{H}'(\mathbf{H}_0\cdot\nabla T)$, and therefore the Maxwell equation takes the form

$$\partial\mathbf{H}'/\partial t = -c\operatorname{rot}\mathbf{E}' = -u_2\operatorname{rot}\mathbf{H}'.$$

If we temporarily denote $\mathbf{H}'$ on the right-hand side by $\tilde{\mathbf{E}}'$, then

$$\frac{\partial\mathbf{H}'}{\partial t} = -u_2\operatorname{rot}\tilde{\mathbf{E}}', \qquad \frac{\partial\tilde{\mathbf{E}}'}{\partial t} = -u_2\operatorname{rot}\mathbf{H}'.$$

This set differs only in sign from the Maxwell equations for the electromagnetic field in a vacuum; therefore the "induced" field has the same sign as the original field, and not the opposite, as is the case in Lenz's law. If we differentiate one of the equations with respect to time, and the other with respect to the coordinate, then we get for the wave traveling along the x axis,

$$\frac{\partial^2\mathbf{H}'}{\partial t^2} + u_2^2\frac{\partial^2\mathbf{H}'}{\partial x^2} = 0,$$

i.e., the electromagnetic field increases or attenuates exponentially in time. The Nernst and Hall terms change this purely exponential growth to an oscillatory one. If the increment ku_2 exceeds the dissipative damping $\nu_m k^2$, instability sets in.

2. NONLINEAR THEORY OF TRAVELING TM-WAVES

In the linear approximation, the TM-waves are unstable in an external magnetic field. We shall find the solution of Eqs. (1.3) and (1.6) in the case of traveling waves of finite amplitude propagating along $\nabla T \parallel \mathbf{H}_0$ (in this case, temperature and density oscillations are absent; only the electric and magnetic fields oscillate). The traveling TM-waves can be created in the core, where $\nabla T \parallel H_0$, for example, by means of two mutually perpendicular transverse coils on one end of the core, in which the oscillating fields are shifted in phase by 90°. A circularly polarized traveling wave will be induced at this end of the core. For $\Omega\tau < 1$ this corresponds to the solution of the dispersion equation $k_1 \approx -\omega/u_1$; the second wave, with $k_2 \approx iu_1/\nu_m$, is rapidly damped if $u_1 d/\nu_m \gg 1$, where d is the length of the core. Here the reflection of waves from the other end is unimportant and one can restrict the consideration to the case of the traveling wave. This traveling wave can be observed by placing a similar set of two mutually perpendicular coils at the other end of the core; the traveling wave induces a current in these coils.

If all the variable quantities depend only on $x \parallel \nabla T$ and t, then it follows from the equation

$$\operatorname{div}\mathbf{j} = 0, \quad \operatorname{div}\mathbf{H} = 0$$

that $j_{\parallel} = 0$ in the case of an open circuit, and $\mathbf{H}_{\parallel} = \mathbf{H}_0 = \text{const}$ ($\parallel$ and $\perp$ designate the longitudinal and transverse components of the field and current). Assuming that all the quantities depend on the combination $x - wt$, where w is the phase velocity of the wave, which must be determined, we get from (1.3):

$$\mathbf{E}_\perp = -\frac{w}{c}[\mathbf{n}\mathbf{H}_\perp],$$

$$\mathbf{j}_\perp = -\frac{c}{4\pi w}\left[\mathbf{n}\frac{\partial \mathbf{H}_\perp}{dt}\right] = \frac{c^2}{4\pi w^2}\frac{\partial \mathbf{E}_\perp}{dt}, \qquad (2.1)$$

where $\mathbf{n}$ is a unit vector in the direction of the x axis.

By multiplying (1.6) in scalar fashion by $\mathbf{H}_\perp$ and using the equality $(\mathbf{E}_\perp \cdot \mathbf{H}_\perp) = 0$, which follows from (2.1), we obtain

$$(\mathbf{j}_\perp \mathbf{H}_\perp) = -\frac{\alpha_2 H_\perp{}^2(\mathbf{H}_0\nabla T)}{\eta + \eta_2 H_\perp{}^2}. \qquad (2.2)$$

In (1.6), we express $\mathbf{E}_\perp$ and $\mathbf{j}_\perp$ in terms of $\mathbf{H}_\perp$ and $\partial\mathbf{H}_\perp/\partial t$, and solve the equation relative to $\partial\mathbf{H}_\perp/\partial t$:

$$-\frac{c}{4\pi w}(\eta^2 + \eta_1{}^2 H_0{}^2)\frac{\partial \mathbf{H}_\perp}{\partial t} = \eta\left\{\frac{w}{c} + \left(\alpha_1 + \frac{\alpha_2\eta_1 H_0{}^2}{\eta + \eta_2 H_\perp{}^2}\right)\right.$$
$$\left.\times(\mathbf{n}\nabla T)\right\}\mathbf{H}_\perp + \left\{\alpha_1\eta_1(\mathbf{H}_0\nabla T) + \frac{w}{c}\eta_1(\mathbf{n}\mathbf{H}_0)\right.$$
$$\left. - \frac{\alpha_2\eta^2}{\eta + \eta_2 H_\perp{}^2}(\mathbf{H}_0\nabla T)\right\}[\mathbf{n}\mathbf{H}_\perp]. \qquad (2.3)$$

In the stationary state $\partial\mathbf{H}_\perp^2/\partial t = 0$, whence

$$\frac{w}{c} = -(\mathbf{n}\nabla T)\left(\alpha_1 + \frac{\alpha_2\eta_1 H_0{}^2}{\eta + \eta_2 H_\perp{}^2}\right). \qquad (2.4)$$

From the remaining equation we get, for the stationary state,

$$H_y = H_{0\perp}\cos[\omega(t - x/w) + \varphi],$$
$$H_z = -H_{0\perp}\sin[\omega(t - x/w) + \varphi], \qquad (2.5)$$

where

$$\omega = kw = \frac{4\pi w\alpha_2}{c(\eta + \eta_2 H_\perp{}^2)}(\mathbf{H}_0\nabla T). \qquad (2.6)$$

The wave is circularly polarized and the sign of its polarization is determined by the sign of w. As $\mathbf{H}_\perp \to 0$, the value of ω is the frequency for which $\operatorname{Im}\omega = 0$ in the linear theory (1.7). Equations (2.4)–(2.6) give the solution of the problem. The quantities ω and w are determined as functions of the field $H_\perp^2$ (which enters both explicitly and through the kinetic coefficients), which is in turn determined from the boundary conditions that fix the value of the vector $k = \omega/w$.

The stationary TM-waves can exist for $|\nabla T| > |(\nabla T)_{cr}|$ which can be found from (2.6) by taking the limit $\mathbf{H}_\perp \to 0$:

$$(\mathbf{H}_0(\nabla T)_{cr}) = ck\eta(H_0)/4\pi\alpha_2(H_0). \qquad (2.7)$$

Thus, the ∇T for which stationary waves are possible should satisfy the inequality

$$0 < \frac{\mathbf{H}_0(\nabla T - (\nabla T)_{cr})}{(\mathbf{H}_0\nabla T)}$$
$$= 1 - \frac{\eta(H_0)\alpha_2(H)}{\alpha_2(H_0)[\eta(H) + \eta_2(H)H_\perp{}^2]}. \qquad (2.8)$$

In the limiting case of a wave with small amplitude $(\mathbf{H}_\perp \to 0)$, the inequality (2.8) reduces to

$$0 < \frac{\mathbf{H}_0(\nabla T - (\nabla T)_{cr})}{(\mathbf{H}_0\nabla T)}$$
$$= H_\perp{}^2\left(\frac{1}{\eta}\frac{\partial\eta}{\partial H^2} + \eta_2 - \frac{1}{\alpha_2}\frac{\partial\alpha_2}{\partial H^2}\right), \qquad (2.9)$$

or, if we use the equation $\eta(H) + \eta_2(H)H^2 = \eta(0)$, where $\eta(0)$ is the resistivity for $H = 0$, then

$$0 > H_\perp{}^2\left(\frac{1}{\alpha_2}\frac{\partial\alpha_2}{\partial H^2} + \frac{H^2}{\eta}\frac{\partial\eta_2}{\partial H^2}\right). \qquad (2.10)$$

In the case of metals with one type of carrier and isotropic mass, $\eta_2 \propto (T/\zeta)^2$, where ζ is the Fermi energy and the term $\eta_2 H_\perp^2$ can be neglected. In this case, $\eta(H) \approx \eta(0) \gg \eta_2 H_\perp^2$. Then

$$0 < \frac{\mathbf{H}_0(\nabla T - (\nabla T)_{cr})}{(\mathbf{H}_0\nabla T)} = 1 - \frac{\alpha_2(H)}{\alpha_2(H_0)}. \qquad (2.11)$$

Equation (2.11) is satisfied if for any value of $\mathbf{H}$,

$$\frac{1}{\alpha_2}\frac{\partial\alpha_2}{\partial H^2} < 0$$

(it must be kept in mind that $H^2 = H_0^2 + H_\perp^2 > H_0^2$). This inequality, in particular, holds if the relaxation time approximation is valid and the dependence of α_2 on the magnetic field has the form $\alpha_2 = \text{const}/(1 + A^2H^2)$.

The validity of (2.8) is not obvious in the general case. If (2.8) is not satisfied, then a stationary state, produced by a traveling wave with a velocity w independent of the coordinate and of time, does not exist. We shall not construct the general theory of the phenomenon for this case, but shall introduce an equation which describes the growth kinetics of the TM-waves, making it possible to understand the difference between the two cases. We do not assume the stationarity of the state, but limit ourselves to the case of small amplitudes $H_\perp \ll H_0$.

Eliminating $\mathbf{E}$ and $\mathbf{j}$ from (1.3) and (1.6), we get a set of equations for $H_\pm = H_y \pm iH_z$:

$$\frac{\partial H_\pm}{\partial t} = \frac{c^2}{4\pi}\frac{\partial}{\partial x}\left\{(\eta \mp i\eta_1 H_0)\frac{\partial H_\pm}{\partial x}\right\}$$

$$+c\frac{\partial}{\partial x}\left(\alpha_1 H_\pm \frac{\partial T}{\partial x}\right) \mp ic\frac{\partial}{\partial x}\{\alpha_2 H_\pm(\mathbf{H}_0\nabla T)\}$$

$$+\frac{c^2}{4\pi}\frac{\partial}{\partial x}\left\{\eta_2\left(H_\perp^2\frac{\partial H_\pm}{\partial x}-\frac{1}{2}\frac{\partial H_\perp^2}{\partial x}H_\pm\right)\right\}. \quad (2.12)$$

We shall solve (2.12) by iteration in $H_\pm$. In the approximation linear in $H_\pm$, (2.12) has a solution for $\nabla T = (\nabla T)_{cr}$:

$$H_\pm = H_{0,\pm}\exp[\pm i(kx-\omega t)], \quad H_{0,+}^* = H_{0,-};$$

$$\omega = -\frac{u_2}{\nu_m}\left(u_1 + u_2\frac{\eta_1 H_0}{\eta}\right). \quad (2.13)$$

As a boundary condition, a fixed value of the wave vector **k** is assumed as before. Let

$$\left|\frac{\nabla T - (\nabla T)_{cr}}{\nabla T}\right| \ll 1,$$

only in this case can we iterate (2.12). With accuracy up to terms of third order in $H_\pm$, (2.12) takes the form

$$\left\{\frac{\partial}{\partial t} - c(u_{1,cr}\mp iu_{2,cr})\frac{\partial}{\partial x} - \frac{c^2}{4\pi}[\eta(H_0)\mp i\eta_1(H_0)H_0]\frac{\partial^2}{\partial x^2}\right\}$$

$$\times H_\pm = c(\Delta u_1 \mp i\Delta u_2)\frac{\partial}{\partial x}H_\pm$$

$$+\frac{c^2}{4\pi}\left(\frac{\partial\eta}{\partial H^2}\mp iH_0\frac{\partial\eta_1}{\partial H^2}+\eta_2\right)\times\frac{\partial}{\partial x}\left(H_\perp^2\frac{\partial}{\partial x}H_\pm\right)$$

$$-\frac{c^2}{8\pi}\eta_2\frac{\partial}{\partial x}\left(H_\pm\frac{\partial}{\partial x}H_\perp^2\right)$$

$$+c\frac{\partial T}{\partial x}\left(\frac{\partial\alpha_1}{\partial H^2}\mp iH_0\frac{\partial\alpha_2}{\partial H^2}\right)\frac{\partial}{\partial x}(H_\perp^2 H_\pm), \quad (2.14)$$

where

$$u_{1,\text{кр}} = c\alpha_1(H_0)(\partial T/\partial x)_{cr}, \quad u_2 = c\alpha_2(H_0)(\mathbf{H}_0(\nabla T)_{cr}),$$

$$\Delta u_1 = c\alpha_1\left(\frac{\partial T}{\partial x}-\left(\frac{\partial T}{\partial x}\right)_{cr}\right)\Delta u_2 = c\alpha_2\mathbf{H}_0(\nabla T-(\nabla T)_{cr}).$$

On the right hand side of (2.14) one can substitute (2.13) (terms of second order in $H_\pm$ are absent), while on the left hand side of (2.14), it is necessary to consider a slow dependence of the amplitude of $H_{0\pm}$ on the time in the terms linear in $H_\pm$. Then

$$dH_{0+}/dt = -i\Delta\omega H_{0+} + \beta H_{0+}|H_{0+}|^2, \quad (2.15)$$

where

$$\Delta\omega = -k\Delta u_1 + ik\Delta u_2,$$

$$\beta = ck\left[\frac{ck}{4\pi}\left(-\frac{\partial\eta}{\partial H^2}+iH_0\frac{\partial\eta_1}{\partial H^2}-\eta_2\right)\right.$$

$$\left.+i\frac{\partial\alpha_1}{\partial H^2}\frac{\partial T}{\partial x}+\frac{\partial\alpha_2}{\partial H^2}(\mathbf{H}_0\nabla T)\right].$$

Multiplying (2.15) by H_{0+}^* and its conjugate by H_{0+}, and adding, we get an expression for $|H_{0+}|^2$:

$$\frac{1}{2}\frac{d}{dt}|H_{0+}|^2 = \gamma|H_{0+}|^2 + \text{Re}\,\beta|H_{0+}|^4, \quad (2.16)$$

where $\gamma = \text{Im}\,\Delta\omega = k\Delta u_2$.

The solution of (2.15) has the form[4]

$$\frac{1}{|H_{0+}(t)|^2} = \frac{e^{-2\gamma t}}{|H_{0+}(0)|^2} + \frac{\text{Re}\,\beta}{\gamma}(e^{-2\gamma t}-1). \quad (2.17)$$

The amplitude of $|H_{0+}(t)|$ increases when $\gamma > 0$, which is the result of the linear theory. The amplitude of the solution (2.17) approaches a limiting value

$$|H_{0+,\,\text{stat}}|^2 = -\frac{\gamma}{\text{Re}\,\beta} = -\frac{\Delta u_2}{c}$$

$$\times\left[\frac{\partial\alpha_2}{\partial H^2}(\mathbf{H}_0\nabla T)-\frac{ck}{4\pi}\left(\frac{\partial\eta}{\partial H^2}+\eta_2\right)\right]^{-1}$$

$$=\frac{\mathbf{H}_0(\nabla T-(\nabla T)_{cr})}{(\mathbf{H}_0\nabla T)}\left[-\frac{1}{\alpha_2}\frac{\partial\alpha_2}{\partial H^2}+\frac{1}{\eta}\frac{\partial\eta}{\partial H^2}+\frac{\eta_2}{\eta}\right]. \quad (2.18)$$

We obtained the last expression by expressing k in terms of $(\nabla T)_{cr}$ according to (2.7). The quantity $|H_{0+,\,\text{stat}}|$ in (2.18) is identical with the value found earlier of the amplitude of the stationary wave (2.9) (in the small amplitude limit), which is thus the final result of the developing instability. Equation (2.17) correctly describes the behavior of H_{0+} at any moment of time for Re $\beta < (\gamma > 0)$, which indicates the softness of the excitation regime, i.e., the continuous dependence of $|H_{0+,\,\text{stat}}|$ on $|\nabla T - (\nabla T)_{cr}|$. In the opposite case of Re $\beta > 0$, the excitation regime is hard, i.e., for some suitably small positive difference $|\nabla T - (\nabla T)_{cr}|$, the amplitude $|H_{0+,\,\text{stat}}|$ and the other quantities undergo a finite jump. The solution (2.17) applies only over a limited interval of time. The condition of the positive nature of the limiting amplitude of (2.18) or, what amounts to the same thing, the condition of the softness of the excitation regime, is identical with the inequality (2.9).

3. NONLINEAR THEORY OF TM-WAVES IN THE NONISOTHERMAL CORE OF AN INDUCTANCE COIL

In [5] we proposed another possible experiment for finding TM-waves. Let an inductance coil be connected in the circuit, having a core in the shape of a plate of height h (z axis), width 2a perpendicular to the axis of the coil (y axis), and thickness 2d (x axis), $d \ll \min(h, a)$. There is a gradient ∇T along the x axis and a constant external magnetic field, such that TM-waves are possible in the plate. Then, as was shown in [5], the additional impedance Z introduced by the plate reveals a number of singularities. In particular, $R = \mathrm{Re}\, Z$ can be negative, which means the possibility of the generation of undamped oscillations. A linear theory of this phenomenon was given in [5]. It is of interest to find the amplitude of these oscillations in the limit of a small excess of $|\nabla T|$ over $|(\nabla T)_{cr}|$:

$$\left|\frac{\nabla T-(\nabla T)_{\mathrm{cr}}}{\nabla T}\right| \ll 1,$$

when only a single frequency is excited; otherwise, a turbulent mixing of the oscillations is produced. Upon satisfaction of this inequality, Eq. (2.12), which the field in the plate satisfies, can be solved by the iteration method. However, in contrast with the previous section, the solution does not have the form of a traveling wave, since it is necessary to satisfy at $x = \pm d$ the boundary conditions

$$H_y = 0, \quad H_z = 4\pi NI / c, \tag{3.1}$$

where N is the number of turns per unit length of the coil, and I is the current in the circuit of the coil, which is determined from the condition that the total emf is equal to zero:

$$IZ_0 = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{N}{c}\frac{d}{dt}\int H_z dx\, dy\, dz; \tag{3.2}$$

Z_0 is the external load impedance (in Z_0, we have included the active resistance of the coil and that part of its reactance which is connected with the magnetic flux that does not pass through the plate). Generation takes place when $|\nabla T| > |(\nabla T)_{cr}|$. The critical temperature gradient is determined from the condition that for a real frequency ($\mathrm{Im}\, \omega = 0$), we have $Z(\omega) + Z_0(\omega) = 0$. For $|\nabla T| < |(\nabla T)_{cr}|$, all the zeroes of the total resistance $Z(\omega) + Z_0(\omega) = 0$ lie in the lower half-plane of the complex ω: for $\mathrm{Im}\, \omega < 0$, which corresponds to damping. For $|\nabla T| = |(\nabla T)_{cr}|$, the solution of Eq. (2.12) that is linear in the excitation is conveniently written in the form

$$\begin{aligned} H_+(x, t) &= (A_1 e^{ik_1 x} + A_2 e^{ik_2 x})\, e^{-i\omega t} \\ &+ (A_3^* e^{-ik_3^* x} + A_4 e^{-ik_4^* x})\, e^{i\omega t}, \\ H_-(x, t) &= H_+^*(x, t). \end{aligned} \tag{3.3}$$

k_1 and k_2 are the roots of the dispersion equation (1.7), corresponding to the upper sign, and k_3 and k_4 are the roots corresponding to the lower sign in (1.7). The frequency ω is determined from the condition $Z + Z_0 = 0$ and for $|\nabla T| = |(\nabla T)_{cr}|$ it is real. The boundary conditions (3.1) connect the coefficients A_1, A_2, A_3, and A_4 with the amplitude of the current in the external circuit:

$$I(t) = I_0 e^{-i\omega t} + I_0^* e^{i\omega t}. \tag{3.4}$$

The problem of nonlinear theory is that of finding I_0 for a given difference $|\nabla T - (\nabla T)_{cr}|$. One can substitute (3.3), in the terms nonlinear in H_y and H_z of Eq. (2.14), which is valid with accuracy up to terms of third order in H_y and H_z. In terms that are linear in H_y and H_z it is necessary, as we shall see below, to regard A_1, A_2, A_3, and A_4 as slowly changing functions of x and t (in contrast with the case of a traveling wave, in which the amplitude was a slow function of t only). I_0 in (3.4) must also be regarded as a slowly changing function of time.

The solution of (2.14)

$$H_\pm(x, t) = H_\pm^{(1)}(x, t) + H_\pm^{(2)}(x, t) \tag{3.5}$$

contains $H_\pm^{(1)}(x, t)$, which represents the solution of (2.14), in which the nonlinear terms are canceled out and the dependence of the amplitudes A_1, A_2, A_3, and A_4 on x, and t is taken into account, and $H_\pm^{(2)}(x, t)$, which is the solution of (2.14) in which (3.3) is substituted in the terms which are nonlinear in H_y and H_z. Inasmuch as the wave vectors k_1, k_2, k_3, and k_4 are complex, the secular terms do not arise in finding $H_\pm^{(2)}(x, t)$. $H_\pm^{(2)}(x, t)$ is the sum of terms of third order in I_0, the components of which depend on the time as $e^{\pm i\omega t}$, $e^{\pm 3i\omega t}$:

$$\begin{aligned} H_\pm^{(2)}(x, t) &= H_{\pm,\omega}^{(2)}(x)\, e^{-i\omega t} + H_{\pm,-\omega}^{(2)}(x)\, e^{i\omega t} \\ &+ H_{\pm,3\omega}^{(2)}(x)\, e^{-3i\omega t} + H_{\pm,-3\omega}^{(2)}\, e^{3i\omega t}. \end{aligned} \tag{3.6}$$

Because of the inequality $\partial \ln A_j / \partial x \ll k_j$ (slow dependence on x), the equation for the amplitudes A_1, A_2, A_3, and A_4 reduces to

$$\frac{\partial A_j}{\partial t} - v_j \frac{\partial A_j}{\partial x} - ik_j \Delta u_\pm A_j = 0, \tag{3.7}$$

where

$$v_{1,2} = u_+ + 2\nu_m k_{1,2}(i + \eta_1 H_0 / \eta),$$

$$v_{3,4} = u_- + 2\nu_m k_{3,4}(i - \eta_1 H_0 / \eta).$$

$u_\pm$ enter into the equations for $A_{1,2}$ and $A_{3,4}$, respectively. Equation (3.7) has the solution

$$A_j(x, t) = A_j(x + v_j t, 0) \exp\{ik_j t \Delta u_\pm\}. \tag{3.8}$$

The boundary conditions (3.1) give the relation between I_0 and A_j:

$$A_1(\pm d + v_1t, 0) \exp\{\pm ik_1d + ik_1t\Delta u_+\} + A_2(\pm d + v_2t, 0)\exp\{\pm ik_2d + ik_2t\Delta u_+\} = i4\pi Nc^{-1}I_0(t) - H^{(2)}_{+,\omega}(\pm d). \tag{3.9}$$

The equations for A_3 and A_4 are similar. One can solve the set of different equations (3.9) by the method of transformation into a Fourier integral:[6]

$$A_1(x+v_1t, 0) = \frac{i}{\pi}\exp\left[-ik_1\Delta u_+\left(t+\frac{x}{v_1}\right)\right]$$
$$\times\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dt'\exp\left[-iq\left(t-t'+\frac{x}{v_1}\right)\right]$$
$$\times\{f(t',q,-d)-f(t',q,d)\}\cdot$$
$$\times\left\{\sin\left[k_2d\left(1-\frac{\Delta u_+}{v_2}\right)-k_1d\left(1-\frac{\Delta u_+}{v_2}\right)\right.\right.$$
$$\left.\left.+qd\left(\frac{1}{v_1}-\frac{1}{v_2}\right)\right]\right\}^{-1};$$
$$f(t,q,d) = \left[i\frac{4\pi N}{c}I_0(t) - H^{(2)}_{+,\omega}(d)\right]$$
$$\times\exp\left[\frac{iqd}{v_2}-ik_2d\left(1-\frac{\Delta u_+}{v_2}\right)\right]. \tag{3.10}$$

The expression for $A_2(x + v_2t, 0)$ is similar. In the same way, one can find $A_3(x, t)$ and $A_4(x, t)$. Formulas (3.3), (3.5), (3.6), (3.8), and (3.10) allow us in principle to find the dependence of H_y and H_z on x and t with accuracy up to terms of third order in I_0. The only unknown parameter is I_0, which is determined from (3.2). We substitute (3.3)–(3.6), (3.8), and (3.10) in (3.2) and equate to zero those terms containing the "fast" dependence on the time, $e^{-i\omega t}$:

$$Z_0I_0 + L(I_0) - i\omega c^{-1}\Phi_{nl}(\omega) = 0, \tag{3.11}$$

where the linear term in I_0, $L(I_0)$ has the form

$$L(I_0) = i\frac{2N^2s}{c^2}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dt'\,e^{-iq(t-t')}(\omega+q)I_0(t')$$
$$\times\left[\frac{1}{k_2(1-\Delta u_+/v_2)-q/v_2}\right.$$
$$\left.-\frac{1}{k_1(1-\Delta u_+/v_1)-q/v_1}\right]\sin\left[k_1d\left(1-\frac{\Delta u_+}{v_1}\right)-\frac{qd}{v_1}\right]$$
$$\times\sin\left[k_2d\left(1-\frac{\Delta u_+}{v_2}\right)-\frac{qd}{v_2}\right]\left\{\sin\left[k_2d\left(1-\frac{\Delta u_+}{v_2}\right)\right.\right.$$
$$\left.\left.-k_1d\left(1-\frac{\Delta u_+}{v_1}\right)+qd\left(\frac{1}{v_1}-\frac{1}{v_2}\right)\right]\right\}^{-1}$$

+ a similar component in which the substitution

$$k_1\to k_3,\quad k_2\to k_4,\quad v_1\to v_3,\quad v_2\to v_4,\quad \Delta u_+\to\Delta u_-$$

has been introduced. Here, $s = 2ah$ is the area of the plate.

The quantity $\Phi_{nl}(\omega)$ is the contribution to the flux of those terms of third order in I_0 which are proportional to $e^{-i\omega t}$. It consists of a component which arises from integration of $H_z^{(2)}$ over x, y, and z, and that part of $H^{(1)}$ which, in accord with (3.10), is also proportional to I_0^3. The contribution from the term $\partial H^{(2)}/\partial t$ lies outside of our approximation.

This is the general solution of our problem. In the case of a very small excess of $|\nabla T|$ over $|(\nabla T)_{cr}|$, when not only $\partial \ln I_0/\partial t \ll k_j v_j$ (this is equivalent to the slow dependence of A_j on x and t, assumed above), but also $(d/v_j)\partial \ln I_0/\partial t \ll 1$ (which is a more rigid condition, since $k_jd \gtrsim 1$), $L(I_0)$ in (3.11) can be expressed in terms of $I_0(t)$ and dI_0/dt, making use of the slowness of the function $I_0(t)$. Then

$$i\frac{\partial Z}{\partial\omega}\frac{dI_0(t)}{dt} + \frac{\partial Z}{\partial(\partial T/\partial x)}\left(\frac{\partial T}{\partial x}-\left(\frac{\partial T}{\partial x}\right)_{cr}\right)$$
$$\times I_0(t) - \frac{i\omega}{c}\Phi_{nl}(\omega) = 0, \tag{3.12}$$

where the additional impedance contributed by the plate[5] is

$$Z(\omega) = \frac{4\pi i}{c^2}N^2s\omega\left\{\left(\frac{1}{k_2}-\frac{1}{k_1}\right)\frac{\sin k_1d\sin k_2d}{\sin(k_2-k_1)d}\right.$$
$$\left.+\left(\frac{1}{k_4}-\frac{1}{k_3}\right)\frac{\sin k_3d\sin k_4d}{\sin(k_4-k_3)d}\right\}.$$

In (3.12) we took account of the fact that $Z(\omega) + Z_0 = 0$. Equation (3.12) can be written in the form

$$dI_0/dt = -i\Delta\omega I_0 + \beta|I_0|^2I_0, \tag{3.13}$$

where

$$\Delta\omega = -\frac{\partial Z}{\partial(\partial T/\partial x)}\left[\frac{\partial T}{\partial x}-\left(\frac{\partial T}{\partial x}\right)_{cr}\right]\left(\frac{\partial Z}{\partial\omega}\right)^{-1},$$
$$\beta = \frac{\omega}{cI_0|I_0|^2}\Phi_{nl}(\omega)\left(\frac{\partial Z}{\partial\omega}\right)^{-1}.$$

It follows from (3.13) and the complex conjugate equation that

$$\frac{d}{dt}\left|I_0\right|^2 = 2\gamma|I_0|^2 + 2\,\mathrm{Re}\beta\,|I_0|^4, \tag{3.14}$$

where $\gamma = \mathrm{Im}\,\Delta\omega$. Equation (3.14) has the form of Eq. (2.16) which was analyzed in Sec. 2. The positiveness of $\gamma > 0$ is the necessary condition in the linear theory that the instability take place. The softness or hardness of the excitation depends on the sign of Re β. In the case of a soft regime (Re $\beta < 0$), it makes sense to determine the stationary value of the current amplitude

$$|I_{0,\,\mathrm{stat}}|^2 = -\gamma^{-1}\,\mathrm{Re}\,\beta, \qquad (3.15)$$

Re $\Delta\omega$ denotes the renormalization of the frequency of the stationary current, which is equal to ω + Re $\Delta\omega$. Thus the frequency of the current depends on the amplitude of the current in the circuit.

Equation (3.14) solves the proposed problem. The specific form of the quantities β and $\Delta\omega$ can be found for the limiting values of the parameters, since the computations in the general case are too cumbersome. We shall give the results for the case of a weak external magnetic field $eH_0\tau/mc \ll 1$. The roots of the dispersion equation (1.7) and the form of the impedance $Z(\omega)$ were found in [5]. The investigation based on the results of this research shows that for $Z_0 \ll 4\pi N^2 c^{-2} s u_1$ generation begins at a frequency $\omega \approx \pi u_1/d$ (if, as in Sec. 2, $u_1/\nu_m d \gg 1$), $(\nabla T)_{cr}$ satisfies the equation $(u_2/u_1)^2 = \nu_m/u_1 d$; in order of magnitude, this is equivalent to

$$\Omega\tau \sim \sqrt{\nu_m/u_1 d} \ll 1. \qquad (3.16)$$

We shall write out only the results of the calculations, which are rather involved but present no difficulties. The coefficients in (3.13) are

$$\Delta\omega = \frac{\pi\Delta u_1}{d}\left(1 + \pi i\frac{u_2^2}{u_1^2}\right),$$

$$\gamma = \mathrm{Im}\,\Delta\omega = \frac{\pi^2}{d}\,\Delta u_1\left(\frac{u_2}{u_1}\right)^2,$$

$$\beta = -\frac{3}{4d}\left(\frac{4\pi N}{c}\right)^2\frac{\partial u_1}{\partial H^2}(1 + 4\pi i),$$

$$\left(\frac{4\pi N}{c}\right)^2|I_{0,\,\mathrm{stat}}|^2 = \frac{4\pi^2}{3}\left(\frac{u_2}{u_1}\right)^2\frac{\Delta u_1}{\partial u_1/\partial H^2}. \qquad (3.17)$$

The quantity $\gamma > 0$ if $\Delta u_1 > 0$ (we recall that, by assumption, $u_1 > 0$). This means that only for $|\nabla T| > |(\nabla T)_{cr}|$ can the system be a generator. The excitation regime is soft if $\partial u_1/\partial H^2 > 0$; in the opposite case, it is hard. The condition for softness of the regime, as a consequence of the fact that $u_1 > 0$, can be written in the form $\alpha_1^{-1}\,\partial\alpha_1/\partial H^2 > 0$.

It follows from (3.17) that the slow dependence of the amplitude A_j on x and t ($d\partial \ln A_j/\partial t \ll v_j$) holds for

$$\frac{\Delta\omega}{\omega} \sim \frac{\Delta u_1}{u_1} \sim \left|\frac{\nabla T - (\nabla T)_{cr}}{\nabla T}\right| \ll 1.$$

In order of magnitude, the amplitude of the oscillations of the magnetic field intensity is equal to

$$|H_{\mathrm{stat}}| \sim H_0\left\{\left|\frac{\nabla T - (\nabla T)_{cr}}{\nabla T}\right|\right\}^{1/2}. \qquad (3.18)$$

In conclusion, we shall compare the magnetic fields for which instability is possible in the case of a medium of infinite extent when the solution has the form of a traveling wave, and for a plate.

In the first case, for $\Omega\tau \ll 1$, we have

$$|H_{01}| = \frac{ck\eta}{4\pi|\alpha_2\nabla T|}. \qquad (3.19)$$

In the plate, under the condition (3.16), we find

$$|H_{02}| = \left\{\frac{c\eta|\alpha_1|}{4\pi d\alpha_2^2|\nabla T|}\right\}^{1/2}. \qquad (3.20)$$

If we assume that $k \sim \pi/d$, then the ratio is

$$H_{02}/H_{01} \sim \sqrt{u_1/\nu_m d} \gg 1.$$

For the second of the experiments considered, stronger fields are necessary than for the first.

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