MOTION OF ABRIKOSOV LINES IN AN ELECTRIC FIELD AND THE ENERGY DISSIPATION MECHANISM IN HOMOGENEOUS SUPERCONDUCTORS

I. O. KULIK

Physico-technical Institute of Low Temperatures, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor January 15, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 1617–1630 (June, 1966)

A theory is developed for "resistive" effects in superconductors of the second kind near the upper critical field H_{c2}. The analysis is based on an application of time-dependent Ginzburg-Landau equations; these equations are derived phenomenologically. If the normal (nonsuperconductive) current is neglected, the equations permit the existence of quasiequilibrium superconducting states in crossed electric and magnetic fields; these states correspond to the motion of Abrikosov flux lines in a direction perpendicular to both **E** and **H**. The upper critical field H_{c2} is a function of the electric field strength; however, for superconducting metals and alloys the effect of the variation of H_{c2} with E is very small. It is shown that the resistive effects in superconductors should be accompanied by the emission of radiofrequency energy (due to the motion of the periodic line structure). Irradiation of a superconductor in the mixed state with external rf power should change the shape of the volt–ampere characteristic when $H < H_{c2}$.

1. INTRODUCTION

THE present paper is an attempt to account for the so-called "resistive" effects due to the existence of a finite resistance) in superconductors. These effects are observed in superconducting alloys (superconductors of the second kind) that are in a mixed state when an applied magnetic field has a component perpendicular to the transmitted current ([1, 2] etc.). To account for the observed potential difference (or the electric field) it is usually considered that magnetic flux lines penetrating the superconductor are set into motion by the Lorentz force in a direction perpendicular to both the current and the magnetic field. The velocity of this motion depends on defects—inhomogeneities, internal strains, etc., in a given sample, which impede the motion of the lines. In the case of small currents (and therefore a small Lorentz force) the lines remain motionless (pinned). A completely superconducting current can then flow, having a critical value that increases with the concentration of defects. Supercritical currents are accompanied by resistance, but when $j \ll j_{cr}$ this resistance is independent of the defect concentration.[2]

The present work examines the causes of resistance in homogeneous superconductors of the second kind that contain no macroscopic defects obstructing the motion of lines; in this case resistance is always found.[3] 1) The mechanism of energy dissipation that is manifested by the resistance is associated with the motion of normal electrons in an electric field against the background of completely "superfluid" motion of Abrikosov lines. The resistance of the superconductor in the mixed state is then expressed in terms of the normal metallic resistance. We note that a similar energy-dissipation mechanism exists in connection with the Josephson effect[5] in superconductors at $V \neq 0$.[6] This effect is very similar physically to the investigated dissipative processes in bulk superconductors, and many qualitative results can be derived from an analogy with the Josephson effect, particularly the conclusion that dissipative phenomena in superconductors should be accompanied by radio-frequency emission (Sec. 3).

The method of solution used here is based on modified, time-dependent, Ginzburg-Landau equations. The equations are "derived" phenomenologically in Sec. 2, and are applied in Sec. 3 to the motion of Abrikosov lines in an electric field with-

1) The existence of resistance in the mixed state when $j \perp H$ can also be derived from the hydrodynamic equations given in[4]. In [4], however, these equations pertain to the case in which the characteristic dimension of the inhomogeneities is much greater than the distance between the flux lines. The present work investigates the situation near H_{c2} when these quantities are of the same order of magnitude.

out dissipation ($\sigma = 0$). Finally, in Sec. 4 the dissipative current in the mixed state and the resistance of a superconductor in fields near the critical field H_{c2} are calculated.

2. PHENOMENOLOGICAL DERIVATION OF TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

Anderson and Dayem[7] have suggested an "other" Ginzburg-Landau equation to describe nonsteady-state processes in superconductors:

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)-2e\Phi|\psi|^2\approx 0 \qquad (2.1)$$

with Φ denoting the scalar potential.[2)]

The left-hand term is proportional to the fourth component of the 4-current, i.e., to the charge. Anderson et al. showed in [8] that this equation is the direct consequence of the fact that, according to Gor'kov,[9] the wave function of a Cooper pair contains a phase factor $e^{-2i\mu t/\hbar}$, where μ is the chemical potential, which equals $\mu_0(T, N) + e\Phi$. In [10] and [11] equations were derived which govern the complex energy gap of superconductors for slow processes; however, these results apply only near absolute zero, and the gap equation is very complicated. It is also evident physically that at temperatures near T_c (or in other cases governed by the Ginzburg-Landau equation) slow nonsteady-state processes should be governed by an equation that becomes the ordinary Ginzburg-Landau equation in the steady state; an equation of this kind is derived phenomenologically in the present section.

We have derived a complete system of nonsteady-state Ginzburg-Landau equations by means of the variational principle for a quantity

$$G=\int d\mathbf{r}\,dt\,L(\mathbf{r},t).$$

The validity of this variational procedure and the form of $L(\mathbf{r}, t)$ that will be given are, of course, phenomenological postulates. However, these ideas possess physical likelihood, since a superconducting condensate bears some resemblance to a mechanical system whose behavior is governed by the principle of least action, with the temperature T appearing in $L(\mathbf{r}, t)$ only as a parameter.[3)] This approach generalizes the derivation by Ginzburg and Landau,[12] who also suggested that the equilibrium-state equation could be derived by minimizing the energy (the free energy at $T \neq 0$) of a superconductor in the presence of a magnetic field.

We now proceed to determine the form of $L(\mathbf{r}, t)$, which contains, first of all, the customary term for the theory of phase transformations:

$$-\alpha|\psi|^2+{}^1/_2\beta|\psi|^4. \qquad (2.2)$$

Since very rapid change of the order parameter ψ cannot occur, Ginzburg and Landau added the term

$$\frac{1}{2m}\left|\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathbf{A}\right)\psi\right|^2, \qquad (2.3)$$

to L;[13] the presence of the vector potential $\mathbf{A}$ in this formula follows from gauge invariance.

It is reasonable to assume that in the nonsteady-state case, since ψ cannot vary too rapidly in time, we must add to L a term similar to (2.3) containing the derivative of ψ with respect to time instead of with respect to the coordinates. This gauge invariant expression is

$$\frac{1}{\varepsilon_0}\left|\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\Phi\right)\psi\right|^2, \qquad (2.4)$$

where ϵ_0 is a phenomenological parameter having the dimensions of energy. It is also evident that ϵ_0 must be a characteristic of the normal metal [like m in (2.3)] and should therefore be of the order of the Fermi energy μ.

The terms (2.2) and (2.3) represent the "potential" energy of the given system. The added nonstationary term (2.4) corresponds to the "kinetic" energy; its sign is therefore probably opposite to that of (2.3), so that ϵ_0 should be a negative quantity.[4)] We shall see that this leads to hyperbolic equations for ψ in agreement with [10]. (See the analogous equations for the Josephson effect in [6].)

Finally, L must contain terms representing the energies of the external electric and magnetic fields; we can obviously add

$$(\mathbf{H}^2-\mathbf{E}^2)/8\pi. \qquad (2.5)$$

We note that the value of ϵ_0 in (2.4) can be calculated as follows. The thermodynamic potential of a metal in an external field is

$$\Omega=\Omega_0(\mu_0+e\Phi), \qquad (2.6)$$

where $\Omega_0(\mu) = E - \mu N$ and E is the energy. In a

2)More accurately, in accordance with [7], in this equation and all succeeding equations Cooper pairs should be represented by twice the charge and mass of an electron. For the sake of simplicity, however, we shall retain the notation e and m to denote twice the electronic charge and mass.

3)A similar approach based on the extremizing of a functional, has been used in[11] to derive an equation for slow processes from a microscopic theory. I am grateful to the authors of this paper for making their results available before publication.

4)Roughly speaking, L equals V – K, where K is kinetic energy and V is potential energy, and is therefore the "Lagrangian" with reversed sign.

weak field Φ we can use the expansion

$$\Omega = \Omega_0(\mu_0) + \frac{\partial\Omega_0}{\partial\mu_0} e\Phi + \frac{1}{2}\frac{\partial^2\Omega_0}{\partial\mu_0^2}(e\Phi)^2 + \dots \quad (2.7)$$

The linear term of this expansion must be dropped. We have $\partial\Omega_0/\partial\mu_0 = -N$, where N is the total concentration of (normal and superconducting) electrons. Since the metal is electrically neutral this must be the constant concentration N_i of positive charge in the lattice. In reality, (2.6) should include an additional term that represents the energy of ion cores in the external field Φ and that is canceled by $-Ne\Phi$. Calculation of the next, quadratic, term of the expansion, utilizing the expression $E = {}^3/_5\, N\mu$ for the energy of a free electron gas and v_0 to represent the Fermi velocity, we obtain

$$\Omega \approx \Omega_0(\mu_0) - \frac{3Ne^2}{2mv_0^2}\Phi^2.$$

Consequently the change of energy per particle is $3e^2\Phi^2/2mv^2$. (We note that the small corrections to all thermodynamic potentials, expressed in terms of the corresponding variables, coincide.) A comparison of this expression with (2.4) shows that $\epsilon_0 = -{}^2/_3\, mv_0^2$. We can also write this negative quantity as

$$\varepsilon_0 = -2mv^2, \quad (2.8)$$

where

$$v = v_0/\sqrt{3} \quad (2.9)$$

is the velocity of collective excitations in an ideal Fermi gas[14] (see also [8, 10]). In the purely phenomenological approach v must be regarded as a parameter equal to the Fermi velocity in order of magnitude.

By hypothesis the equations describing the state of a superconductor can be derived from $\delta G = 0$, where

$$G = \int d\mathbf{r}\,dt \Big\{ -\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m}\Big|\frac{\hbar}{i}\nabla\psi - \frac{e}{c}\mathbf{A}\psi\Big|^2 - \frac{1}{2mv^2}\Big|\frac{\hbar}{i}\frac{\partial\psi}{\partial t} + e\Phi\psi\Big|^2 + \frac{1}{8\pi}(\mathbf{H}^2 - \mathbf{E}^2)\Big\}. \quad (2.10)$$

Taking $\mathbf{A}$, Φ, ψ, and ψ^* as the independent dynamic variables, we obtain

$$\frac{1}{2m}\Big(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\Big)^2\psi - \frac{1}{2mv^2}\Big(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\Phi\Big)^2\psi - \alpha\psi + \beta|\psi|^2\psi = 0, \quad (2.11)$$

$$\Delta\Phi + \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{A} = -4\pi\rho, \quad (2.12)$$

$$\Delta\mathbf{A} - \nabla\operatorname{div}\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi = -\frac{4\pi}{c}\mathbf{j}, \quad (2.13)$$

where $\mathbf{j}$ and ρ are the superconductive current and charge, defined by

$$\mathbf{j} = \frac{ie\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{e^2}{mc}\mathbf{A}|\psi|^2, \quad (2.14)$$

$$\rho = \frac{1}{v^2}\Big[\frac{ie\hbar}{2m}\Big(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\Big) - \frac{e^2}{m}\Phi|\psi|^2\Big]. \quad (2.15)$$

With the Lorentz gauge of the potentials (2.12) and (2.13) become $\Box\Phi = -4\pi\rho$ and $\Box\mathbf{A} = -4\pi c^{-1}\mathbf{j}$, where $\Box$ is the d'Alembertian operator. However, we shall also make use of another gauge (Sec. 4).

With regard to the foregoing ''derivation'' it must, of course, be mentioned that Eqs. (2.11), (2.14), and (2.15) are actually postulates, which in the steady-state case follow from the microscopic theory.[15] We can hope to derive these equations also from the microscopic theory of superconductivity. The properties of (2.11)–(2.15) that prove their internal consistency are, first, their obvious gauge invariance and, second, the fact that the equation of continuity $\operatorname{div}\mathbf{j} + \partial\rho/\partial t = 0$ follows automatically from (2.11) for ψ. We note that the foregoing generalization of the Ginzburg-Landau equations to apply to the nonsteady-state case is accomplished by introducing only one additional parameter (v). With this limitation the form of the nonsteady-state equations satisfying the aforesaid properties is essentially unique. However, the phenomenological theory cannot determine the sign of the second term in (2.11). It has been shown in [10] that at T = 0 the equation for Δ is hyperbolic and can assume the same for nonzero T.

3. MOTION OF ABRIKOSOV LINES IN AN ELECTRIC FIELD WITHOUT ENERGY DISSIPATION

We shall here use (2.11)–(2.15) to describe the properties of the Abrikosov mixed state in an external electric field $\mathbf{E} \perp \mathbf{H}$. In the simplest case the magnetic field H is close to the upper critical value H_{c2}. At the same time $\psi \to 0$; consequently, we can substitute $\mathbf{j} = 0$ and $\rho = 0$ in the zeroth approximation in (2.12) and (2.13). We then chose the potentials $\mathbf{A}$ and Φ in the forms

$$\mathbf{A}_0 = (0, Hx, 0), \quad \Phi_0 = -Ex \quad (3.1)$$

where H is the magnetic field in the z direction and E is the electric field in the x direction.

Because of the assumption $\psi \to 0$ in this case, the nonlinear term $\beta|\psi|^2\psi$ in (2.11) can also be dropped, permitting the solution

$$\psi = \text{const}\cdot e^{iky+i\omega t}\varphi(x), \qquad (3.2)$$

for this equation; k and ω are constants and $\varphi(x)$ satisfies the equation

$$-\frac{d^2\varphi}{dx^2} + \left(k - \frac{eH}{\hbar c}x\right)^2\varphi - \left(\frac{\omega}{v} - \frac{eE}{\hbar v}x\right)^2\varphi = \frac{2ma}{\hbar^2}\varphi. \quad (3.3)$$

This becomes the equation of a linear harmonic oscillator with the solution

$$\varphi(x) = \varphi_n\left[\left(x - \frac{k\alpha_0^{-2} - \beta_0^{-2}\omega/v}{\alpha_0^{-4} - \beta_0^{-4}}\right)(\alpha_0^{-4} - \beta_0^{-4})^{1/4}\right],$$

$$n = 0, 1, 2, \ldots \qquad (3.4)$$

in which $\varphi_n(z)$ are Hermite functions and

$$a = \frac{\hbar^2}{2m}\left\{(2n+1)(\alpha_0^{-4} - \beta_0^{-4})^{1/2} + \frac{(k\beta_0^{-2} - \alpha_0^{-2}\omega/v)^2}{\alpha_0^{-4} - \beta_0^{-4}}\right\}. \qquad (3.5)$$

We have here introduced the notation

$$\alpha_0^2 = \hbar c / eH, \quad \beta_0^2 = \hbar v / eE. \qquad (3.6)$$

The quantity α_0 is the smallest quantum-mechanical radius of an electron trajectory in the magnetic field;[16] β_0 is an analogous quantum length for the electric field when crossed fields are present.

The highest magnetic field permitting superconductivity corresponds to n = 0 in (3.5).[17] Since the electric and magnetic field are taken as fixed, Eq. (3.5) provides a relationship between k and ω:

$$\omega = kv(\alpha_0/\beta_0)^2, \qquad (3.7)$$

which in conjunction with (3.6) yields

$$\omega = ckE/H. \qquad (3.8)$$

It is important to note that v drops out of this relationship, which thus holds true independently of the model.

The solution (3.4) corresponds to decrease of the order parameter with increasing distance from the point $x = x_0 = k\alpha_0^2$ and vanishing for $x \to \pm\infty$. Since this point is not actually distinguished in any way, we must, as in the case E = 0,[17] superpose solutions such as (3.2):

$$\psi = \text{const}\cdot\sum_{n=-\infty}^{\infty} C_n e^{in(ky+\omega t)}\varphi_0\left(\frac{x - nx_0}{(\alpha_0^{-4} - \beta_0^{-4})^{-1/4}}\right). \quad (3.9)$$

Furthermore, $|\psi|$ must be a periodic function of x. It follows[17] that $C_{n+N} = C_n$, where N is a number. With N = 1 we obtain an Abrikosov line structure possessing the symmetry of a square lattice;[17] with N = 2 the symmetry is triangular.[18] In both cases we obtain the same qualitative picture of all effects that interest us. However, N = 1 is simpler mathematically and will be the case considered in the present work. For ψ we then have the form[17]

$$\psi = C\sum_{n=-\infty} e^{ink(y+v_0 t)}\varphi_0\left(\frac{x - nx_0}{\gamma_0}\right), \qquad (3.10)$$

where

$$v_0 = cE/H, \quad \gamma_0 = (\alpha_0^{-4} - \beta_0^{-4})^{-1/4}, \quad \varphi_0(z) = \pi^{-1/4}e^{-z^2/2}. \qquad (3.11)$$

Here v_0 is the electron drift velocity in the crossed electric and magnetic fields. We shall now consider some consequences of the derived formulas.

1. It follows, first of all, that in the presence of an electric field superconductivity disappears as H increases while differing from the upper critical field for E = 0. In other words, H_{c2} is a function of E; $H_{c2}(E)$ is determined from (3.5) for n = 0 [the second term in the curly brackets vanishes by virtue of (3.7)]. We obtain

$$H_{c2}^2 = (H_{c2}^0)^2 + (cE/v)^2. \qquad (3.12)$$

The presence of an electric field perpendicular to the magnetic field thus elevates H_{c2}, the upper critical field for superconductivity.[5)] However, in practically all hitherto investigated cases the second term on the right-hand side of (3.12), which represents the increase of H_{c2} as a function of E, is negligibly small when compared with the first term. For example, in the experiments of [1] we have $H_{c2} = 3.5 \times 10^3$ Oe and $E = 1.3 \times 10^{-3}$ V/cm, yielding $cE/vH \sim 0.5 \times 10^{-6} \ll 1$.

Elevation of the upper critical field is thus found to be practically nonexistent in the described experiments, but we can expect an observable dependence of H_{c2} on E in some cases. A possibility of this kind is presented by superconducting semiconductors,[19] where v can be much smaller than the Fermi velocity in metals. At the same time, because of the small concentration of carriers, i.e., the poor conductivity, considerably higher field strengths E can be obtained. In addition, with E = 0 we also find small values of H_{c2} for these materials, thus increasing the relative contribution of the second term in (3.12).

The value of the constant C in (3.10) should be obtainable by taking into account the nonlinear terms in (2.11). However, for $cE \ll Hv$, i.e.,

5)We note that H_{c2} will be a decreasing function of the electric field for elliptic equations. The other conclusions do not depend on the type of equation.

$\beta_0 \gg \alpha_0$ (which case will be considered hereafter) C should coincide with the value obtained by Abrikosov in the absence of an electric field.[17] Using $|\overline{\psi}|^2 = C^2/\sqrt{2\pi}$, where the bars denote averages, we obtain, on the basis of [17],

$$C^2 = \frac{mc}{e\hbar}\,\frac{H_{c2}-H}{\beta\sqrt{2\pi}(2\varkappa^2-1)},$$

$$\beta = \Big(\sum_{n=-\infty}^{\infty} e^{-\pi n^2}\Big)^2 = 1.18, \quad \varkappa > \frac{1}{\sqrt{2}}, \qquad (3.13)$$

where the Ginzburg-Landau parameter κ equals the ratio of H_{c2} to $\sqrt{2}\,H_{cm}$ (H_{cm} is the thermodynamic critical field).

2. The most important difference between the presently considered case and that of $E = 0$ is the time dependence of the order parameter ψ. Equation (3.10) describes the motion of the Abrikosov line structure in the direction perpendicular to **E** and **H** with the velocity $v_0 = CE/H$; this agrees formally with the electron drift velocity in crossed fields. However, this is only a "wave" motion that does not result in charge transport, as can easily be proved by calculating the mean density (in space and time) of the current **j**. Substituting (3.10) into (2.14), we easily obtain $\overline{\mathbf{j}} = 0$.

Equation (3.10) shows that at every point in space ψ is an oscillating function of time with the frequency given by (3.8):

$$\nu = \frac{c}{x_0}\,\frac{E}{H}, \qquad (3.14)$$

where x_0 is the period of the Abrikosov structure. When $E \to 0$ the value of x_0 is derived subject to the condition that the period $k\alpha_0^2$ of $|\psi|$ in the x direction coincides with the period in the y direction, which is $2\pi/k$ according to (3.10). Then $k = k_0 = \sqrt{2\pi}/\alpha_0$, or

$$x_0 = (hc/eH_{c2})^{1/2}. \qquad (3.15)$$

The frequency is easily computed by means of (3.14) and (3.15). Using the aforementioned experimental data,[1] we obtain $x_0 \sim 1.1 \times 10^{-5}$ cm and $\nu \sim 3 \times 10^6$ cps. A very wide range of ν is associated with variation of E.

The oscillations of ψ (which are oscillations in the density of the "superconductive electrons" and superconductive current at each point in space) resemble the oscillations of the Josephson tunneling current [5] in the presence of a potential difference between metals. In the absence of energy dissipation these are only "virtual" oscillations in the sense that they do not lead to the emission of real photons. However, when the oscillating current interacts with a field of electromagnetic oscillations so that energy is dissipated, electromagnetic energy can be emitted with the frequency given by (3.14) or (3.8). In this case we obtain the Josephson frequency relation[5]

$$2eV = \hbar\omega \qquad (3.16)$$

where V is the potential difference between the metals.[6)]

4. ENERGY DISSIPATION IN A SUPERCONDUCTOR FOR H ⊥ E NEAR THE UPPER CRITICAL FIELD

We assume in this section that dissipative processes in a superconductor can be accounted for by adding to the right-hand side of (2.13) the "dissipative" current

$$\mathbf{j}_{\rm diss} = \sigma\mathbf{E} = \sigma\Big(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\Phi\Big),$$

where $\sigma = N_n e^2\tau/m$ is the effective conductivity.[7)] This method of considering relaxation effects in type II superconductors can be justified by the fact that there are few superconductive electrons near the upper critical field and their influence on normal conductivity can be neglected. (Specifically, σ can be regarded as coinciding with the conductivity σ_N of a normal metal.) Under these conditions the superconducting and normal systems exist independently of each other. The influence of the first system on the second can be neglected because for $H \to H_{c2}$ we have $\psi \to 0$. The influence of the normal system on the superconducting system will also be weak in the case of small damping. It will be shown here that this situation is realized when the product $\sigma\omega$ is much smaller than a certain quantity:

$$\sigma\omega \ll c^2 k_0^2/4\pi. \qquad (4.1)$$

This inequality is ordinarily satisfied in a very wide range of field strengths far from H_{c2} [see [1] and Eq. (4.22)], but the condition $N_s \ll N_n$ holds

6)From a rigorous point of view, our analysis thus far, which pertains to an unbounded metal, shows only that energy loss occurs through rf radiation accompanying the motion of flux lines. This radiation is absorbed within the metal, whereas in reality the radiation would be observed only outside of the metal. To evaluate the radiated power we require an analysis that allows for the existence of boundaries; this would go beyond the present paper.

7)Moreover, Eq. (2.12) should also include an analogous additional term $\rho_{\rm diss} = \rho'$ derived from the condition $\partial\rho'/\partial t + {\rm div}\,\mathbf{j}_{\rm diss} = 0$. This term, which represents an addition to normal electron density, is ordinarily very small in metals, since in all sufficiently slow variable processes practically complete charge compensation exists.

true only near H_{c2}. Rigorous inclusion of relaxation effects for arbitrary H is extremely complicated and requires a separate discussion.

On the basis of the foregoing hypotheses the calculation of the dissipative processes is reduced to the following. The oscillating current and charge, expressed in terms of $\psi = \psi_0$ through (2.14) and (2.15) (in which we can assume $\mathbf{A} = \mathbf{A}_0$ and $\Phi = \Phi_0$) produce oscillating additions to the vector and scalar potentials $\mathbf{A}$ and Φ. We denote these additions by $\mathbf{A}_1$ and Φ_2. Choosing the gauge of the potentials such that $\Phi_1 = 0$, we obtain from (2.3) the following equation for $\mathbf{A}_1$ (the term $-c^{-2}\partial^2\mathbf{A}_1/\partial t^2$ being negligible):

$$\Delta\mathbf{A}_1 - \nabla\,\mathrm{div}\,\mathbf{A}_1 - \frac{4\pi\sigma}{c^2}\frac{\partial\mathbf{A}_1}{\partial t} = -\frac{4\pi}{c}\mathbf{j}_0, \tag{4.2}$$

where j is defined by

$$\mathbf{j}_0 = \frac{ie\hbar}{2m}(\psi_0\nabla\psi_0^* - \psi_0^*\nabla\psi_0) - \frac{e^2}{mc}\mathbf{A}_0|\psi_0|^2,$$

and where, in accordance with (3.10) and $cE \ll Hv$,

$$\psi_0 = C\sum_{n=-\infty}^{\infty} e^{ink_0(y+v_0t)}\varphi_0\left(\frac{x-nx_0}{a_0}\right). \tag{4.3}$$

Substituting into (2.11) the expression for $\mathbf{A}_1$ derived from (4.2), we obtain an addition to ψ. According to (2.11) the equation for ψ_1 is (with $cE \ll Hv$)

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}_0\right)^2 - \alpha\right]\psi_1 = \frac{e}{mc}\left[\mathbf{A}_1\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}_0\right) + \frac{\hbar}{2i}\mathrm{div}\,\mathbf{A}_1\right]\psi_0. \tag{4.4}$$

Finally, ψ_1 is used to calculate the correction to $\mathbf{j}_1$, which has a nonvanishing average ($\bar{\mathbf{j}}_1 \neq 0$):

$$\bar{\mathbf{j}}_1 = \frac{2e\hbar}{m}\mathrm{Re}\,\overline{\psi_1^*\left(\frac{\nabla}{i} - \frac{e}{\hbar c}\mathbf{A}_0\right)\psi_0} + \frac{e^2}{mc}\overline{\mathbf{A}_1|\psi_0|^2} \tag{4.5}$$

representing a current that in the presence of a potential difference leads to energy emission in the given sample. In the first step of the calculation we obtain the addition to the vector potential $\mathbf{A}_1$ from (4.2). Using (4.3), we easily obtain[8)]

$$j_{0x} = -j_0\sum_{p,q=-\infty}^{\infty}\exp\{ipk_0x + iqk_0(y+v_0t)\}\,iqC(p,q),$$

$$j_{0y} = j_0\sum_{p,q=-\infty}^{\infty}\exp\{ipk_0x + iqk_0(y+v_0t)\}\,ipC(p,q),$$

$$j_{0z} = 0, \quad k_0 = \frac{2\pi}{x_0}, \quad j_0 = \frac{e\hbar k_0}{m}C^2, \tag{4.6}$$

where

$$C(p,q) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\pi ipz}\varphi_0(z\sqrt{2\pi})\varphi_0((z-q)\sqrt{2\pi})\,dz. \tag{4.7}$$

Elementary integration yields

$$C(p,q) = \frac{1}{2\sqrt{2\pi}}e^{-\pi(p^2+q^2)/2}e^{-\pi ipq}. \tag{4.8}$$

The solution of (4.2) is

$$A_{1x} = -\frac{4\pi j_0}{ck_0^2}\sum_{p,q=-\infty}^{\infty}\exp\{ipk_0x + iqk_0(y+v_0t)\}\frac{iqC(p,q)}{p^2+q^2+iq\varepsilon},$$

$$A_{1y} = \frac{4\pi j_0}{ck_0^2}\sum_{p,q=-\infty}^{\infty}\exp\{ipk_0x + iqk_0(y+v_0t)\}\frac{ipC(p,q)}{p^2+q^2+iq\varepsilon},$$

$$A_{1z} = 0 \tag{4.9}$$

(showing that $\mathrm{div}\,\mathbf{A}_1 = 0$), where we have introduced the notation

$$\varepsilon = 4\pi\sigma v_0 / c^2 k_0. \tag{4.10}$$

The Fourier component of $\mathbf{A}_1$ with $p = q = 0$ cannot be derived from (4.2), but we can assume that this component vanishes because it clearly is not essential for determining $\mathbf{E}_1$ and $\mathbf{H}_1$ and cannot alter the value of $\mathbf{j}_1$.

Equations (4.9) enable us to obtain the alternating parts of the electric and magnetic fields associated with the motion of flux lines in superconductors:

$$E_{1x} = -\frac{4\pi j_0 v_0}{c^2k_0}\sum_{p,q=-\infty}^{\infty}{}'\exp\{ipk_0x + iqk_0(y+v_0t)\}\times\frac{q^2C(p,q)}{p^2+q^2+iq\varepsilon}, \tag{4.11}$$

$$E_{1y} = \frac{4\pi j_0 v_0}{c^2k_0}\sum_{p,q=-\infty}^{\infty}{}'\exp\{ipk_0x + iqk_0(y+v_0t)\}\times\frac{pqC(p,q)}{p^2+q^2+iq\varepsilon}, \tag{4.12}$$

$$H_{1z} = -\frac{4\pi j_0}{ck_0}\sum_{p,q=-\infty}^{\infty}{}'\exp\{ipk_0x + iqk_0(y+v_0t)\}\times\frac{(p^2+q^2)C(p,q)}{p^2+q^2+iq\varepsilon}, \tag{4.13}$$

$$H_{1x} = H_{1y} = E_{1z} = 0$$

where the primed summations denote, as henceforth, that the terms with $p = q = 0$ have been dropped.

We calculate the correction to ψ by substituting the derived expression for $\mathbf{A}_1$ into (4.4), which can be solved as a series:

[8)]It follows from the given equations that $\mathrm{div}\,\mathbf{j}_0 = 0$ as an approximation when $cE \ll Hv$.

$$\psi_1 = \sum_{n=-\infty}^{\infty} e^{ink_0(y+v_0t)} \sum_{m=0}^{\infty} a_m \varphi_m\left(\frac{x - nx_0}{\alpha_0}\right), \quad (4.14)$$

where $\varphi_m(z) = (2^m m! \sqrt{\pi})^{-1/2} H_m(z) e^{-z^2/2}$ are normalized Hermite functions and $H_m(z)$ are Hermite polynomials. It will become evident that a contribution to the mean current comes only from the coefficient a_1 in this expansion. Substituting (4.14) into (4.4), we obtain

$$a_1 = \frac{4e\alpha_0^3 \sqrt{\pi} j_0}{\hbar c^2} C \sum_{p,\,q=-\infty}^{\infty}{}' \frac{(q - ip) C(p, q)}{p^2 + q^2 + iq\varepsilon} D(p, q), \quad (4.15)$$

where

$$D(p, q) = \frac{1}{2} \int_{-\infty}^{\infty} e^{2\pi i p z} \varphi_1(z\sqrt{2\pi}) \varphi_1((z + q)\sqrt{2\pi})\, dz. \quad (4.16)$$

Comparing (4.16) with (4.7), we obtain (with the substitutions p for $-p_z$ and q for $-q$)

$$D(p, q) = -\frac{1}{\pi i} \frac{\partial^2 C(p, q)}{\partial p\, \partial q} = [1 - \pi(p^2 + q^2)] C(p, q). \quad (4.17)$$

The substitution of (4.9) and (4.14) into (4.5) yields

$$\bar{j}_{1x} = \frac{C}{\pi\sqrt{2}} \frac{e\hbar k_0}{m} \operatorname{Im} a_1 + \frac{8\pi e^2 j_0}{mc^2 k_0^2} C^2 \sum_{p,\,q=-\infty}^{\infty}{}' \frac{iqC^2(p, q)}{p^2 + q^2 + iq\varepsilon}, \quad (4.18)$$

$$\bar{j}_{1y} = -\frac{C}{\pi\sqrt{2}} \frac{e\hbar k_0}{m} \operatorname{Re} a_1 - \frac{8\pi e^2 j_0}{mc^2 k_0^2} C^2 \sum_{p,\,q=-\infty}^{\infty}{}' \frac{ipC^2(p, q)}{p^2 + q^2 + iq\varepsilon}. \quad (4.19)$$

Finally, the expression for a_1 in (4.15) must be substituted here. It is easily verified that the mean value of j_{1y} in the given approximation vanishes, while $\bar{j}_{1x}$ is given by

$$\bar{j}_{1x} = \frac{8\pi e x_0 j_0^2}{\hbar c^2 k_0^2} \varepsilon \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} \frac{(p^2 + q^2) q^2}{(p^2 + q^2)^2 + q^2 \varepsilon^2} C^2(p, q), \quad (4.20)$$

and j_0 is obtained from (4.6):

$$j_0 = C^2 e\hbar k_0 / m, \quad (4.21)$$

where C is given by (3.13) when $H \to H_{c2}$.

We note that the mean current $\bar{j}_{1x}$ represented by (4.20) vanishes in the two limiting cases $\epsilon \to 0$ (the absence of dissipation) and $\epsilon \to \infty$ (infinitely large dissipation). The Josephson current exhibits similar behavior as a function of the single-particle resistance.[20]

The parameter ϵ is found to be very small in actual experiments.[1,2] Thus for the already cited experiment in [1] we have $\sigma \sim 2 \times 10^{17}\ \text{sec}^{-1}$ and $v_0 \sim 40$ cm/sec, yielding $\epsilon \sim 2 \times 10^{-7}$. The condition $\epsilon \ll 1$ is equivalent to the inequality (4.1) postulated at the beginning of the present section (with $\omega = k_0 v_0$). This condition can be rewritten as

$$k_0^2 \delta^2 \gg 1, \quad (4.22)$$

where $\delta = (c^2/4\pi\sigma\omega)^{1/2}$ is the skin depth for a normal metal; we thus have the condition for a weak skin effect.

Assuming $\epsilon \ll 1$, we obtain from (4.20)

$$\bar{j}_{1x} = \sigma_1 E, \quad (4.23)$$

where

$$\sigma_1 = (\gamma x_0 j_0 / \lambda_0^2 k_0^2 c H_{c2}) \sigma, \quad (4.24)$$

$$\gamma = \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} \frac{q^2}{p^2 + q^2} e^{-\pi(p^2+q^2)} = 0.046, \quad (4.25)$$

and λ_0 is obtained from

$$\lambda_0^2 = mc^2 / 4\pi e^2 C^2. \quad (4.26)$$

The quantities j_0 and λ_0 have the qualitative meanings of the "critical current" and the "London penetration depth." Far from H_{c2} they actually are of the same order of magnitude as the critical current density in a superconductor and the London penetration depth, although for $H \to H_{c2}$ we have $j_0 \ll j_{cr}$ and $\lambda_0 \gg L_L$.

Our result shows that $\sigma_1 \ll \sigma_N$ near H_{c2}, i.e., the resistance of a superconducting metal then differs insignificantly from its normal resistance. On the other hand, far from the upper critical field σ_1 can become of the same order or substantially larger than σ_N, so that the resistance of the metal will be substantially smaller than R_N. This agrees qualitatively with the experimental dependence of R/R_N on H that was observed in [1,2] and elsewhere. The theory also leads to a linear relationship, agreeing with experiment, between the potential difference and the transmitted current [see Eq. (4.23)]. A more detailed calculation of the function R(H) for magnetic field strengths not limited to the vicinity of H_{c2} and a comparison with experiment will be published later.

In conclusion, we shall discuss the rf power radiated when the flux lines move in an electric field (see Sec. 3). Unfortunately, it is difficult to calculate the intensity of this radiation because the conditions for its observation are not fully understood. As a general rule, the power of interest here is determined from $\sigma_{tot} - \sigma_N = \sigma_1$, i.e., $w \sim w_0(R_N - R)/R_N$, where $w_0 = V^2/R_N$. However, this determines only the energy absorbed inside the sample. For radiation into exterior space the energy is obtained from (3.11)–(3.13), in which we can assume $\epsilon = 0$, for the time-dependent portions of the fields $\mathbf{E}_1$ and $\mathbf{H}_1$.[9)] The energy also depends

[9)]From (4.11)–(4.13) we easily derive $E_1/H_1 \approx E/H \ll 1$, i.e., the variable fraction of the electric field is considerably smaller than the variable fraction of the magnetic field.

on the parameters of the device (resonator) with which it is registered. Moreover, boundary effects are important in this calculation (see footnote [6]).

We note, finally, that in principle there are additional effects inverse to rf emission when a dissipative current flows through a semiconductor. Indeed, the irradiation of a semiconductor with rf power should alter the form of its current-voltage characteristic; singularities of the function I(E) are associated with values of E that are related to the irradiating frequency ν by (3.14). This effect is entirely analogous to the Shapiro effect for the Josephson tunneling current.[21]

The observation of the aforesaid effects—the emission of electromagnetic energy and the reaction of the irradiation on the dc resistance—is rendered difficult by the additional circumstance that under real conditions the structure of the mixed state is not a completely regular system of Abrikosov lines. The irregularities result in ''smearing'' of the emitted frequency bands and in a diminished influence of irradiation on resistance.

In conclusion I wish to thank Corresponding Member of the U.S.S.R. Academy of Science A. A. Abrikosov, I. E. Dzyaloshinskiĭ, and L. P. Pitaevskiĭ for discussions of this work and for valuable comments that assisted me in writing this paper. I also wish to acknowledge my indebtedness to L. P. Pitaevskiĭ and M. P. Kemoklidze for their close interest in this work.

[1] C. F. Hempstead and Y. B. Kim, Phys. Rev. Letters **12**, 145 (1964).

[2] A. R. Strnad, C. F. Hempstead, and Y. B. Kim, Phys. Rev. Letters **13**, 794 (1964).

[3] R. A. Kamper, Phys. Letters **5**, 9 (1963).

[4] A. A. Abrikosov, M. P. Kemoklidze, and I. M. Khalatnikov, JETP **48**, 765 (1965), Soviet Phys. JETP **21**, 506 (1965).

[5] M. D. Fiske, Rev. Mod. Phys. **36**, 221 (1964).

[6] I. O. Kulik, JETP Pis'ma **2**, 134 (1965), JETP Letters **2**, 84 (1965); R. E. Eck, D. J. Scalapino, and B. N. Taylor, Preprint, 1965.

[7] P. W. Anderson and A. H. Dayem, Phys. Rev. Letters **13**, 195 (1964).

[8] P. W. Anderson, N. R. Werthamer, and J. M. Luttinger, Phys. Rev. **138**, A1157 (1965).

[9] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962 (Engl. transl., Prentice-Hall, 1963).

[10] M. Stephen and H. Suhl, Phys. Rev. Letters **13**, 797 (1964).

[11] M. P. Kemoklidze and L. P. Pitaevskiĭ, JETP **50**, 243 (1966), Soviet Phys. JETP **23**, 160 (1966).

[12] V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950).

[13] E. A. Lynton, Superconductivity, Methuen, London, 1962 (Russian transl., Mir, 1964).

[14] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, Novyĭ metod v teorii sverkhprovodimosti (A New Method in the Theory of Superconductivity), AN SSSR, 1958; V. M. Galitskiĭ, JETP **34**, 1011 (1958), Soviet Phys. JETP **7**, 698 (1958); P. W. Anderson, Phys. Rev. **112**, 1900 (1959).

[15] L. P. Gor'kov, JETP **36**, 1918 (1959), Soviet Phys. JETP **9**, 1364 (1959).

[16] G. E. Zil'berman, JETP **32**, 296 (1957), Soviet Phys. JETP **5**, 208 (1957).

[17] A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957).

[18] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. **133**, A1226 (1964).

[19] M. L. Cohen, Revs. Modern Phys. **36**, 240 (1964); Phys. Rev. **134**, A511 (1964).

[20] I. O. Kulik, ZhTF (in press).

[21] S. Shapiro, Phys. Rev. Letters **11**, 80 (1963).

Translated by I. Emin
194