IONIZATION OF ATOMS IN AN ALTERNATING ELECTRIC FIELD

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A method is developed for calculating the probability of ionization of a bound state under the action of an alternating electric field. The method can be applied under the conditions $F \ll F_0$, $\omega \ll \omega_0$ (ω and F are the frequency and amplitude of the external field; $\omega_0 = \kappa^2/2$ and $F_0 = \kappa^3$ are appropriate atomic quantities). It uses the quasiclassical character of the motion of a particle in a homogeneous electric field, and is a generalization of the usual quasiclassical approximation to the nonstationary case. Preliminary attention is given to the adiabatic approximation in the problem of ionization (the range of frequencies $\omega \ll \omega_t$ where $\omega_t = F/\kappa$ is the tunneling frequency), and Eqs. (7) and (9) are derived for the probability of ionization of an arbitrary atom by the fields of waves with plane and elliptical polarizations. A detailed study is made of a model problem of the ionization of a bound level in a homogeneous δ-potential for arbitrary values of the "adiabaticity parameter" $\gamma = \omega/\omega_t$. In Sec. 4 formulas are derived for the probability of ionization of an arbitrary bound state under the action of an alternating electric field. The cases treated are those of plane [Eq. (54)] and of circular [Eq. (72)] polarization of the electromagnetic wave. [The Coulomb interaction between the electron and the atomic residue at large distances ($\kappa r \gg 1$) is neglected.] The total probability of ionization is a sum over many-photon processes. In the low-frequency limit ($\omega \ll \omega_t$) the formulas derived here go over into those of the adiabatic approximation.

1. INTRODUCTION

In recent years, owing to the development of laser techniques, light beams with field intensities $F \sim 10^6$–10^7 V/cm have become attainable. When light beams of such large intensities pass through matter, phenomena of excitation and ionization of atoms are observed (cf. e.g., the review articles [1,2]).

There are a number of papers[3-5] on the theoretical treatment of these effects, the most complete being the work of Keldysh.[4] The formulas derived for the probability of ionization of an atom in the radiation field correctly represent the main features of the effect: the exponential dependence of the probability of ionization on the amplitude of the field, and threshold peculiarities at frequencies corresponding to thresholds for absorption of n quanta. The coefficient before the exponential in [4] is incorrect, however, in particular because at low frequencies ($\omega \to 0$) there is no connection with the well known formula[6] for the ionization of a hydrogen atom in a constant field.

As can be seen from the solution of the corresponding problem for a constant field (cf. [6], page 327), to get the correct coefficient before the exponential it is necessary to know the exact quasiclassical wave function of the electron, taking into account both the effect of the external field and the interaction of the electron with the atomic residue. Therefore we need to generalize the usual quasiclassical method to the case of alternating fields. It is clear, however, that in this way we cannot get such general and convenient formulas as in the stationary case, since the solution of the classical Hamilton-Jacobi equation for the action S cannot be expressed in terms of quadratures in the case of an arbitrary alternating field. It is possible to find the quasiclassical solution of the problem of ionization of an atom by an alternating field only owing to the following simplifying circumstances: 1) we are confining ourselves to the case of not too high frequencies ω and field strengths F ($\omega \ll \omega_0$, $F \ll F_0$, where ω_0 and F_0 are quantities characteristic of the atom, see the next section), for which the ionization occurs slowly in comparison with atomic times; 2) the wavelength of the light is much larger than the radius of the atom, so that the electric field can be treated as uniform; 3) since the speed of the electron is nonrelativistic, the effect of the magnetic field of the wave can be neglected.

In Sec. 2 of the present paper we consider the region of small frequencies [$\omega \ll \omega_t$, where ω_t is the so-called tunneling frequency, see (2)], in which the adiabatic approximation is correct.

In Sec. 3 we study a model of extreme simplicity (ionization of a bound level in a one-dimensional δ-potential), which admits of asymptotically exact solution (for $F \to 0$) for arbitrary frequencies of the external field. In spite of the extreme simplification of this model, it reflects the main features of the effect [in particular, the formula for the ionization probability $wF(\omega)$ contains the same exponential as in the three-dimensional problem]. Moreover, the method developed in Sec. 3 for the one-dimensional case can also be directly transferred to the more complicated three-dimensional problem.

In Sec. 4 this method is used to derive asymptotically correct (in the limit $F \ll F_0$) formulas (54) and (72) for the probability of ionization of an atom in which the electron undergoing ionization is in a state with binding energy $\kappa^2/2$ and orbital angular momentum l [Eq. (54) is for the case of ionization by a plane-polarized wave, and Eq. (72) for the case of circular polarization]. The case considered is that in which the interaction between the electron and the rest of the atom falls off at large distances ($\kappa r \gg 1$) more rapidly than $1/r$ (this occurs, for example, for singly charged negative ions of the type of He^-, or for the process of dissociation of the hydrogen molecular ion, $H_2^+ \to H + p$). To get the correct factor before the exponential in the case of ionization of neutral atoms it is necessary to take into account the Coulomb interaction, which decidedly distorts the electron wave function at large distances from the nucleus. We shall treat this case in a subsequent paper.

2. IONIZATION BY A CONSTANT FIELD AND THE ADIABATIC APPROXIMATION

Let us consider an atom which is in the field of a plane electromagnetic wave of intensity much smaller than the intraatomic electric field strength F_0. Under these conditions the mean time of ionization is large in comparison with characteristic atomic times, and the probability of ionization is determined by the behavior of the distant "tail" of the wave function of the valence electron. Let us denote[1)] the binding energy of the electron by $\omega_0 = \kappa^2/2$ and its orbital angular momentum by l. Far from the nucleus ($\kappa r \gg 1$) the field acting on the electron reduces to a Coulomb field $V_C = -Z/r$ (Z is the charge of the atomic residue) and the electric field of the wave, $F(t) = F \cos \omega t$. We introduce distances characteristic for the problem:

$$r_1 = \frac{1}{\kappa}\left(\frac{F_0}{F}\right)^{1/2}, \quad r_2 = \frac{F_0}{\kappa F} \quad (F_0 = \kappa^3). \tag{1}$$

At the point $r \sim r_1$ the external field is of the same order as the Coulomb "tail," and at $r \sim r_2$ the electron comes out from under the potential barrier; F_0 is a typical intraatomic field. In ionization of the atom (for $F \ll F_0$) there is leaking out of the electron through a very broad potential barrier, and moreover its width varies periodically from r_2 to infinity.

For sufficiently small frequency ω the change of the field during the time of passage of the electron through the barrier can be neglected; according to [4] the condition for validity of the adiabatic approximation is of the following form:

$$\gamma = \frac{\omega}{\omega_t} \ll 1 \quad \left(\omega_t = \frac{F}{\kappa} = \frac{2F}{F_0}\omega_0\right) \tag{2}$$

(here $1/\omega_t$ is the time of free flight of an electron of momentum κ through a barrier of width r_2). The parameter γ characterizes the degree of adiabaticity of the motion through the barrier. The calculation of the probability of ionization of the atom in the case $\gamma \ll 1$ reduces to averaging the probabilities of ionization in constant fields over a period of the external field.

The ionization of a hydrogen atom in a constant field has been treated by Oppenheimer[7] and by Lanczos;[8] a simpler formula, based on the quasiclassical approximation and asymptotically valid for weak fields ($F \ll F_0$) has been given by Landau and Lifshits (see [6], page 328):

$$w_{\text{stat}}(F) = 8\omega_H \frac{F_H}{F} \exp\left\{-\frac{2}{3}\frac{F_H}{F}\right\}, \tag{3}$$

where $F_H = m^2e^5\hbar^{-4} = 5.142 \times 10^9$ V/cm is the electric field strength at the first Bohr orbit, and $\omega_H = 2.07 \times 10^{16}$ sec^{-1} = 13.6 eV is the ionization energy of the hydrogen atom.

Analogous calculations for an arbitrary atom have been made recently by Smirnov and Chibisov.[9] For the case in which the electron is in a state with orbital angular momentum l and its projection along the direction of the field is m, the formula for the probability of ionization can be put in the following form:[2)]

$$w_{\text{stat}}(F) = \omega_0 |C_{\kappa l}|^2 (2l+1) \frac{(l+|m|)!}{2^{|m|}(|m|)!(l-|m|)!} \times \left(\frac{2F_0}{F}\right)^{2\kappa-|m|-1} \exp\left\{-\frac{2}{3}\frac{F_0}{F}\right\}, \tag{4}$$

[1)] In this paper we use atomic units: $\hbar = m = e = 1$.

[2)] There is an inaccuracy in the derivation of the numerical coefficient in Eq. (11) of [9]. The corrected value of the coefficient is used in our Eq. (4).

where $\omega_0 = \kappa^2/2$, $F_0 = \kappa^3$, $\lambda = \kappa_C/\kappa$ (κ_C is the Coulomb momentum; $\kappa_C = me^2Z/\hbar$—in atomic units $\kappa_C = Z$). It is convenient to express the field F_0 and the parameter λ for an arbitrary atom in terms of the quantities for the hydrogen atom:

$$F_0 = \left(\frac{\omega_0}{\omega_H}\right)^{3/2} F_H, \quad \lambda = Z\left(\frac{\omega_0}{\omega_H}\right)^{-1/2}. \tag{5}$$

The dimensionless constant $C_{\kappa l}$ and the power-law index λ that appear in (4) have simple physical meanings: in the region $\kappa^{-1} \ll r \ll r_1$ the intra-atomic field is already small and reduces to $V_C(r)$, and the external field can still be neglected. Therefore in this region the ψ function is the same as the asymptotic wave function for the free atom:

$$\psi(\mathbf{r}) = C_{\kappa l}\kappa^{3/2}(\kappa r)^{\lambda-1}e^{-\kappa r}Y_{lm}(\mathbf{r}/r). \tag{6}$$

If the interaction between the electron and the atomic residue falls off more rapidly than the Coulomb interaction for $\kappa r \gg 1$ (for example, in the case of negative ions of the type of He^-), then $\lambda = 0$ and the asymptotic form of $\psi(r)$ is $\sim e^{-\kappa r}/r$—the usual form for potentials with a finite range. The presence of a Coulomb "tail" at infinity changes the exponent of r in (6) (for example, for a hydrogen atom in a state with principal quantum number n we have $\lambda = n$). The exact value of $C_{\kappa l}$ can be found only in the simplest cases. For the hydrogen atom in the ground state $C_0 = 2$; for an s level in a three-dimensional potential well of radius r_0 and depth V_0 we have

$$C_{\kappa 0} = e^{\kappa r_0}\left[\frac{2}{1+\kappa r_0}\left(1-\frac{\kappa^2}{2V_0}\right)\right]^{1/2}.$$

Determination of the constant $C_{\kappa l}$ requires the exact solution of the Schrödinger equation in the entire region $0 \le r < \infty$.

We denote the probability of ionization of the atom in an alternating electric field with amplitude F and frequency ω by $w(F, \omega)$. The value of $w(F, \omega)$ for $\gamma \ll 1$ can be found by averaging (4) over a period of the external field. The result is the following formula for the probability of ionization in the field of a plane-polarized wave:

$$w(F, \omega) = (3F/\pi F_0)^{1/2} w_{stat}(F) \quad (\omega \ll \omega_t). \tag{7}$$

Comparing (4) and (7), we see that the only change is in the coefficient, and the main factor $\exp(-2F_0/3F)$ remains unchanged. This means that in the adiabatic case the ionization of an atom occurs mainly at the times when the field reaches its maximum values, and the ionization current consists of sharp peaks.

Let us consider the general case of a monochromatic wave with elliptical polarization:

$$\mathbf{F}(t) = F(\mathbf{e}_x \cos\omega t \pm \varepsilon \mathbf{e}_y \sin\omega t), \quad 0 \leqslant \varepsilon \leqslant 1 \tag{8}$$

(for $\epsilon = 0$ the wave is plane-polarized, and for $\epsilon = \pm 1$, circularly polarized). By the same method as before we get

$$w(F, \omega, \varepsilon) = A(F, \omega, \varepsilon) w_{stat}(F) \quad (\omega \ll \omega_t), \tag{9}$$

where

$$A(F, \omega, \varepsilon) = \left[\frac{\varepsilon(1+\varepsilon)}{2}\right]^{-1/2} a\left(\frac{1-\varepsilon}{3\varepsilon}\frac{F_0}{F}\right),$$
$$a(x) = e^{-x}I_0(x) \tag{10}$$

[here $I_0(x)$ is the Bessel function of imaginary argument]. a(x) is a monotonically decreasing function: $a(0) = 1$, $a(x) \sim (2\pi x)^{-1/2}$ for $x \gg 1$.

It follows from (10) that in cases in which the polarization of the wave is not too nearly circular the law of dependence of the probability of ionization on the field strength F is the same as for a plane-polarized wave:

$$w(F, \omega, \varepsilon) = \left(\frac{3}{\pi}\frac{1}{1-\varepsilon^2}\frac{F}{F_0}\right)^{1/2} w_{stat}(F) \tag{11}$$

for $(1 - \epsilon) \gg F/F_0$. For a circularly polarized wave we have from (10)

$$w(F, \omega, \pm 1) = w_{stat}(F),$$

as is natural, since in this case the amplitude of the wave is constant. Finally, in the narrow transition region $(1 - \epsilon) \sim F/F_0 \ll 1$ the dependence of $w(F, \omega, \epsilon)$ on F is a complicated function and cannot be put in a simple power-law form. Accordingly, the case of a circularly polarized wave is in a sense an exceptional one, and is characterized by an "instability": for small deviations of the degree of polarization ϵ from unity there is a rapid change of the coefficient of the exponential.

3. A ONE-DIMENSIONAL MODEL

For $\omega \gg \omega_t$ the adiabatic approximation does not apply, and to find $w(F, \omega)$ one must solve the time-dependent Schrödinger equation. To avoid complications which are of no importance in principle, we first consider a model of the simplest sort—the one-dimensional motion of a particle in the field of short-range forces (in the limit we replace them by a δ-potential). As is well known, in the potential $V(x) = -\kappa\delta(x)$ there is one bound state with energy $\omega_0 = \kappa^2/2$. Let us consider the ionization of this level by a uniform electric field f(t).

It is not hard to find the probability of ioniza-

tion of this level by a constant field and in the adiabatic case (for $\gamma \ll 1$):

$$w_{\text{stat}}(F) = 2\omega_0 \exp\{-2F_0/3F\},$$
$$w(F,\omega) = 2\omega_0(3\pi^{-1}F/F_0)^{1/2}\exp\{-2F_0/3F\} \quad (12)$$

($\omega \ll \omega_t$). We now go to the case of an alternating field. The Schrödinger equation is then

$$i\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2}\frac{\partial^2}{\partial x^2} - \varkappa\delta(x) - f(t)x\right\}\psi, \quad (13)$$

and the initial condition ($t = t_0$) is

$$\psi(x, t_0) = \sqrt{\varkappa}\exp\{-\varkappa|x| + i\varkappa^2 t/2\}. \quad (14)$$

Equation (13) can be reduced to an integral equation. To do so we rewrite it in the form

$$\left\{i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + f(t)x\right\}\psi(x,t) = -\varkappa\delta(x)\psi(x,t). \quad (15)$$

The solution that satisfies the initial condition (14) can be written formally in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, G(x,t;x',t_0)\psi(x',t_0)$$
$$+ i\varkappa\int_{t_0}^{t} dt'\, G(x,t;0,t')\psi(0,t'), \quad (16)$$

where the Green's function G(x, t; x', t') corresponds to the motion of the particle in the uniform field depending on the time, and can be found easily by going over to the momentum representation:

$$G(x,t;x',t') = \frac{\theta(t-t')}{2\pi}\int_{-\infty}^{\infty} dp\exp\left\{i\left[\pi(t)x - \pi(t')x' - \frac{1}{2}\int_{t'}^{t}\pi^2(\tau)\,d\tau\right]\right\}. \quad (17)$$

Here

$$\pi(t) = p - A(t), \qquad A(t) = -\int_{t_0}^{t} f(t')\,dt',$$

and $\pi(t)$ is the generalized momentum for the motion in the uniform electric field; in what follows we also use the quantity

$$\xi(t) = -\int_{t_0}^{t} A(t')\,dt',$$

which is the classical trajectory of the particle in the field f(t).

Equation (16) is the exact integral equation for $\psi(x, t)$. For our purposes it suffices to find an approximate solution valid under the condition that the mean time of ionization is much larger than atomic times. In this case, in calculating the current we can neglect the damping of the ψ-function in the region $|x| \lesssim \kappa^{-1}$; that is, we insert in the right member of (16) instead of the exact wave function $\psi(0, t)$ its unperturbed value

$$\psi(0,t') = \sqrt{\varkappa}\exp\{i\varkappa^2 t'/2\}.$$

The first term in (16) describes the smearing out of the initial state; it falls off as $[\kappa^2(t - t_0)]^{-1/2}$ and does not contribute to the current. Since we are interested only in the stationary part of the current, we let t_0 go to $-\infty$ and assume that the field was turned on adiabatically. Then

$$\psi(x,t) = i\varkappa^{3/2}\int_{-\infty}^{t} dt'\, G(x,t;0,t')\exp\left\{\frac{i\varkappa^2 t'}{2}\right\}. \quad (18)$$

Using (17), we get from this the expression for the current:

$$j(x,t) = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$
$$= \frac{\varkappa^3}{8\pi^2}\int_{-\infty}^{\infty} dp_1\,dp_2[\pi_1(t) + \pi_2(t)]\exp\left\{i(p_2 - p_1)(x - \xi(t))\right.$$
$$\left. + \frac{i}{2}(p_1^2 - p_2^2)t\right\}\int_{-\infty}^{t} dt_1\,dt_2\exp\{i[\chi(p_2,t_2) - \chi(p_1,t_1)]\}, \quad (19)$$

where

$$\chi(p,t) = \frac{1}{2}\left[(p^2 + \varkappa^2)t + 2p\xi(t) + \int_0^t A^2(\tau)\,d\tau\right]. \quad (20)$$

We make the further calculations for an alternating field which oscillates according to the harmonic law

$$f(t) = F\cos\omega t. \quad (21)$$

The values of A(t) and ξ(t) are then[3)]

$$A(t) = -F\frac{\sin\omega t}{\omega}, \qquad \xi(t) = -F\frac{\cos\omega t}{\omega^2}. \quad (22)$$

Substituting in (19) the expansion of $\exp\{i\chi(p, t)\}$ in a Fourier series

$$\exp\{i\chi(p,t)\} = \sum_{n=-\infty}^{\infty} f_n(p)\exp\left\{\frac{i}{2}\left[p^2 + \varkappa^2\left(1 + \frac{1}{\gamma}\right) - 2n\omega\right]t\right\} \quad (23)$$

(here we have introduced the coefficients $f_n(p)$, which are important for what follows), we calculate the integrals over t_1 and t_2 and find ($\delta \to +0$):

[3)] To remove ambiguities arising in the integrals for A(t) and ξ(t) for $t_0 \to -\infty$, one must affix to f(t) a factor $e^{\epsilon t}$, $\epsilon \to +0$. This corresponds to turning the field on adiabatically.

$$j(x,t)=\frac{\varkappa^3}{8\pi^2}\int_{-\infty}^{\infty}dp_1\,dp_2[p_1+p_2-2A(t)]\sum_{n_1,n_2}f_{n_1}^{*}(p_1)f_{n_2}(p_2)$$
$$\times\exp\{i[(p_2-p_1)(x-\xi(t))+(n_1-n_2)\omega t]\}$$
$$\times\{[1/2(p_1^2+\varkappa^2(1+1/2\gamma^2))-n_1\omega+i\delta]$$
$$\times[1/2(p_2^2+\varkappa^2(1+1/2\gamma^2))-n_2\omega-i\delta]\}^{-1}. \quad (24)$$

The integrand has poles at the points $p=\pm p_n$, where

$$p_n=[2n\omega-\varkappa^2(1+1/2\gamma^2)]^{1/2}. \quad (25)$$

We set

$$\nu=\frac{\varkappa^2}{2\omega}\left(1+\frac{1}{2\gamma^2}\right); \quad (26)$$

for $n<\nu$ the pole $p=p_n$ lies on the imaginary axis, and for $n\gtrsim\nu$ it lies near the real axis.

The probability of ionization $w(F,\omega)$ is determined by the current $j(x,t)$ at infinity. Setting $|x|\to\infty$ in (24) and using the relation

$$\lim_{x\to+\infty}[e^{ipx}/(p^2-K-i\delta)]$$
$$=\begin{cases}\pi iK^{-1/2}\exp\{iK^{1/2}x\}\delta(p-K^{1/2}), & K>0\\ 0, & K<0\end{cases}, \quad (27)$$

we get

$$\lim_{x\to+\infty}j(x,t)=\frac{\varkappa^3}{2}\int_{-\infty}^{\infty}dp_1dp_2(p_1+p_2-2A(t))$$
$$\times\sum_{n_1n_2\geqslant\nu}^{\infty}\frac{f_{n_1}^{*}(p_1)f_{n_2}(p_2)}{p_1p_2}\exp\{i(p_{n_2}-p_{n_1})(x-\xi(t))$$
$$+i(n_1-n_2)t\}\delta(p_1-p_{n_1})\delta(p_2-p_{n_2}). \quad (28)$$

The integration over p_1 and p_2 in (28) can be carried out by means of the δ functions, and the result is a double sum over n_1 and n_2. For $x\to\infty$ the nondiagonal terms of this sum ($n_1\neq n_2$) oscillate rapidly and cancel each other; there is a finite contribution only for $n_1=n_2$. Averaging $j(x,t)$ over a period of the external field and multiplying by two (the flux of electrons goes in both directions from the "atom"), we find that the probability of ionization is a sum of the probabilities of many-photon processes:

$$w(F,\omega)=\sum_{n\geqslant\nu}^{\infty}w_n(F,\omega),\quad w_n(F,\omega)=\omega_0\frac{|2f_n(p_n)|^2}{p_n/\varkappa}. \quad (29)$$

$w_n(F,\omega)$ is the probability of ionization with the absorption of n quanta of frequency ω; the law of conservation of energy holds,

$$1/2p_n^2=-1/2\varkappa^2(1+1/2\gamma^2)+n\omega$$

[the term $\varkappa^2/4\gamma^2=1/2\overline{A^2(t)}$ is the mean kinetic energy of the oscillatory motion of the electron in the field $F\cos\omega t$]. For the concrete calculation of $w(F,\omega)$ we must know the function $f_n(p_n)$. As is clear from (23), it can be put in the form of a single integral. In what follows we confine ourselves to the case $\omega\ll\omega_0=\varkappa^2/2$, for which this integral can be calculated by the method of steepest descents.[4] This gives

$$|f_n(p_n)|^2=\frac{\omega}{\pi\varkappa^2}\left(\frac{\gamma^2}{1+\gamma^2}\right)^{1/2}\exp\left\{-\frac{\varkappa^2}{\omega}\left[\left(1+\frac{1}{2\gamma^2}\right)\mathrm{Arsh}\,\gamma\right.\right.$$
$$\left.\left.-\frac{(1+\gamma^2)^{1/2}}{2\gamma}+\left(\mathrm{Arsh}\,\gamma-\frac{\gamma}{(1+\gamma^2)^{1/2}}\right)\frac{p_n^2}{\varkappa^2}\right]\right\}$$
$$\times\left[1+(-1)^n\cos\left(\frac{2\varkappa p_n}{\omega\gamma}(1+\gamma^2)^{1/2}\right)\right]. \quad (30)^*$$

We present the formula for $w(F,\omega)$ in final form:

$$w(F,\omega)=2\omega_0\left(\frac{3}{\pi}\frac{F}{F_0}\right)^{1/2}\exp\left\{-\frac{2}{3}\frac{F_0}{F}g(\gamma)\right\}R(\omega,\gamma), \quad (31)$$

$$R(\omega,\gamma)=\left(\frac{2}{3\pi}\frac{\gamma^3}{1+\gamma^2}\right)^{1/2}$$
$$\times\sum_{n\geqslant\nu}^{\infty}\frac{e^{-\alpha(n-\nu)}}{(n-\nu)^{1/2}}[1+(-1)^n\cos(b(n-\nu)^{1/2})]. \quad (32)$$

Here we have introduced the following notations:

$$g(\gamma)=\frac{3}{2\gamma}\left[\left(1+\frac{1}{2\gamma^2}\right)\mathrm{Arsh}\,\gamma-\frac{(1+\gamma^2)^{1/2}}{2\gamma}\right]$$
$$=\begin{cases}1-1/10\gamma^2+9/280\gamma^4, & \gamma\ll1\\ 3/2\gamma^{-1}(\ln2\gamma-1/2), & \gamma\gg1\end{cases} \quad (33)$$

[the function $g(\gamma)$ decreases monotonically with increasing γ, see Fig. 1];

$$\alpha(\gamma)=2\left[\mathrm{Arsh}\,\gamma-\frac{\gamma}{(1+\gamma^2)^{1/2}}\right]=\begin{cases}2/3\gamma^3, & \gamma\ll1\\ 2(\ln2\gamma-1), & \gamma\gg1\end{cases}; \quad (34)$$

$$b=4\left[\frac{\omega_0}{\omega}\left(1+\frac{1}{\gamma^2}\right)\right]^{1/2},\quad \nu=\frac{\omega_0}{\omega}\left(1+\frac{1}{2\gamma^2}\right),\quad F_0=\varkappa^3. \quad (35)$$

Let us discuss the physical meaning of Eq. (31). For $\gamma\to0$ the main contribution to the sum in (32) comes from large n. Replacing the summation over n by an integration, we get $R(\omega,\gamma)=1+o(\gamma^{3/2})$, so that [compare this with (12)]

$$w(F,\omega)=2\omega_0\left(\frac{3F}{\pi F_0}\right)^{1/2}\exp\left\{-\frac{2}{3}\frac{F_0}{F}\left(1-\frac{1}{10}\gamma^2+\ldots\right)\right\}. \quad (36)$$

[4] For a detailed exposition of the calculations see [10]. Actually the integral for f_n had already occurred in a paper by Nikishov and Ritus, [11] but the formula obtained in [11] is not entirely convenient for our purposes.

*$\mathrm{Arsh}\equiv\sinh^{-1}$.

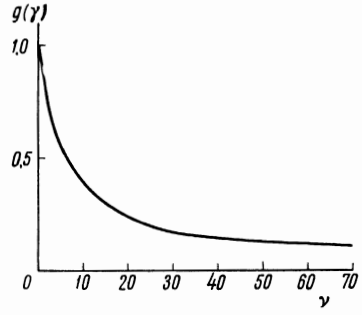

FIG. 1. Curve of the function $g(\gamma)$ which determines the frequency dependence of the exponent in the formula for the probability of ionization

Now let ω increase (for a fixed field strength F). For $\gamma \gg 1$, as can be seen from Fig. 1, $\exp\{-(2F_0/3F)g(\gamma)\}$ increases rapidly, i.e., in general the ionization by an alternating field with $\omega \gg \omega_t$ is much greater than for a constant field of the same amplitude. The increase of $w(F, \omega)$ is not monotonic, however, because of the uneven behavior of the factor $R(\omega, \gamma)$ before the exponential. For $\gamma \gg 1$ the sum in (32) reduces to the first term (i.e., the probabilities w_n fall off rapidly with increase of n). The coefficient $R(\omega, \gamma)$ has threshold singularities at the points $\omega = \omega_n$ ($\nu = n$) which correspond to thresholds for absorption of n quanta, and in addition it oscillates rapidly as ω varies (for details see [10]). We note that these oscillations are characteristic of the one-dimensional case only.

4. IONIZATION OF A LEVEL BOUND BY SHORT-RANGE FORCES. CASES OF LINEAR AND CIRCULAR POLARIZATION OF THE INCIDENT ELECTROMAGNETIC WAVE

Proceeding to the consideration of the actual three-dimensional case, we note first that the general course of the calculations is the same as in the case of the one-dimensional model. This allows us to avoid repetition of the developments of Sec. 3, and point out only some new points which are of importance in principle.

The integral equation for the quasistationary process in the alternating field F(t) is of the form [compare with (16) and (18)]

$$\psi(\mathbf{r}, t) = -i\int_{-\infty}^{t} dt' \int d\mathbf{r}'\, G(\mathbf{r}, t;\ \mathbf{r}', t')\, V(\mathbf{r}')\psi(\mathbf{r}', t'), \quad (37)$$

where it is assumed that the field was turned on adiabatically at $t_0 \to -\infty$. Here $G(\mathbf{r}, t;\ \mathbf{r}', t')$ is the Green's function of the electron for motion in the uniform field $\mathbf{F}(t)$:

$$G(\mathbf{r}, t;\ \mathbf{r}', t') = \frac{\theta(t-t')}{(2\pi)^3} \times \int d\mathbf{p} \exp\left\{i\left[\boldsymbol{\pi}(t)\mathbf{r} - \boldsymbol{\pi}(t')\mathbf{r}' - \frac{1}{2}\int_{t'}^{t} \pi^2(\tau)\, d\tau\right]\right\}, \quad (38)$$

where $\pi(t)$ is the generalized momentum:

$$\boldsymbol{\pi}(t) = \mathbf{p} - \mathbf{A}(t), \qquad \mathbf{A}(t) = -\int_{-\infty}^{t} \mathbf{F}(t')\, dt'. \quad (39)$$

Our main approximation is that in (37) the function $\psi(\mathbf{r}', t')$ is replaced by the wave function of the bound state for the free atom:

$$\psi(\mathbf{r}, t) \to \varphi_{lm}(\mathbf{r}) \exp\{i\varkappa^2 t/2\} \equiv \psi^{(0)}(\mathbf{r}, t),$$
$$V(r)\psi(\mathbf{r}, t) \to V(r)\psi^{(0)}(\mathbf{r}, t) = {}^1/_2(\nabla^2 - \varkappa^2)\varphi_{lm}(\mathbf{r})\exp\{i\varkappa^2 t/2\}. \quad (40)$$

The justification for this is as follows. Since $F \ll F_0$, the difference between the exact wave function $\psi(\mathbf{r}, t)$ and $\psi^{(0)}(\mathbf{r}, t)$ for $\varkappa r \lesssim 1$ is negligibly small,[5] and owing to the rapid decrease of $V(r')\psi(\mathbf{r}', t')$ large values of r' do not contribute to the integral (37). We emphasize that it is essential here that $V(r)$ falls off more rapidly than $1/r$ for $r \to \infty$; for potentials with a Coulomb "tail," as can be seen from the calculation of $w_{stat}(F)$ (cf. [6]), the integration extends to the region $r \sim F_0/F\varkappa$, and to get the correct factor before the exponential one must take the behavior of the ψ function at large distances into account more accurately.

Now let $\mathbf{F}(t) = \mathbf{F}\cos\omega t$ be the electric field of a plane-polarized wave; we choose the x axis along the direction of $\mathbf{F}$. The probability of ionization is determined by the total flux of particles through a plane perpendicular to the x axis:

$$w(F, \omega) = 2\lim_{x\to\infty} \overline{J(x, t)}$$

(the bar indicates averaging over a period of the field), and for the flux $J(x, t)$ we have an expression analogous to (24):

$$J(x, t) = \int_{-\infty}^{\infty} dy\, dz\, j_x(\mathbf{r}, t) = \frac{1}{4\pi}\int d\mathbf{p}_1\, d\mathbf{p}_2\, \delta(\mathbf{p}_{1\perp} - \mathbf{p}_{2\perp}) \times (p_{1x} + p_{2x} - 2A(t)) \sum_{n_1 n_2 = -\infty}^{\infty} F_{n_1}^{*}(\mathbf{p}_1) F_{n_2}(\mathbf{p}_2)$$

[5] Calculations with perturbation theory show [10] that the first-order correction $\psi^{(1)}(\mathbf{r}, t)$ is comparable in size with $\psi^{(0)}(\mathbf{r}, t)$ only for $\varkappa r \sim (F_0/F)^{1/2} f(\gamma)$; here $f(\gamma) \sim 1$ as long as $\gamma \leq (F_0/F)^{1/2}$, and $f(\gamma) \sim (\ln \gamma)/\gamma$ for $\gamma \gg (F_0/F)^{1/2}$. For all frequencies that satisfy the condition $\omega \ll \omega_0$ this distance is much larger than the atomic radius $\varkappa^{-1}$.

$$\times \exp\{i[(p_{2x}-p_{1x})(x-\xi(t))+(n_1-n_2)\omega t]\}$$
$$\times [1/2(p_1^2+\varkappa^2+\varkappa^2/2\gamma^2)-n_1\omega+i\delta]^{-1}$$
$$\times [1/2(p_2^2+\varkappa^2+\varkappa^2/2\gamma^2)-n_2\omega-i\delta]^{-1}, \quad (41)$$

where

$$\xi(t)=-\int_{-\infty}^{t} A(t')\,dt',$$

$\mathbf{p}_\perp$ denotes the transverse momentum, and the quantities $F_n(\mathbf{p})$ are the coefficients of the following Fourier series:

$$\sum_{n=-\infty}^{\infty} F_n(\mathbf{p})e^{-in\omega t}=1/2[\pi^2(t)+\varkappa^2]\varphi_{lm}(\boldsymbol{\pi}(t))$$
$$\times \exp\left\{-i\frac{\omega_0}{\omega}\left[\frac{2p_x}{\varkappa\gamma}\cos\omega t+\frac{1}{4\gamma^2}\sin 2\omega t\right]\right\}. \quad (42)$$

In (42) $\varphi_{lm}(\pi)$ is the wave function of the electron in the momentum representation:

$$\varphi_{lm}(\mathbf{p})=(2\pi)^{-3/2}\int d\mathbf{r}\, e^{-i\mathbf{pr}}\varphi_{lm}(\mathbf{r})=\frac{r_{lm}(p)}{p^2+\varkappa^2}Y_{lm}(\mathbf{p}/p),$$

where $r_{lm}(p = i\kappa) = (2\kappa/\pi)^{1/2} C_{\kappa l}$ [here we have used the fact that the potential $V(r)$ falls off more rapidly than r^{-1}].

After the same manipulations as used in Sec. 3, we arrive at the formula for the probability of ionization:

$$w=\sum_{n\geqslant\nu}^{\infty} w_n(F,\omega), \quad \nu=\frac{\omega_0}{\omega}\left(1+\frac{1}{2\gamma^2}\right); \quad (43)$$

$w_n(F, \omega)$ is the probability of ionization with the absorption of n quanta:

$$w_n(F,\omega)=2\pi\int d\mathbf{p}\,\delta\left(\frac{1}{2}\left[\mathbf{p}^2+\varkappa^2+\frac{\varkappa^2}{2\gamma^2}\right]-n\omega\right)|F_n(\mathbf{p})|^2. \quad (44)$$

Accordingly, the task reduces to the calculation of the quantities $F_n(p)$ for $p = p_n$: p_n is defined in (25). It follows from (42) that for this we need to know the exact expression for $\varphi_{lm}(\mathbf{p})$, which cannot be found for an arbitrary atom without numerical calculations. We shall show, however, that in the most interesting case $\omega \ll \omega_0$ (when the absorption of a large number of quanta is needed for ionization of the atom) it is sufficient for the determination of $F_n(\mathbf{p})$ to know only the asymptotic form of the wave function, Eq. (6).

In fact, we have from (42)

$$F_n(\mathbf{p})\,|_{p=p_n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{lm}(\boldsymbol{\pi}(\alpha))$$
$$\times \exp\left\{-i\frac{\omega_0}{\omega}\int_0^{\alpha}\left[\frac{\pi^2(y)}{\varkappa^2}+1\right]dy\right\}d\alpha, \quad (45)$$

where we have set

$$\chi_{lm}(\boldsymbol{\pi}(\alpha))=1/2(\pi^2(\alpha)+\varkappa^2)\varphi_{lm}(\boldsymbol{\pi}(\alpha)). \quad (46)$$

Making the replacement $\alpha \to \beta = \pi/2 - \alpha$, expressing n in terms of p_n^2 by using (25), and introducing the primed variables

$$\gamma'=\gamma r, \quad \omega'=\omega/r^2,$$
$$p'=p_x/r, \quad r=(1+p_\perp^2/\varkappa^2)^{1/2}, \quad (47)$$

we put (45) in the form

$$F_n(\mathbf{p})\,|_{p=p_n}=\frac{i^n}{2\pi}\int_{-\pi}^{\pi}d\beta\,\chi_{lm}(\boldsymbol{\pi}(\beta)) \quad (48)$$
$$\times \exp\left\{-i\frac{\omega_0}{\omega'}\left[\left(\frac{p'^2}{\varkappa^2}+1+\frac{1}{2\gamma'^2}\right)\beta\right.\right.$$
$$\left.\left.+\frac{2p'}{\varkappa\gamma'}\sin\beta+\frac{1}{4\gamma'^2}\sin 2\beta\right]\right\}.$$

Owing to the factor $\omega_0/\omega' \gg 1$ the exponential in (28) is a rapidly oscillating function, which enables us to calculate this integral by the method of steepest descents (cf. analogous calculations in [4, 11]).

The equation for the saddle points can be put in the form

$$\boldsymbol{\pi}^2(\beta)\equiv\mathbf{p}_\perp^2+\left(p_x+\frac{\varkappa}{\gamma}\cos\beta\right)^2=-\varkappa^2, \quad (49)$$

which shows that the quantity $F_n(\mathbf{p})\,|_{p=p_n}$ is determined for $\omega \ll \omega_0$ only by the behavior of the wave function $\varphi_{lm}(\pi)$ in the neighborhood of the pole $\pi^2 = -\kappa^2$; the exact form of $\varphi_{lm}(\pi)$ for all values of π is unimportant. This means that in the x space it suffices to know the asymptotic behavior of $\varphi_{lm}(\mathbf{r})$ for $\kappa r \gg 1$. Using (6) (with $\lambda = 0$), and also (46), we get

$$\chi_{lm}(\boldsymbol{\pi})\,|_{\pi^2=-\varkappa^2}=\left(\frac{\varkappa}{2\pi}\right)^{1/2}C_{\varkappa l}\xi_l Y_{lm}(\mathbf{n}), \quad (50)$$

$$\xi_l=\begin{cases}1 & \text{for } \sqrt{\pi^2}=i\varkappa.\\ (-1)^l & \text{for } \sqrt{\pi^2}=-i\varkappa\end{cases} \quad (50a)$$

The unit vector $\mathbf{n} = \pi/(\pi^2)^{1/2}$ becomes complex at the pole $\pi^2(\beta) = -\kappa^2$. From (49) we get (under the condition $p_\perp^2/\kappa^2 \ll 1$)

$$\mathbf{n}_{1,2}=\frac{\boldsymbol{\pi}(\beta_{1,2})}{(\pi^2(\beta_{1,2}))^{1/2}}=\mathbf{e}_x\left(1+\frac{\mathbf{p}_\perp^2}{\varkappa^2}\right)^{1/2}\mp i\frac{\mathbf{p}_\perp}{\varkappa}. \quad (51)$$

(The indices − and + refer to the two saddle points β_1 and β_2.) From this we have

$$Y_{lm}(\mathbf{n}_{1,2})\approx C_{lm}\left(\mp\frac{p_\perp}{\varkappa}\right)^{|m|}e^{im\varphi},$$
$$C_{lm}=\frac{1}{2^{|m|}|m|!}\left[\frac{(2l+1)}{4\pi}\frac{(l+|m|)!}{(l-|m|)!}\right]^{1/2}. \quad (52)$$

In the integration over β in (48) we must allow for the fact that the effective values of p_x and $p_\perp$ are much smaller than κ [this can be seen from the result, see (53)], and owing to this all quantities that occur in the argument of the exponential in (48) must be expanded in powers of p_x/κ, $p_\perp/\kappa$ to and including quadratic terms. The final expression for $|F_n(\mathbf{p})|^2_{\mathbf{p}=\mathbf{p}_n}$ is [6)]

$$|F_n(\mathbf{p})|^2_{\mathbf{p}=\mathbf{p}_n} = \frac{1}{4}\frac{\varkappa}{\pi^2}|C_{\varkappa l}|^2 C_{lm}{}^2 \frac{\omega\gamma}{\omega_0(1+\gamma^2)^{1/2}} \times \exp\left\{-\frac{2}{3}\frac{F_0}{F}g(\gamma)\right\} \exp\left\{-\frac{2\omega_0}{\omega}\left[\frac{p_\perp{}^2}{\varkappa^2}\operatorname{Arsh}\gamma + \frac{p_x{}^2}{\varkappa^2}\left(\operatorname{Arsh}\gamma - \frac{\gamma}{(1+\gamma^2)^{1/2}}\right)\right]\right\}\left(\frac{p_\perp}{\varkappa}\right)^{2|m|} \times\left[1+(-1)^{n+|m|}\cos\left(4\frac{\omega_0}{\omega}\frac{(1+\gamma^2)^{1/2}}{\gamma}\frac{p_x}{\varkappa}\right)\right]. \quad (53)$$

The function $g(\gamma)$ is defined in (33); $p_\perp$ and p_x are connected by the condition $p_\perp^2 + p_x^2 = p_n^2$.

The expression (53) is to be substituted in (44), and integrated over the angles; in doing this we can neglect the term that contains the rapidly oscillating factor

$$\cos\left(\frac{4\omega_0}{\omega}\frac{(1+\gamma^2)^{1/2}}{\gamma}\frac{p_x}{\varkappa}\right).$$

The result is the following formula for the ionization in the field of a plane-polarized wave, from a level with binding energy $\kappa^2/2$, orbital angular momentum l, and projection m in the direction of the field:

$$w_{lm}(F,\omega) = \omega_0|C_{\varkappa l}|^2\left(\frac{6}{\pi}\right)^{1/2}\frac{(2l+1)(l+|m|)!}{2^{|m|}|m|!(l-|m|)!} \times\left(\frac{F(1+\gamma^2)^{1/2}}{2F_0}\right)^{|m|+3/2} A_m(\omega,\gamma)\exp\left\{-\frac{2}{3}\frac{F_0}{F}g(\gamma)\right\}. \quad (54)$$

Here

$$A_m(\omega,\gamma) = \frac{4}{\sqrt{3\pi}}\frac{1}{|m|!}\frac{\gamma^2}{1+\gamma^2} \times\sum_{n\geqslant\nu}^{\infty}\exp\{-\alpha(n-\nu)\}\, w_m(\sqrt{\beta(n-\nu)}), \quad (55)$$

$$w_m(x) = e^{-x^2}\int_0^x e^{y^2}(x^2-y^2)^{|m|}\,dy = \frac{x^{2|m|+1}}{2}\int_0^1\frac{e^{-x^2t}t^{|m|}}{(1-t)^{1/2}}dt. \quad (56)$$

The parameters α, β, γ are functions of the frequency ω and the field strength F; $\alpha(\gamma)$ is defined in (34), and $\beta = 2\gamma(1+\gamma^2)^{-1/2}$.

Let us discuss the physical meaning of the formula (54). As has already been pointed out in [4], the probability of ionization is the sum of the probabilities of many-photon processes, each of which corresponds to the absorption of a whole number ($n \geq \nu$) of quanta. The main factor in $w(F,\omega)$ is the exponential, which increases rapidly for $\gamma \gg 1$. A comparison with (31) shows that the exponential is the same here as in the one-dimensional case. This exponential in the expression for $w_{lm}(F,\omega)$ was first derived by Keldysh.[4] [7)]

Let us now examine how the coefficient of the exponential in (54) varies with the frequency. For $\gamma \ll 1$ there is a large number of important terms in the sum (55), and owing to this the representation of $A_m(\omega,\gamma)$ in the form of a series is inconvenient. Using the formula (56) for the function $w_m(x)$, we can put $A_m(\omega,\gamma)$ in the following form:

$$A_m(\omega,\gamma) = \frac{1}{2\sqrt{3\pi}}\frac{1}{|m|!}\left(\frac{2\gamma}{(1+\gamma^2)^{1/2}}\right)^{|m|+s/2} \times\int_0^1 dx\,\frac{x^{|m|}\exp\{-(\alpha+\beta x)\delta\}}{(1-x)^{1/2}}\Phi(e^{-(\alpha+\beta x)\delta} -|m|-1/2,\delta), \quad (57)$$

where $\delta = [\nu] + 1 - \nu$ ($0 \leq \delta \leq 1$), $[\nu]$ means the integer part of the number ν, and $\Phi(z,s,v)$ is a generalized ζ function which has been studied in detail in the mathematical literature:[12]

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}\frac{z^n}{(n+v)^s}, \quad |z|<1. \quad (58)$$

For $\gamma \ll 1$, $A_m(\omega,\gamma) \to 1$, and (54) goes over into the corresponding formula of the adiabatic approximation:

$$w_{lm}(F) = \omega_0|C_{\varkappa l}|^2\left(\frac{6}{\pi}\right)^{1/2}\frac{(2l+1)(l+|m|)!}{2^{|m|}|m|!(l-|m|)!}\left(\frac{F}{2F_0}\right)^{|m|+3/2} \times\exp\left\{-\frac{2F_0}{3F}\left(1-\frac{1}{10}\gamma^2\right)\right\}. \quad (59)$$

6)For a more detailed exposition of the calculations see [10].

7)We note that the formula for $w_n(F,\omega)$ derived in [4] differs from the exact formula (44), in that in the integral (45) for $F_n(\mathbf{p})$ instead of the function $\chi_{lm}(\mathbf{p}) = 1/2(p^2+\varkappa^2)\varphi_{lm}(\mathbf{p})$ it has the matrix element

$$V_{lm}(\mathbf{p}) = \int e^{-i\mathbf{pr}}\mathbf{F}\mathbf{r}\varphi_{lm}(\mathbf{r})\,d^3\mathbf{r} \sim \mathbf{F}\nabla_{\mathbf{p}}\varphi_{lm}(\mathbf{p}).$$

Unlike $\chi_{lm}(\mathbf{p})$, which in the case of short-range potentials is finite at the saddle point, $V_{lm}(\mathbf{p})$ has a pole of second order there. The result of this is that the coefficient of the exponential caculated in [4] is incorrect for high frequencies ($\gamma \gtrsim 1$). If $\gamma \to 0$, however, the result becomes correct. The Keldysh method is a direct extension of the method of Oppenheimer [7] to the case of an alternating field, and gives the correct value of w_{stat} for short-range potentials.

In the opposite case of high frequencies ($\gamma \gg 1$)

$$(F/2F_0)(1+\gamma^2)^{1/2} = \omega/4\omega_0,$$

and the function $A_m(\omega, \gamma)$ reduces to the first term of the series:

$$A_m(\omega,\gamma) = \frac{4}{\sqrt{3\pi}}\frac{1}{|m|!}\left(\frac{2\gamma}{e}\right)^{-2\delta} w_m(\sqrt{2\delta}); \qquad (60)$$

$\delta = [\nu] + 1 - \nu$, $e = 2.718\ldots$. In this frequency range the coefficient of the exponential is a rapidly varying function, with singularities at the thresholds for absorption of n quanta.

As is seen from (54) and (60), the probability of ionization changes rapidly with increase of $|m|$; the largest probability of ionization is that for the state with $m = 0$. For the mean probability of ionization of unpolarized atoms we get

$$\bar{w}_l = \frac{1}{2l+1}\sum_{m=-l}^{l} w_{lm}(F,\omega) = \omega_0 |C_{\varkappa l}|^2 \left(\frac{6}{\pi}\right)^{1/2}\left(\frac{F(1+\gamma^2)^{1/2}}{2F_0}\right)^{3/2} A_0(\omega,\gamma) \times \exp\left\{-\frac{2F_0}{3F}g(\gamma)\right\}. \qquad (61)$$

The function $w_0(x) \equiv w(x)$ which appears in $A_0(\omega, \gamma)$ is related to the error integral, and there are tables for it.[13] A plot of the function w(x) is given in Fig. 2.

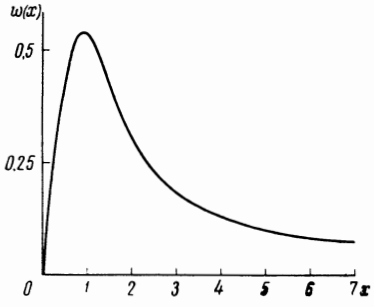

FIG. 2. Curve of the function w(x)

The nature of the variation with frequency of the coefficient of the exponential in (61) is shown qualitatively in Fig. 3 (for the case $\gamma \gg 1$). The points $\omega = \omega_n$ at which this coefficient has singularities of fractional-exponent type, correspond to thresholds for absorption of n quanta. We have $n\omega_n = \kappa^2/2 + \kappa^2/4\gamma^2$, that is, an n-th order resonance ($\kappa^2/4\gamma^2$ is the mean kinetic energy of the electron moving in the field of the plane wave). The maximum values are $\sim(\gamma^3/\ln\gamma)^{1/2}$, and occur near the threshold frequencies: $(\omega_n^{max} - \omega_n) \sim \omega_n(4n\ln\gamma)^{-1}$.

Let us now consider ionization by a circularly polarized wave

$$\mathbf{F}(t) = F(\mathbf{e}_x \cos\omega t + \mathbf{e}_y \sin\omega t) \qquad (62)$$

(as is clear from the results of Sec. 2, this is the most exotic case). To obtain $w(F, \omega)$ it is necessary to integrate the radial component of the particle flux over a cylinder of radius R ($R \to \infty$) with its axis along the z axis (the direction of propagation of the wave). The result is[8] a formula for $w(F, \omega)$ which is of the same form as (43). Now, however, the threshold for ionization (the minimum number of quanta) is $\nu = (\omega_0/\omega)(1 + 1/\gamma^2)$. The difference from (43) is due to the fact that the mean kinetic energy of the electron in the field of the circularly polarized wave is twice its mean energy in the field of the plane-polarized wave.

The quantities $F_n(p)$ are now the coefficients of the following Fourier series [cf. Eq. (42)]:

$$\sum_{n=-\infty}^{\infty} F_n(\mathbf{p}) e^{-in\omega t} = 1/2(\pi^2(t) + \varkappa^2)\varphi_{lm}(\pi(t)) \exp\left\{-i\frac{\omega_0}{\omega}\left[\frac{2p_x}{\varkappa\gamma}\cos\omega t + \frac{2p_y}{\varkappa\gamma}\sin\omega t\right]\right\}. \qquad (63)$$

Confining ourselves for simplicity to the case $l = m = 0$, we get

$$|F_n(\mathbf{p})|_{p=p_n} = 1/2\left(\frac{\varkappa}{2\pi^2}\right)^{1/2} |C_{\varkappa 0}| J_n\left(\frac{F_0}{F\gamma^2}\frac{p_\perp}{\varkappa}\right), \qquad (64)$$

where $p_\perp$ denotes the component of the momentum $\mathbf{p}$ which lies in the xy plane. When (64) is substituted in (44) we encounter the integral

$$\int_{-p_n}^{p_n} dp_z J_n^2\left(\frac{F_0}{F\gamma^2}\frac{(p_n^2 - p_z^2)^{1/2}}{\varkappa}\right) = 2p_n\int_0^1 \frac{v}{(1-v^2)^{1/2}} J_n^2(xv)\,dv, \qquad (65)$$

where $x = (F_0/F\gamma^2)(p_n/\kappa)$. Substituting here the expansion of $|J_n(xv)|^2$ in power series (see [14],

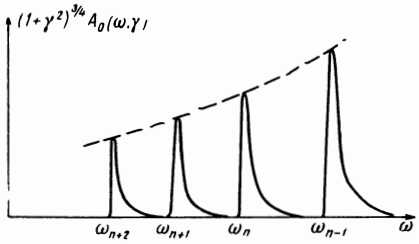

FIG. 3. Frequency dependence of the factor before the exponential in (61); the field strength F is assumed constant

[8] A more detailed exposition of the calculations will be found in the next paper of the writers, which is devoted to a treatment of the general case of elliptical polarization.

page 161) and integrating term by term, we get

$$\int_0^1 \frac{vJ_n^2(xv)}{\sqrt{1-v^2}}dv = \frac{1}{2x}\int_0^{2x} J_{2n}(y)\,dy = \frac{n}{x}\int_0^{x/n} J_{2n}(2ny)\,dy. \quad (66)$$

By the use of (66) the formula for the probability of n-quantum ionization can be reduced to the form

$$w_n = 2\omega_0 |C_{\varkappa 0}|^2 \frac{1+\gamma^2}{\gamma(1+t)} \int_0^{[(1-t^2)/(1+\gamma^2)]^{1/2}} J_{2n}(2ny)\,dy$$

$$(n \geqslant \nu_c = \omega_0\omega^{-1}(1+1/\gamma^2) \gg 1). \quad (67)$$

Here we have introduced the variable $t = 2\nu_c/n - 1$, which is convenient in our further work and varies from 1 (at the threshold, $n = \nu_c$) to -1 (for $n \to \infty$). For $\omega \ll \omega_0$ we have $n \gg 1$, and the Bessel function in (67) has large values of the argument and index. Using an asymptotic formula derived for this case by Watson (see [14], page 283), we transform (67) to the form

$$w_n = \frac{\omega_0 |C_{\varkappa 0}|^2}{2\sqrt{\pi}(2\nu_c)^{3/2}}(1+t)\left[\left(1+\frac{1}{\gamma^2}\right)(1-t)\right]^{1/2}\left(\frac{1+\gamma^2}{t^2+\gamma^2}\right)^{3/4}$$

$$\times \exp\{-4\nu_c\varphi(t,\gamma)\}, \quad (68)$$

where

$$\varphi(t,\gamma) = \frac{1}{1+t}\left\{\mathrm{Arth}\left(\frac{t^2+\gamma^2}{1+\gamma^2}\right)^{1/2} - \left(\frac{t^2+\gamma^2}{1+\gamma^2}\right)^{1/2}\right\}. \quad (69)^*$$

For given ν the function $\varphi(t, \gamma)$ has a unique minimum in the range $-1 \leq t \leq 1$, at the point $t = t_0(\gamma)$ whose position is given by the equation

$$\mathrm{Arth}\left(\frac{t_0^2+\gamma^2}{1+\gamma^2}\right)^{1/2} = \frac{1}{1-t_0}\left(\frac{t_0^2+\gamma^2}{1+\gamma^2}\right)^{1/2}. \quad (70)$$

It is easy to see that $0 \leq t_0(\gamma) \leq 1$; the curve of the function $t_0(\gamma)$ is shown in Fig. 4. For $\gamma \gg 1$ and $\gamma \ll 1$ Eq. (70) can be solved approximately:

$$t_0(\gamma) = \begin{cases} {}^1/_3\gamma^2(1 - {}^{28}/_{45}\gamma^2 + \cdots) & \text{for } \gamma \ll 1 \\ 1 - 1/\ln\gamma + \cdots & \text{for } \gamma \gg 1. \end{cases} \quad (71)$$

The probability w_n, considered as a function of the number n of absorbed quanta, has a flat maximum at $n = n_0 = 2\nu_c/(1+t_0)$, the width of this maximum being of the order $\delta n \sim \nu_n^{1/2} \gg 1$. Replacing the summation over n with an integration, we arrive at the following formula for the total probability of ionization of an s level by a circularly polarized wave:

$$w_c(F, \omega) = \omega_0 |C_{\varkappa 0}|^2 \frac{F}{2F_0} h(\gamma) \exp\left\{-\frac{2}{3}\frac{F_0}{F} g_c(\gamma)\right\}; \quad (72)$$

*Arth ≡ $\tanh^{-1}$.

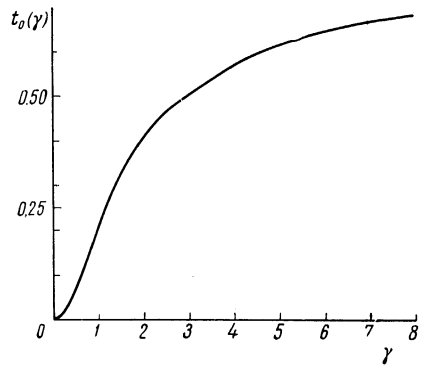

FIG. 4. The root $t_0(\gamma)$ of the transcendental equation (70).

$$g_c(\gamma) = \frac{3t_0}{\gamma^2(1-t_0^2)}\left[(1+\gamma^2)\left(1+\frac{t_0^2}{\gamma^2}\right)\right]^{1/2}, \quad (73)$$

$$h(\gamma) = (1-t_0)\left[\frac{(1+\gamma^2)(1-t_0^2)}{(1+t_0^2/\gamma^2)(1+t_0^2+2t_0^2/\gamma^2)}\right]^{1/2}. \quad (74)$$

By means of (71) one can easily verify that for $\gamma \to 0$ we have $g_c(\gamma) \to 1 - \gamma^2/15$, $h(\gamma) \to 1$, and $w_c(F, \omega)$ goes over into $w_{stat}(F)$. For $\gamma \gg 1$ we have $g_c(\gamma) \sim (3\gamma/2)\ln\gamma$, which leads to a rapid increase of the probability of ionization.

A comparison of (33) and (73) shows that the argument of the exponential in the formula for $w(F, \omega)$ has different frequency dependences for plane and circular polarization—i.e., the exponential is not a universal function of γ. Therefore to obtain the complete picture it is necessary to consider the general case of ionization in the field of an elliptically polarized wave; this will be done in the authors' next paper.

There are always many values of n which contribute to the probability of ionization $w_c(F, \omega)$. Therefore the threshold singularities in the coefficient $h(\gamma)$, unlike those in the case of linear polarization, merge together and disappear.

Lasers existing at present have a fixed frequency; therefore for comparison with experiment it is convenient to rewrite the exponential in the formula for $w(F, \gamma)$ in the form

$$\exp\{-2\omega_0\omega^{-1}f(\gamma)\}, \quad f(\gamma) = {}^2/_3\gamma g(\gamma). \quad (75)$$

The functions $f(\gamma)$ for linear and for circular polarization are shown in Fig. 5. For the light of a ruby laser the parameter γ has a value of 30 to 50 for $F \sim 10^7$ V/cm (see table in [10]). Noting that under these conditions $2\omega_0/\omega$ is of the order of a small multiple of ten, we see that linearly polarized light should ionize atoms much more strongly than circularly polarized light.

In conclusion we note that the formulas derived here for $w(F, \omega)$ apply to the calculation of ionization not only from the ground state, but also from

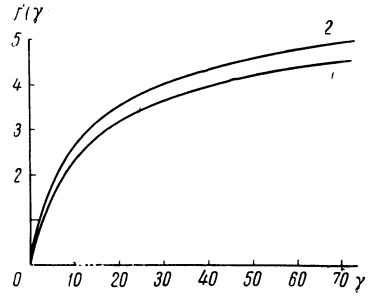

FIG. 5. The functions $f(\gamma)$ for the case of linear polarization (curve 1) and circular polarization (curve 2) of the light.

excited states. In fact, the method developed above for calculating $w(F, \omega)$ is applicable under the condition $\tau \gg T_t$ (τ is the lifetime of the excited state, and $T_t \sim 1/\omega_t$) or $\Gamma/E \ll F/F_0$. For not too high excited states $\Gamma/E \sim 10^{-8}-10^{-10}$ (cf. table in [15] for the hydrogen atom), from which it follows that this condition is satisfied for $F \gtrsim 1000$ V/cm—that is, in the very range of values of F where the ionization effect is experimentally observable.

Note added in proof (March 30, 1966). After this paper had been completed the authors learned that analogous questions had been investigated by another method by Nikishov and Ritus [16]. In the formulas (23), (23′), and (29) found in their paper for the probabilities of ionization of an s level by the fields of waves with linear and circular polarizations there should be an additional factor ½. When this factor is included these equations agree with Eqs. (54) (for $l = 0$) and (72) (for $\gamma \ll 1$) of the present paper. In comparing them it must be kept in mind that the parameter ξ used by Nikishov and Titus is equal to γ^{-1}.

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