CONTRIBUTION TO THE THEORY OF DIFFRACTION SCATTERING OF PARTICLES BY NUCLEI, BASED ON THE METHOD OF COMPLEX ANGULAR MOMENTA

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Expressions for elastic and inelastic diffraction scattering of particles by nuclei, involving the excitation of collective states, are derived by the technique of complex angular momenta. It is assumed that the modulus as well as phase shift of the S matrix may possess poles in the complex angular momentum plane. It is shown that the presence of poles in the S-matrix phase shift near the poles of its modulus and Coulomb interaction permit one to explain a number of interesting features of the behavior of the differential scattering cross sections, such as the decrease of the oscillation amplitudes of the cross sections with growth of the nuclear charge, the possible presence of inelastic-scattering cross-section oscillations in the case when the oscillations are absent in elastic scattering, and decrease of the oscillation amplitude with growth of the scattering angle. It is shown that "competition" between the Coulomb and nuclear phases can explain the "cross section drop" (presence of one or two cross section minima which are much lower than the neighboring ones). It is noted in this connection that the quantity $\delta(l_0)$ (where l_0 is the limiting nuclear angular momentum) can readily be estimated.

1. INTRODUCTION

THE theory of diffraction scattering of particles by nuclei has been intensively developed in recent years. Thus, in a recent paper by Inopin[1], the method of complex angular momenta was proposed to obtain the cross sections for elastic diffraction scattering of particles by nuclei. Subsequently this method was employed[2] to calculate the cross sections for inelastic scattering with excitation of collective states of the nuclei.

In [1,2] the scattering matrix was written in the form

$$\eta(l) = A(l)e^{2i\delta(l)} \quad (1)$$

and it was assumed that in the plane of the complex angular momenta the function $A(l)$ has poles. Thus, for example, at the points $l_n = l_0 \pm in\pi$ ($n = 1, 3, \ldots$) there are poles of the function

$$A(l) = \left[1 + \exp\left(\frac{l_0 - l}{\lambda}\right)\right]^{-1}, \quad (2)$$

which is frequently used in the analysis of the experimental data. The limiting momentum is

$$l_0 = kR(1 - B/E)^{1/2}, \quad (3)$$

where B is the Coulomb barrier and E the c.m.s. energy of the incident particle.

In the earlier papers[1,2] no account was taken of the fact that the phase $\delta(l)$ can have singularities near the poles of the function $A(l)$. One can advance arguments, however, in favor of the fact that such singularities can exist. Indeed, in the presence of a finite interaction radius R, and if $kR \gg 1$ in the region $l \sim l_0$, one can expect rapid variation not only of the absorption function $A(l)$ from zero to unity, but also of the scattering phase $\delta(l)$ from a certain value δ_0 to 0. Such a behavior of $\delta(l)$ can be described by a system of poles located near the poles of $A(l)$. We note in this connection that many authors (see [3,4] and others), in a numerical analysis of the diffraction scattering, take besides the function $A(l)$ in the form (2) also the function $\delta(l)$ in the form

$$\delta(l) = \delta_0\left[1 + \exp\left(\frac{l - l_\delta}{\mu}\right)\right]^{-1} \quad (4)$$

with values of the parameters l_δ and μ close to l_0 and λ, respectively.

By virtue of the foregoing it is of interest to consider diffraction scattering of particles by nuclei under the assumption that the functions $A(l)$ and $\delta(l)$ can have closely-lying poles in the plane of the complex angular momenta.

2. CROSS SECTIONS FOR ELASTIC AND INELASTIC SCATTERING

We assume that both $A(l)$ and $\delta(l)$ can have poles in complex-conjugate points of the complex l plane. The poles of these functions generate singularities of the S matrix.

We assume further that the main contribution to the cross section is made by one pair of singularities of the S matrix at the points $l = l_0 \pm i\beta$. Of great significance in this case is the relative arrangement of the poles of the functions $A(l)$ and $\delta(l)$. In the case when two poles of the functions $A(l)$ at complex-conjugate points (we are referring to poles closest to the real axis) are located close to the corresponding poles of the function $\delta(l)$, we shall assume that the positions of these poles coincide. On the other hand, if these poles are located at essentially different distances from the real axis, the singularities of the S-matrix are determined by the poles of only one of the functions, $A(l)$ or $\delta(l)$.

Under these assumptions, we consider the diffraction scattering of spinless particles by zero-spin nuclei. Austern and Blair [5] have shown that in this case the amplitude of inelastic scattering with excitation of n-phonon state of the nucleus can be represented in the form

$$f_{IM,00}^{(n)}(\vartheta) = i(-1)^{n-1}\frac{k^{n-1}}{2(n!)}(2I+1)^{1/2}C_n(I)[I:M]$$

$$\times\sum_{l=M}^{\infty}(2l+1)^{1/2}e^{2i\sigma(l)}\frac{\partial^n\eta(l)}{\partial l^n}Y_l^{-M}(\vartheta,0), \tag{5}$$

where ϑ is the scattering angle in the c.m.s., I and M are the spin and the spin projection of the nucleus in the excited state, and $C_n(I)$ is the spectroscopic factor. The coefficients [I:M] are of the form

$$[I:M] = i^{-M}(4\pi/(2I+1))^{1/2}Y_I^M(\pi/2,0), \tag{6}$$

hence

$$\sum_M |[I:M]|^2 = 1. \tag{7}$$

We note that the coefficients (6) vanish if I and M have different parity. When $n = 0$ formula (5) describes elastic scattering.

We expand the S-matrix in a series:

$$S(l) = \eta(l)e^{2i\sigma(l)} = A(l)e^{2i\sigma(l)}\sum_{k=0}^{\infty}\frac{(2i\delta(l))^k}{k!}. \tag{8}$$

If $|\delta(l)| \ll 1$, then, naturally, we can confine ourselves to a small number of expansion terms.

The Coulomb phase $\sigma(l)$ in the plane of the complex angular momenta will be defined by the formula

$$\sigma(l) = \varkappa\ln(l+1), \tag{9}$$

which follows from the relation

$$\exp[2i\sigma(l)] = \frac{\Gamma(l+1+i\varkappa)}{\Gamma(l+1-i\varkappa)}, \quad \varkappa = \frac{Z_1Z_2\mu e^2}{\hbar^2 k}$$

when the following conditions are satisfied

$$|l| \gg 1, \quad \vartheta_c = 2\varkappa/l_0 \approx B/E < 1. \tag{10}$$

Using these relations we obtain, with the aid of the procedure developed in [1,2], the following expression for the amplitude of the diffraction scattering:

$$f_{IM,00}^{(n)}(\vartheta) = i(-1)^{M+1}\frac{k^{n-1}}{2(n!)}[2(2I+1)l_0]^{1/2}C_n(I)[I:M]$$

$$\times e^{2i\varkappa\ln l_0}(\sin\vartheta)^{-1/2}e^{-\beta\vartheta}$$

$$\times\{\theta_+^nF(\theta_+)e^{i\Omega_{nM}} + \theta_-^nG(\theta_-)e^{-i\Omega_{nM}}\}, \tag{11}$$

where

$$\Omega_{nM} = (l_0 + 1/2)\vartheta + \gamma_{nM},$$

$$\gamma_{nM} = \frac{n+M}{2}\pi + \frac{\pi}{4} + i\beta\vartheta_c, \tag{12}$$

$$\theta_\pm = \vartheta \pm \vartheta_c. \tag{13}$$

The polynomials $F(x)$ and $G(x)$ are of the form

$$F(x) = \sum_{p=0}^{\infty}\frac{i^p}{p!}Q_p(x), \quad G(x) = \sum_{p=0}^{\infty}\frac{i^p}{p!}Q_p^*(x), \tag{14}$$

where

$$Q_p(x) = \sum_{s=0}^{p}\frac{(ix)^s}{s!}q_{ps}. \tag{15}$$

The quantity

$$q_{ps} = 2^p\sum{}' a_{k_0}b_{k_1}\ldots b_{k_p} \tag{16}$$

is the sum of different products of $p+1$ coefficients of the expansion of the functions $A(l)$ and $\delta(l)$ in a Laurent series:

$$A(l) = \sum_{k=-1}^{\infty}a_k(l-l_1)^k, \quad \delta(l) = \sum_{k=-1}^{\infty}b_k(l-l_1)^k \tag{17}$$

near the poles of these functions at the point $l_1 = l_0 + i\beta$. The summation in (16) is carried out under conditions

$$k_0 + k_1 + \ldots + k_p = -(s+1),$$

$$-1 \leqslant k_i \leqslant p-s-1. \tag{18}$$

If some of the terms of the sum (16) repeat, they all must be taken into account.

We now write an expression for the cross sections for elastic and inelastic scattering of particles by nuclei with excitation of states of the n-phonon type:

$$\sigma_I^{(n)}(\vartheta) = \sum_M |f_{IM,00}^{(n)}(\vartheta)|^2$$
$$= B_n(I)\,|F(\theta_+)G(\theta_-)|\,|\vartheta^2 - \vartheta_c^2|^n (\sin\vartheta)^{-1} e^{-2\beta\vartheta}$$
$$\times \{\cos^2[(l_0 + 1/2)\vartheta + \gamma_{nI}(\vartheta)] + \mathrm{sh}^2 z_n\}, \tag{19}$$

where

$$B_n(I) = 2(2I+1)\frac{k^{2n-2}}{(n!)^2} l_0 |C_n(I)|^2, \tag{20}$$
$$\gamma_{nI}(\vartheta) = \frac{n+I}{2}\pi + \frac{\pi}{4} - \frac{1}{2}\arg\left[\frac{\theta_-^n G(\theta_-)}{\theta_+^n F(\theta_+)}\right], \tag{21}$$
$$z_n = \beta\vartheta_c + \frac{n}{2}\ln\frac{|\theta_-|}{\theta_+} + \frac{1}{2}\ln\frac{|G(\theta_-)|}{|F(\theta_+)|}. \tag{22}$$

The series (14) and (15) in the cross section (19) converge rapidly and their structure does not depend on the form of the functions $A(l)$ and $\delta(l)$. This allows us to draw definite conclusions concerning the influence of the presence of poles of the function $\delta(l)$ near the poles of the function $A(l)$ on the diffraction scattering, without the need for stipulating a specified form of these functions, and to confine ourselves in concrete calculations to a small number of terms of the expansion (8).

It must be noted that Inopin presented in a recent paper[6] a more consistent derivation of the amplitude of the scattering than given by Austern and Blair[5]. If one uses in lieu of formula (5) the relation obtained in [6], then the cross section (19) retains its form, but acquires an additional common factor $\frac{1}{2}(1 + P_I(\cos\vartheta))$.

3. DISCUSSION OF RESULTS

We stop first to investigate the influence of the Coulomb field on the diffraction scattering in the particular case when the poles of the function $\delta(l)$ are much farther from the real axis than the poles of the function $A(l)$, so that we can assume that the S-matrix singularities are determined only by the poles of the function $A(l)$. Then $b_{-1} = 0$ and

$$F(\theta_+) = a_{-1}\sum_{p=0}^{\infty}\frac{(2ib_0)^p}{p!} = a_{-1}e^{2i\delta(l_1)}, \tag{23}$$
$$G(\theta_-) = a_{-1}^* e^{2i\delta(l_1^*)},$$

whence

$$\sigma_I^{(n)}(\vartheta) = B_n(I)\,|a_{-1}|^2 |\vartheta^2 - \vartheta_c^2|^n (\sin\vartheta)^{-1} e^{-2\beta\vartheta}$$
$$\cdot\{\cos^2[(l_0 + 1/2)\vartheta + \gamma_{nI}] + \mathrm{sh}^2 z_n\}, \tag{24*}$$

where

$$\gamma_{nI} = \frac{n+I}{2}\pi + \frac{\pi}{4} + \arg\left(a_{-1}\frac{\theta_-^n}{\theta_+^n}\right), \tag{25}$$
$$z_n = \beta\vartheta_c + 2\,\mathrm{Im}\,\delta(l_1) + \frac{n}{2}\ln\frac{|\theta_-|}{\theta_+}. \tag{26}$$

Formula (24) is the analog of the corresponding expressions obtained in.[1,2]

We see that if the phase $\delta(l)$ has no poles, then the period and the amplitude of the oscillations of the cross section of elastic scattering do not depend on the scattering angle. With increasing charge of the target nucleus, the magnitude of the amplitude of the oscillations should decrease, this being connected with the increase in ϑ_c, and consequently, as a rule, also of the quantity $\sinh^2 z_0$. This phenomenon is noted in the paper by Inopin and Kresnin[7], who have analyzed the experimental data on elastic scattering of α particles with energy $E \sim 40$ MeV by different nuclei, and indicate that the oscillations of the cross section for scattering by lead are already practically nonexistent.

More complicated is the influence of the Coulomb interaction on inelastic scattering. Mathematically this is connected with the fact that when account is taken of the Coulomb interaction, the contributions to the amplitude of the scattering (11) from the two poles at the points l_1 and l_1^* are not of equal sign, and the closer the scattering angle ϑ is located near the region of angles $\sim\vartheta_c$ the larger the difference. Formula (24) can be rewritten in the form

$$\sigma_I^{(n)}(\vartheta) = 1/4 B_n(I)\,|a_{-1}|^2 \theta_+^{2n}(\sin\vartheta)^{-1}$$
$$\times \exp\{-2\beta\theta_+ - 4\,\mathrm{Im}\,\delta(l_1)\}$$
$$\times\{(1-\chi_n)^2 + 4\chi_n\cos^2[(l_0 + 1/2)\vartheta + \gamma_{nI}]\}, \tag{27}$$

where

$$\chi_n = \frac{|\theta_-|^n}{\theta_+^n}\exp[2\beta\vartheta_c + 4\,\mathrm{Im}\,\delta(l_1)]. \tag{28}$$

When $\vartheta \gg \vartheta_c$, we have $|\theta_-|/\theta_+ \approx 1$ and formula (27) gives an oscillating picture with a constant oscillation amplitude. As $\vartheta \to \vartheta_c$ we get $\chi_n \to 0$ $(n \geq 1)$, that is, the Coulomb interaction may result in a sharp decrease or even total absence of oscillations when scattering is through angles $\sim\vartheta_c$. When α particles with energy $E \sim 40$ MeV are scattered by nuclei of medium atomic weight we have $\vartheta_c \sim 15-30°$. As is well known, a regular variation of the oscillations of the diffraction cross section is observed starting with somewhat larger scattering angles. Account

*sh ≡ sinh.

must be taken, however, of the fact that if the energy of the incoming particle is much higher than the Coulomb barrier, so that $\vartheta_c \ll 1$, then formulas (24) and (27) are not applicable in the region $\vartheta \sim \vartheta_c$, since our analysis includes the region of very small scattering angles.

We have already noted that, owing to the strong Coulomb interaction, the cross sections for elastic scattering of particles by heavy elements do not experience oscillations. However, the cross sections for inelastic scattering can in this case have a diffraction character. Indeed, in (26) the term $\frac{1}{2}n \ln(|\theta_-|/\theta_+)$ is negative for all scattering angles, and therefore $\sinh^2 z_n < \sinh^2 z_0$ (for $n \geq 1$). This phenomenon was observed in experiments [8] on the scattering of α particles by $Pb^{207,208}$ and Bi^{209}.

All the foregoing results are valid also when the phase $\delta(l)$ has poles, but δ_0 is close to zero.

Let us see now how the diffraction scattering picture changes if the functions $A(l)$ and $\delta(l)$ have closely-lying poles. As already noted, some deductions can be drawn without resorting to concretization of the form of these functions. By comparing (24) with (19) we see that the presence of poles of $\delta(l)$ does not violate the most important conclusion in [1,2] that the cross section decreases exponentially with decreasing angle θ. The diffraction character of the differential cross section is likewise retained, but its form may change appreciably. Indeed, if the phase $\delta(l)$ has poles, then the quantities $\gamma_{nI}(\vartheta)$ and z_n (formulas (21)—(22)) depend now on the scattering angle, that is, the period and amplitude of the oscillations of the differential cross section change with the scattering angle.

The dependence of the period and of the amplitude of the oscillations on the scattering angle was observed experimentally in several cases. Thus, for example, in experiments on elastic and inelastic scattering of 43-MeV α particles by Ni^{58} nuclei it was noted that the period increased and the oscillation amplitude decreased with increasing scattering angle [9]. In some cases the interesting phenomenon of "cross-section drop," was observed, consisting in the fact that one or two minima of the diffraction cross section were much deeper (by one order of magnitude or more) than the neighboring minima, for both larger and smaller scattering angles. This phenomenon was observed, for example, in experiments [4] on the scattering of 42-MeV α particles by Sr^{88} and Y^{89} nuclei, where the "drop angle" is $\vartheta_0 \approx 58-59°$.

It should be noted that the presence of poles of the phase $\delta(l)$ and the associated change in the period of the amplitude of the oscillations does not change the Blair phase rule [10], which can be formulated here as follows: the cross section for inelastic scattering with excitation of the n-phonon state with spin I is in phase with the cross section for elastic scattering if $n+I$ is even, and in counterphase if $n+I$ is odd. To be sure, owing to the presence of the factor $\vartheta^{2n}e^{-2\beta\vartheta}$ in the cross section, this correspondence may be somewhat violated.

As seen from (19), the cross sections for elastic and inelastic scattering are determined by the same parameters, if we disregard the matrix elements $C_n(I)$. The number of these parameters will not be very large even if the functions $A(l)$ and $\delta(l)$ are not concretely specified, since, as a rule, we can confine ourselves to a small number of terms in the expansion (8). One can expect, however, that the obtained results will not change appreciably even when the form of these functions changes in a sufficiently wide range. This allows us to limit the problem, by specifying a definite form of the functions $A(l)$ and $\delta(l)$. In this case the number of parameters will not depend on the number N. Many authors used the functions (2) and (4) in their numerical calculations of the cross sections. If it is assumed here that $\mu = \lambda$ and $l_\delta = l_0$, then the number of parameters entering in the theory will be equal to three: l_0, $\beta = \pi\lambda$, and δ_0. Let us consider the particular case of formula (19), taking the functions $A(l)$ and $\delta(l)$ and the form (2) and (4), and confining ourselves to quadratic terms in $\delta(l)$ in the expansion (8). Then

$$\sigma_I^{(n)}(\vartheta) = B_n(I)\lambda^2|\vartheta^2 - \vartheta_c^2|^n(\sin\vartheta)^{-1}e^{-2\beta\vartheta} \times[(1+2\lambda\delta_0\vartheta)^2 - 2\lambda^2\delta_0^2(\vartheta^2-\vartheta_c^2)] \times\{\cos^2[(l_0+1/2)\vartheta+\gamma_{nI}] + \mathrm{sh}^2 z_n\}, \tag{29}$$

where

$$\gamma_{nI} = 1/2(n+I)\pi + \pi/4 + \lambda\delta_0^2\vartheta, \tag{30}$$

$$z_n = \beta\vartheta_c - 2\lambda\delta_0\vartheta + \frac{n}{2}\ln\frac{|\theta_-|}{\theta_+} + 2\lambda^2\delta_0^2\vartheta\vartheta_c. \tag{31}$$

Formulas (29)—(31) enable us to trace the "competition" between the nuclear and Coulomb phases and to explain some characteristic features of the differential cross sections. It becomes possible here to leave out the terms quadratic in δ_0, since the effects of interest to us are due already to linear terms of the expansion (8). An exception is the effect of the change of the period of the oscillations, which apparently is an effect of higher order and will not be considered here. We confine ourselves also to examination of elastic scattering only, since we have already ascertained the changes that result from allowance for the term $\frac{1}{2} n \ln (|\theta_-|/\theta_+)$ in the formula.

We express z_0 in the form

$$z_0 = \beta\vartheta_c(1 - x\vartheta), \quad x = 2\delta_0/\pi\vartheta_c. \tag{32}$$

We see that z_0, and consequently also the amplitude of the oscillations of the scattering cross section, will depend on the angle ϑ. The character of this dependence is determined by the sign of δ_0 and by the relation between the quantity δ_0 and ϑ_c, that is, the relation between the jump in phase at $l \sim l_0$ and the magnitude of the Coulomb interaction.

Let $\delta_0 > 0$. If $x \ll 1$, that is, the Coulomb interaction is large, and the phase jump is small, then the amplitude of the oscillations of the cross section is practically independent of the scattering angle. In the case of weak Coulomb interaction and a large phase jump $(x \gg 1)$ the amplitude of the oscillations of the cross section decreases with increasing scattering angle.

Of greatest interest is the case when the jump in phase and in ϑ_c are comparable in magnitude $(x \sim 1)$. This case is apparently frequently encountered in experiments on the scattering of α particles by nuclei with medium atomic weights.

If $x \sim 1$, then for the angles $\vartheta = \vartheta_0$, where

$$\vartheta_0 = \pi\vartheta_c/2\delta_0, \tag{33}$$

z_0 vanishes. If for $\vartheta \approx \vartheta_0$ the diffraction cross section has a minimum, that is, the cosine in formula (29) vanishes, then this minimum will be much deeper than the neighboring minima; we observe the "cross section drop" phenomenon. In the already mentioned experiments [4], the minima of the elastic scattering of alpha particles by Sr^{88} and Y^{89} nuclei at angles 58—59° lie 10—20 times lower than the neighboring minima. An analogous phenomenon can be observed in many other cases. It is obvious that the collapse should not greatly change the magnitude of the maxima of the cross section, as is indeed observed in the experiments.

Measurement of the collapse angle makes it possible to estimate very simply the value of the constant δ_0. Let us estimate, for example, this quantity for the nuclei Sr^{88} and Y^{89} using the results of Alster et al. [4] on the scattering of 42-MeV α particles. We have $\vartheta_0 \sim 58$—59°. The radius of interaction of the α particles with the Sr^{88} nuclei is assumed equal to 7.93 F, as follows from an analysis given in [2]. Then $\vartheta_c \approx 0.33$ and we obtain from (33) $\delta_0 \approx 0.5$. A similar value will be obtained also for Y^{89}. Alster et al. [4] assumed in their calculations value 0.40 and 0.44 respectively for δ_0.

If $\delta_0 < 0$, then the amplitude of the oscillations of the cross section will decrease monotonically with increasing scattering angle. The drop in the cross section will, of course, not occur here. This circumstance enables us to draw conclusions concerning the sign of δ_0 from the general form of the cross section, since the relatively strong Coulomb interaction for nuclei with medium atomic weights makes it possible to exclude the case $x \gg 1$ when $\delta_0 > 0$.

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