A QUANTUM GENERALIZATION OF THE EXPRESSION FOR ENERGY DISSIPATION

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The usual expression for energy dissipation per unit time (formula (3)) under the action of external forces is a quadratic function of the forces. It is shown that this expression is meaningful only in the classical domain when one can neglect the quantum correlations. In the general case, when it is not possible to neglect the quantum correlations, a formula is obtained for the energy dissipation which can be expressed in terms of the average values of the quadratic magnitudes of the external forces and which is valid both in the classical and in the quantum domains.

1. THE well known expression for the Joule losses in a circuit element of resistance R has the form

$$Q = RI^2, \tag{1}$$

where I is the current in the circuit. It can be easily seen that this expression, generally speaking, does not hold in the quantum case (when one should, evidently, interpret I^2 as the average square of the current $\langle I^2 \rangle$). One can easily give an example when this expression is obviously incorrect. Let us take R to be the radiation resistance, and let I denote in this case the derivative of the dipole moment of the system (in the dipole approximation). Then expression (1) can be interpreted as the intensity of spontaneous radiation and is equivalent to the well-known expression

$$Q = 2\ddot{\mathbf{d}}^2 / 3c^3, \tag{2}$$

where $\mathbf{d}$ is the dipole moment. In the quantum case this expression is incorrect, since it differs from zero even in the case when the system is in its ground state and there should be no spontaneous radiation. (In the ground state the mean quadratic fluctuations of the dipole moment, generally speaking, differ from zero.) For the case of radiation the situation has been discussed in detail previously[1,2] and the corresponding quantum generalizations have been obtained.

In the present paper we shall carry out the quantum generalization of the formula for the dissipation of energy per unit time (cf., for example, [2,3])

$$Q = \frac{i\omega}{4} \sum_{ab} (\chi_{ab}^*(\omega) - \chi_{ba}(\omega)) f_{0a} f_{0b}^*. \tag{3}$$

In deriving this classical formula it was assumed that the system is acted upon by forces $f_a(t)$, the energy of interaction with which has the form

$$V = -\sum_a f_a(t) x_a, \tag{4}$$

where x_a are quantities conjugate to f_a, and in the linear approximation

$$\langle x_a(t) \rangle = \mathrm{Re} \sum_b \chi_{ab} f_b(t) = \frac{1}{2} \sum_b (\chi_{ab}(\omega) f_{0b} e^{-i\omega t} + \chi_{ab}^*(\omega) f_{0b}^* e^{i\omega t}), \tag{5}$$

where $\chi_{ab}(\omega)$ is the generalized susceptibility of the system. Formula (3) expresses the energy dissipation per unit time under the action of the sinusoidal force

$$f_a(t) = {}^1/_2 (f_{0a} e^{-i\omega t} + f_{0a}^* e^{i\omega t}). \tag{6}$$

The Hamiltonian operator of the system (Y) upon which the forces (X) are acting can be written in the form

$$\mathcal{H} = H_Y(y_1, y_2 \ldots y_n) + H_X(x_1, x_2 \ldots x_n) - \sum_a x_a y_a. \tag{7}$$

Here the quantities x and y enter in a completely symmetric manner, and one can with equal justification consider the quantities y_a to be the forces, and x_a the quantities conjugate to them. It should also be noted that the Hamiltonian (7) describes a macroscopic system and, generally speaking, depends also on other variables in addition to x and y. We shall define the dissipation of energy per unit time as the average value of the operator $-\dot{H}_X$:

$$Q = -\langle \dot{H}_X \rangle = -\frac{i}{\hbar}\langle [\mathcal{H}, H_X] \rangle = -\sum_a \langle \dot{x}_a y_a \rangle$$
$$= -\sum_a \mathrm{Sp}\,(\rho \dot{x}_a y_a). \quad (8)$$

Here we have made the natural assumption that the forces x commute with y and H_X commutes with H_Y.

In order to determine Q up to the second order in x it is sufficient to find the density matrix ρ in the first approximation with respect to the interaction energy (in the interaction representation)

$$\rho(t) = \rho(0) + \frac{i}{\hbar}\int_0^t \sum_a [x_a(t_1) y_a(t_1), \rho(0)]\, dt_1. \quad (9)$$

We substitute (9) into (8) and assume that the density matrix $\rho(0)$ can be represented in the form of the product $\rho_X(0)\,\rho_Y(0)$, i.e., we assume the statistical independence of the subsystems X and Y at the instant t = 0. As a result we obtain

$$Q = -\frac{i}{\hbar}\sum_{ab}\int_0^t \{\langle \dot{x}_a(t) x_b(t_1)\rangle \langle y_a(t) y_b(t_1)\rangle$$
$$- \langle x_b(t_1)\dot{x}_a(t)\rangle \langle y_b(t_1) y_a(t)\rangle\}\, dt_1. \quad (10)$$

Here the averaging is carried out with the aid of the unperturbed density matrix $\rho(0)$ and, as usual, it is assumed that the quantities y are chosen so that in the absence of the forces x

$$\langle y(t)\rangle = \mathrm{Sp}\,\rho(0) y(t) = 0. \quad (11)$$

The operator $x_a(t)$ in the general case can be represented in the form

$$x_a(t) = \sum_\omega [A_a(\omega) e^{-i\omega t} + A_a^+(\omega) e^{i\omega t}].$$

Since we are going to be interested in the effect of a monochromatic force, then (just as in formula (5)) we shall pick one Fourier component

$$x_a(t) = A_a e^{-i\omega t} + A_a^+ e^{i\omega t}. \quad (12)$$

Further we assume that the unperturbed density matrix $\rho(0)$ describes a stationary state of the system Y, we neglect terms with the double frequency 2ω (the active part of the power) and we let $t \to \infty$. As a result of this we obtain

$$Q = \frac{\omega}{\hbar}\int_0^\infty \sum_{ab} [(\langle A_a^+ A_b\rangle \langle y_a(\tau) y_b(0)\rangle$$
$$- \langle A_b A_a^+\rangle \langle y_b(0) y_a(\tau)\rangle) e^{i\omega t} + (\langle A_b^+ A_a\rangle \langle y_b(0) y_a(\tau)\rangle$$
$$- \langle A_b A_a^+\rangle \langle y_a(\tau) y_b(0)\rangle) e^{-i\omega\tau}]\, d\tau.$$

This expression can be easily transformed into the form

$$Q = \frac{\omega}{2\hbar}\sum_{ab}\int_0^\infty (\langle \{y_a(\tau) y_b(0)\}\rangle \langle [A_a^+ A_b]\rangle e^{i\omega\tau}$$
$$+ \langle [y_a(\tau) y_b(0)]\rangle \langle \{A_a^+ A_b\}\rangle e^{i\omega\tau} + \langle \{y_a(\tau) y_b(0)\}\rangle$$
$$\times \langle [A_b^+ A_a]\rangle e^{-i\omega\tau} - \langle [y_a(\tau) y_b(0)]\rangle \langle \{A_b^+ A_a\}\rangle e^{-i\omega\tau})\, d\tau,$$

where

$$\{AB\} \equiv AB + BA \quad \text{and} \quad [AB] \equiv AB - BA.$$

Further we make use of the expressions for the spectral density of the fluctuations and for the susceptibility of the system (cf., for example, [2,3])

$$(y_a y_b)_\omega = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{2}\langle \{y_a(0) y_b(\tau)\}\rangle e^{i\omega\tau}\, d\tau,$$

$$\chi_{ab}(\omega) = \frac{i}{\hbar}\int_0^\infty \langle [y_a(\tau) y_b(0)]\rangle e^{i\omega\tau}\, d\tau.$$

After straightforward computations we obtain

$$Q = \frac{2\omega\pi}{\hbar}\sum_{ab}\langle [A_a^+ A_b]\rangle (y_b y_a)_\omega - \frac{i\omega}{2}\sum_{ab}(\chi_{ab}(\omega)$$
$$- \chi_{ba}^*(\omega))\langle \{A_a^+ A_b\}\rangle. \quad (13)$$

We pick out from this expression the part associated with the effect of only the forces x. In order to do this we assume that the system Y exists in a state of thermodynamic equilibrium and we utilize the fluctuation-dissipation theorem

$$(y_a y_b)_\omega = \frac{i\hbar}{2\pi}(\chi_{ab}^*(\omega) - \chi_{ba}(\omega))\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1}\right). \quad (14)$$

The dissipation of energy per unit time associated with the effect of only the forces x, can now be obtained from (13) and (14) if we set the temperature of the system Y equal to zero. As a result of this we obtain

$$Q_X = \frac{i\omega}{2}\sum_{ab}(\chi_{ab}^*(\omega) - \chi_{ba}(\omega))\langle \{A_b^+ A_a\} - [A_a A_b^+]\rangle$$
$$= i\omega\sum_{ab}(\chi_{ab}^*(\omega) - \chi_{ba}(\omega))\langle A_b^+ A_a\rangle. \quad (15)$$

An analogous expression could be set up for the power of the losses associated with y if we treat the latter as forces and introduce the corresponding susceptibilities. One can easily verify that in the ground state of the system at T = 0 expression (15) vanishes. In order to do this it is sufficient to establish the relation between the commutators $\langle [A_a A_b^\dagger]\rangle$ and the anti-commutators $\langle \{A_b^\dagger A_a\}\rangle$ in a state of thermodynamic equilibrium. Such a relation can be derived in a manner simi-

lar to the fluctuation-dissipation theorem and has the form

$$\frac{1}{2}\langle\{A_b^+A_a\}\rangle=\left(\frac{1}{2}+\frac{1}{e^{\hbar\omega/kT}-1}\right)\langle[A_aA_b^+]\rangle. \quad (16)$$

For T = 0 it follows from (15) and (16) that also Q = 0.

The energy dissipation associated with the system Y can be introduced both by means of an expression analogous to (15) (we have just referred to that), and also with the aid of (13) and (16) if we assume that the system X is in a state of thermodynamic equilibrium:

$$Q_Y=-\frac{2\pi\omega}{\hbar}\sum_{ab}\langle[A_aA_b^+]\rangle \times\left[(y_ay_b)_\omega-\frac{i\hbar}{4\pi}(\chi_{ab}^*(\omega)-\chi_{ba}(\omega))\right]. \quad (17)$$

From (14) it follows that this expression must vanish for T = 0.

Equations (14) and (16) also guarantee the vanishing of expression (13) for

$$Q=Q_X+Q_Y,$$

if the temperatures of the subsystems X and Y coincide, i.e., if the system X + Y is in a state of thermodynamic equilibrium.

In the case of interaction with a radiation field the quantities A_a and A_a^+ are proportional to the photon annihilation and creation operators. In this case the right-hand side of (15) turns out to be proproportional to the average number of photons of the ν-th mode $\langle n_\nu\rangle=\langle a_\nu^+a_\nu\rangle$. Thus, in this case (15) describes the absorption of photons (or the stimulated emission). (For greater details cf. [1,2].)

We carry out a comparison of expression (15) with the classical expression (3). These expressions formally coincide if the quantities $\frac{1}{2}f_{0a}$ and $\frac{1}{2}f_{0b}^*$ are respectively replaced by the operators A_a and A_b^+, if a definite order is adopted for these operators, and if an average is then taken. In the classical approximation the order of the operators in the product is not important, and (15) goes over into (3). If only one force acts on the system, then (15) goes over into

$$Q_X=\omega\chi''(\omega)(\langle AA^++A^+A\rangle-\langle[AA^+]\rangle)=\omega\chi''(\omega)(\overline{\langle x^2\rangle}-\langle[AA^+]\rangle),$$

where $\overline{\langle x^2\rangle}=\langle AA^++A^+A\rangle$ is the square of the force x averaged over time and over the ensemble, while $\chi''(\omega)=\operatorname{Im}\chi(\omega)$. Thus, the difference from the classical expression of type (1) consists of the additional term $-\omega\chi''(\omega)\langle[AA^+]\rangle$ which, as is well known, vanishes as the Planck constant $\hbar$ tends to zero. At the same time this additional term in the quantum case when $T\to0$ guarantees the vanishing of the quantities Q_X.

In conclusion we make some remarks on the interpretation of the relations obtained here. The question arises as to how valid it is to regard formula (15) as a generalization of formula (3). In the classical domain when the average $\langle A_b^+A_a\rangle$ can be replaced by the product of the averages $\langle A_b^+\rangle\langle A_a\rangle$, formulas (15) and (3) coincide. Thus, when we interpret relation (3) as the energy dissipation under the influence of given forces, then essentially we have in mind that the average values of these forces $\langle A_b\rangle$ are given in the unperturbed system (i.e., without taking into account the interaction between X and Y) (cf., Hamiltonian (7)), while their dispersions $\langle A_b^+A_a\rangle-\langle A_b^+\rangle\langle A_a\rangle$ are negligibly small. Since the energy dissipation is a quadratic function of the acting forces, then in order to be able to utilize the concept of given forces also in the quantum domain, when the dispersions $\langle A_b^+A_a\rangle-\langle A_b^+\rangle\langle A_a\rangle$ are not small, it is necessary to generalize this concept. Such a generalization consists of the fact that we regard as given not the average values of the forces, but their quadratic combinations $\langle A_b^+A_a\rangle$. In case of interaction with the radiation field this means that we regard the number of photons as given (and not the average values of the field operators).

Once again we note that a more detailed illustration of the relations obtained above for the case of interaction with radiation can be found in [1,2].

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