ELECTRON RADIATION STIMULATED IN CROSSED FIELDS

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Stimulated emission and absorption of a relativistic electron in crossed electrostatic and magnetic fields are studied by quantum-mechanical methods. The case when emission is greater than absorption is considered. An expression is deduced for the intensity of the stimulated radiation.

INTEREST in the theory and practice of quantum-mechanical generators and amplifiers whose working media are not atoms or molecules but electron beams controlled by electric and magnetic fields has increased recently.[1-6]

Indeed, an electron executing a finite motion in electromagnetic fields should have discrete energy levels. It is important to note that in all cases of practical interest, such a "macroatom" is a quasiclassical system, whose energy levels are equidistant accurate to terms proportional to $\hbar$. Therefore when radiation of definite frequency interacts with the external field, the electron can absorb a quantum and go over to a higher energy level, or, to the contrary, give up part of its energy and go over to lower levels.

The really observable quantity is the total power radiated (or absorbed) by the electron as a result of the two processes.[7] If conditions are produced, by suitable choice of the corresponding field configuration, whereby the probabilities of the transitions to the lower levels are higher than the probabilities of the transitions to the higher levels, then emission prevails over absorption and the system works like a maser or a laser.

In this paper we examine the radiation of a relativistic electron in light of quantum theory, and show that in the fields used in the analysis of the operation of the magnetron[8] the electron is capable of amplifying external radiation over a wide range of frequencies.

To this end we investigate plane motion of an electron (charge e_0) in a homogeneous magnetic field H (directed along the z axis) with a vector potential

$$\mathbf{A} = \{-\tfrac{1}{2}yH, \tfrac{1}{2}xH, 0\} \tag{1}$$

and in an electrostatic field with potential

$$\varphi = ar^2/2, \quad r^2 = x^2 + y^2, \quad a > 0. \tag{2}$$

To find the wave function of the electron we shall use the Klein equation, since inclusion of the spin properties of the electrons results in an insignificant contribution to the intensity of the radiation:

$$(i\hbar\partial/\partial t + e_0\varphi)^2\psi = (m_0^2c^4 + c^2p^2)\psi, \tag{3}$$

where $\mathbf{p} = -i\hbar\nabla + e_0c^{-1}\mathbf{A}$, and m_0 and $e_- = -e_0$ are the mass and charge of the electron.

We seek the solution of (3) in the form

$$\psi = \frac{1}{\sqrt{2\pi}}\exp\left\{il\varphi - \frac{iEt}{\hbar}\right\}R(r), \tag{4}$$

where E is the total energy and l the orbital number. The function R satisfies the equation

$$R'' + \frac{1}{r}R' + \left[\frac{E^2 - m_0^2c^4}{\hbar^2c^2} - 2\gamma l - \frac{l^2}{r^2} - \gamma^2\left(1 - \frac{4aE}{e_0H^2}\right)r^2\right]R = -\frac{e_0^2a^2}{4c^2\hbar^2}r^4R, \quad \gamma = \frac{e_0H}{2c\hbar}, \tag{5}$$

which has no exact solution. By regarding the term on the right side of (5) as a perturbation (the conditions necessary for this purpose are given later), we obtain the eigenfunctions in the zeroth approximation:

$$R(r) = (2\gamma_1 m_0c^2/E^0_{nl})^{1/2} I_s^{|l|}(\gamma_1 r^2). \tag{6}$$

Here

$$\gamma_1^2 = \gamma^2(1 - 4aE^0_{n,l}/e_0H^2) > 0,$$

$n = 2s + |l|$ (n = 0, 1, 2, ...) is the principal quantum number, s (equal to 0, 1, 2, ...) is the radial quantum number, $I^l_s(x)$ is a function connected with the Laguerre polynomials by the relation

$$I_s^l(x) = \sqrt{\frac{s!}{(s+l)!}}\,e^{-x/2}x^{l/2}L_s^l(x). \tag{6a}$$

To determine the eigenvalues of the energy we obtain the equation

$$E^2 = m_0^2c^4 + e_0Hc\hbar l + e_0Hc\hbar(1 - 4aE/e_0H^2)^{1/2}(n+1), \tag{7}$$

from which we get

$$E_{nl} = E_{nl}^0 - {}^1/_2\hbar(\Omega - \Omega_1)(n+1),$$
$$E_{nl}^0 = [m_0^2c^4 + e_0Hc\hbar(n+l+1)]^{1/2},$$
$$\Omega = e_0Hc/E_{nl}^0, \quad \Omega_1 = \Omega(1 - 4aE_{nl}^0/e_0H^2)^{1/2}. \tag{7a}$$

The condition for applicability of the zeroth approximation is the inequality

$$\left|\frac{e_0^2a^2}{4}\int \psi_{n'l'}\nabla^4\psi_{nl}d^3x\right| \ll |E_{nl}^2 - E_{n'l'}^2|.$$

Let us calculate the frequency and intensity of radiation produced by transition of an electron from the ground state n, l to a state $n' = n - \nu_n$, $l' = l - \nu_l$. The radiation frequency is determined from the energy-momentum conservation law

$$\hbar\omega_{nl}^{n'l'} = E_{nl} - E_{n'l'}, \qquad p_z' = -\hbar\omega_{nl}^{n'l'}c^{-1}\cos\theta, \tag{8}$$

where p_z is the momentum along the z axis. In the vicinity of very large quantum numbers we can put (confining ourselves only to terms proportional to $\hbar$)

$$E_{n'l'} = E_{nl} - {}^1/_2\hbar(\Omega\nu_l + \Omega_1\nu_n). \tag{9}$$

From (9) we see that in a small section of the quasiclassical part of the spectrum the energy levels are equidistant. From (8) and (9) we obtain finally

$$\omega_{nl}^{n'l'} = {}^1/_2(\Omega\nu_l + \Omega\nu_n). \tag{10}$$

In the nonrelativistic case, formula (10) was derived by P. L. Kapitza with the aid of the classical averaging method.[8] Great interest is attached to the conditions under which

$$\nu_l = -\nu_n = \nu_E, \ \omega_{nl}^{n'l'} \equiv \omega_E = \nu_E\omega_E^0, \quad \omega_E^0 = {}^1/_2(\Omega - \Omega_1); \tag{11}$$

$$\nu_l = \nu_n = \nu_H, \quad \omega_{nl}^{n'l'} \equiv \omega_H = \nu_H\omega_H^0, \ \omega_H^0 = {}^1/_2(\Omega + \Omega_1). \tag{12}$$

In the interaction between an electron and a plane electromagnetic wave (frequencies ω, electric field intensity $\mathbf{E}_\lambda$, polarization λ) with a wave vector $\kappa(\theta, \varphi')$, the probability of the induced transitions $nl \to n'l'$ per unit time is determined by the expression

$$W_{nl}^{n'l'}(\omega, \varkappa, \lambda) = 2\pi(e_0/m_0\hbar\omega)^2 |M_{\lambda, nl}^{n'l'}|^2 g_{nl}^{n'l'}(\omega) u_{\varkappa\lambda}, \tag{13}$$

$$g_{nl}^{n'l'}(\omega) = \frac{\tau}{\tau^2(\omega_{nl}^{n'l'} - \omega)^2 + {}^1/_4}, \tag{14}$$

where τ is the lifetime of the initial state, which does not depend on the quantum numbers n, l, n', or l,

$$M_{\lambda, nl}^{n', l'} = \int \psi_{n'l'}^+ e^{-i\varkappa\mathbf{r}}(\mathbf{a}_\lambda\mathbf{p})\psi_{nl}d^3x \tag{15}$$

is the matrix element, $\mathbf{a}_\lambda$ is the polarization vector, and $u_{\kappa,\lambda}$ the spectral density of the external radiation in the frequency interval $d\omega$ and in the solid-angle interval $d\underline{o}$ with polarization λ. The coefficient preceding $u_{\kappa,\lambda}$ in the right side of (13) is the well known Einstein differential coefficient.[9]

To calculate the probability of the induced absorption (transitions n, $l \to n''$, l'') it is necessary to make in (15) the substitutions $n, l' \to n''$, l'' and n', $l' \to n$, l, and go over from the matrix element M_λ to its adjoint. The matrix elements (15) are calculated in the Appendix.

Assuming that the electron is first in a state with quantum numbers n and l, let us consider the system of levels $E_{n'',l''} > E_{n,l} > E_{n',l'}$, the distances between which are equal to $\nu_E\omega_E^0$, that is

$$n'' = n - \nu_E, \quad l'' = l + \nu_E,$$
$$n' = n + \nu_E, \quad l' = l - \nu_E. \tag{16}$$

We write down further an expression for the total power $p_{\nu_E}(\omega, \kappa, \lambda)$ absorbed by the electron in the frequency and angle intervals $d\omega$ and $d\underline{o}$:

$$P_{\nu_E}(\omega, \varkappa, \lambda) = \hbar\omega_{nl}^{n''l''}W_{nl}^{n''l''} - \hbar\omega_{nl}^{n'l'}W_{nl}^{n'l'}. \tag{17}$$

Substituting in (17) the values of the probabilities of the transitions and taking formulas (A.15–A.22), (11), and (16) into account, we obtain the power absorbed in the interaction of an electron with an unpolarized electromagnetic wave:

$$P_{\nu_E}(\omega, \varkappa) = -\frac{\pi e_0^2(\omega_E^0)^3\nu_E}{\omega^2\Omega_1E_{nl}^0}\frac{\tau}{\tau^2(\omega - \nu_E\omega_E^0)^2 + {}^1/_4}u_\varkappa$$
$$\cdot\left\{4x\left[\left(\frac{d}{dx}I_{(n-l)/2}^{\nu_E}\right)^2 - \left(\frac{d}{dx}I_{(n-l)/2-\nu_E}^{\nu_E}\right)^2\right]\right.$$
$$\left. + \frac{\nu_E^2\cos^2\theta}{x}[(I_{(n-l)/2}^{\nu_E})^2 - (I_{(n-l)/2-\nu_E}^{\nu_E})^2]\right\}(I_{(n+l)/2}^0)^2, \tag{18}$$

where

$$x = \frac{\omega^2\sin^2\theta}{4\gamma_1} = \frac{\omega^2\hbar\sin^2\theta}{2\Omega_1E_{n,l}^0}.$$

For further analysis of formula (18), let us estimate the quantum numbers. The solution of the classical equations of motion has in our approximation the form

$$X = R_0\cos(\omega_H^0t + \alpha_H) + r_0\cos(\omega_E^0t + \alpha_E), \tag{19}$$

$$Y = R_0\sin(\omega_H^0t + \alpha_H) + r_0\sin(\omega_E^0t + \alpha_E). \tag{20}$$

Comparing the expressions for the energy and momentum obtained by the quantum and classical theories, we get approximately

$$R_0^2 = \frac{(n+l+1)\hbar c}{e_0H}, \quad r_0^2 = \frac{(n-l+1)\hbar c}{e_0H}. \tag{21}$$

We note that $R_0 > r_0$, when $l > 0$, and $R_0 < r_0$ when $l < 0$. The values of R_0 and r_0 are determined by initial conditions. For example, if an electron with energy E_0 moved along a circle of radius d with frequency ω prior to turning on the electric field, then

$$R_0 = d\frac{\omega_H^0}{\Omega_1}, \qquad r_0 = d\frac{\omega_E^0}{\Omega_1}. \tag{22}$$

For $a = 1.6 \times 10^4$ V/cm^2, $H = 1.6 \times 10^4$ G, $E^0 = 50$ MeV, and $d = 10$ cm we get $\lambda_E = 2\pi c/\omega_E = 1800\ \nu_E^{-1}$ cm, $R_0 = 10.3$ cm, $r_0 = 0.3$ cm, $(n + l)/2 = 10^{13}$, and $(n - l)/2 = 10^{11}$.

Since the quantum numbers are quite large, we can use the asymptotic formulas (A.13) and (A.14), taking into account the fact that $|x| \ll 1$. Expression (18) takes the form

$$P_{\nu_E}(\omega, \varkappa) = -\frac{4\pi e_0^2 c^2 (\omega_E^0)^3 \nu_E^3}{\omega^2 \Omega_1 E_{nl}^0 \eta_0} \frac{\tau u_\varkappa}{\tau^2(\omega - \nu_E\omega_E^0)^2 + 1/4}$$
$$\times\left[\left(1 - \frac{\eta_0^2}{\nu_E^2}\right) + \cos^2\theta\right] J_{\nu_E}(\eta_0)\frac{d}{d\eta_0} J_{\nu_E}(\eta_0),$$
$$\eta_0 = \frac{r_0\omega \sin\theta}{c}. \tag{23}$$

In the nonrelativistic case the maximum power is emitted in the case of dipole transitions with $\nu_E = 1$[3]

$$P_1 = -\frac{\pi e_0^2 (\omega_E^0)^3}{\omega^2 \Omega_1 m\tau} \frac{1 + \cos^2\theta}{(\omega - \omega_E^0)^2 + 1/4\tau^2} u_\varkappa. \tag{24}$$

It follows from (23) and (24) that $P_{\nu_E} < 0$, which corresponds to negative resonant absorption.

We have thus demonstrated in the approximation (9), which leads to an equidistant spectrum of the levels, the possibility of amplification of external radiation at the frequency $\nu_E\omega_E^0$. It is important to note that the intensity of the radiation does not contain Planck's constant, thus pointing to a classical nature of the effect. We can show similarly that at the frequency $\nu_H\omega_H^0$, only resonant absorption is possible.

In the case when $a < 0$ (cylindrical oscillator in a magnetic field), resonant absorption will be observed at frequencies that are multiples of $(\Omega_1 + \Omega)/2$ and $(\Omega_1 - \Omega)/2$.

However, inclusion of terms proportional to $\hbar^2$ in (9) leads to unequal spacing of the levels. In this case the corresponding calculation shows that at the frequencies ω close to resonance an external radiation can be amplified (see also [1, 2]) and that Planck's constant does not enter in the final expression for the radiation intensity.

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APPENDIX

The function $I_s^l(x)$ satisfies the equation

$$\frac{d^2I_s^l}{dx^2} + \frac{1}{x}\frac{dI_s^l}{dx} + \left[\frac{4s + 2l + 2}{4x} - \frac{1}{4} - \frac{l^2}{4x^2}\right] I_s^l = 0 \tag{A.1}$$

We present below several relations which are useful in the calculations:

$$\sqrt{x}\, I_s^l = \sqrt{s + l + 1}\, I_s^{l+1} - \sqrt{s}\, I_{s-1}^{l+1}, \tag{A.2}$$

$$\sqrt{x}\, I_s^l = \sqrt{s + l}\, I_s^{l-1} - \sqrt{s + 1}\, I_{s+1}^{l-1}, \tag{A.3}$$

$$\left(\frac{d}{dx} - \frac{l}{2x} - \frac{1}{2}\right) I_s^l = -\sqrt{\frac{s + l + 1}{x}}\, I_s^{l+1}, \tag{A.4}$$

$$\left(\frac{d}{dx} + \frac{l}{2x} + \frac{1}{2}\right) I_s^l = \sqrt{\frac{s + l}{x}}\, I_s^{l-1}, \tag{A.5}$$

$$\left(\frac{d}{dx} - \frac{l}{2x} + \frac{1}{2}\right) I_s^l = -\sqrt{\frac{s}{x}}\, I_{s-1}^{l+1}, \tag{A.6}$$

$$\left(\frac{d}{dx} + \frac{l}{2x} - \frac{1}{2}\right) I_s^l = \sqrt{\frac{s + 1}{x}}\, I_{s+1}^{l-1}. \tag{A.7}$$

From (A.4)–(A.7) it follows that

$$\sqrt{s + l + 1}\, I_s^{l+1} - \sqrt{s + l}\, I_s^{l-1} = -2\sqrt{x}\frac{dI_s^l}{dx}, \tag{A.8}$$

$$\sqrt{s + l + 1}\, I_s^{l+1} + \sqrt{s + l}\, I_s^{l-1} = \frac{l + x}{\sqrt{x}} I_s^l, \tag{A.9}$$

$$\sqrt{s + 1}\, I_{s+1}^{l-1} - \sqrt{s}\, I_{s-1}^{l+1} = 2\sqrt{x}\frac{dI_s^l}{dx}, \tag{A.10}$$

$$\sqrt{s + 1}\, I_{s+1}^{l-1} + \sqrt{s}\, I_{s-1}^{l+1} = \frac{l - x}{\sqrt{x}} I_s^l, \tag{A.11}$$

$$I_s^{-l}(x) = (-)^l I_{s-l}^l(x). \tag{A.12}$$

For small values of x the following relationships are valid

$$\lim_{n\to\infty} I_s^l(x) = J_l(\sqrt{2(n + 1)x}), \qquad n = 2s + l, \tag{A.13}$$

$$\lim_{x\to 0} I_s^l(x) = \frac{1}{l!}\sqrt{\frac{(s + l)!}{s!}}\, x^{l/2}. \tag{A.14}$$

When investigating the polarization of the radiation we should substitute in (14) in lieu of a_λ the values corresponding to two linear or two circular polarizations.

For the σ-component of the linear polarization (E_σ lies in the xy plane) we have

$$M_\sigma = {}^1/_2 i M_-. \tag{A.15}$$

For the σ-components of the linear polarization (E_π parallel to the vector $\kappa \times E_\sigma$)

$$M_\pi = {}^1/_2 \cos\theta M_+ - \sin\theta M_z. \tag{A.16}$$

For right-hand ($m = 1$) and left-hand ($m = -1$)

circular polarizations we have respectively

$$M_m = (M_\sigma + imM_\pi)/\sqrt{2}. \tag{A.17}$$

In formulas (A.15)–(A.17) we used the following notation

$$M_\pm = \int \psi^+_{n'l'} e^{-ikr} (e^{-i\varphi'} p_+ \pm e^{i\varphi'} p_-) \psi_{nl}\, d^3x, \tag{A.18}$$

$$M_z = \int \psi^+_{n'l'} e^{-ikr} p_z \psi_{nl}\, d^3x, \quad p_\pm = p_x \pm i p_y, \tag{A.19}$$

$$M_- = \frac{-2i\hbar m_0 c^2 \sqrt{x}}{\sqrt{\gamma_1 E_{nl}{}^0 E_{n'l'}{}^0}} \times \Big[(\gamma + \gamma_1) I^{(n-l)/2-(n'-l')/2}_{(n'-l')/2}(x) \frac{d}{dx} I^{(n+l)/2-(n'+l')/2}_{(n'+l')/2}(x) + (\gamma - \gamma_1) \frac{d}{dx} I^{(n-l)/2-(n'-l')/2}_{(n'-l')/2}(x) I^{(n+l)/2-(n'+l')/2}_{(n'+l')/2}(x) \Big], \tag{A.20}$$

$$M_+ = \frac{i\hbar m_0 c^2}{(\gamma_1 x E_{nl}{}^0 E_{n'l'}{}^0)^{1/2}} \Big[(\gamma + \gamma_1) \Big(\frac{n+l}{2} - \frac{n'+l'}{2} + x \Big) - (\gamma - \gamma_1) \Big(\frac{n-l}{2} - \frac{n'-l'}{2} + x \Big) \Big] \times I^{(n-l)/2-(n'-l')/2}_{(n'-l')/2}(x) I^{(n+l)/2-(n'+l')/2}_{(n'+l')/2}(x), \tag{A.21}$$

$$M_z = -i\hbar p_z I^{(n-l)/2-(n'-l')/2}_{(n'-l')/2}(x) I^{(n+l)/2-(n'+l')/2}_{(n'+l')/2}(x), \tag{A.22}$$

Where the functions I^l_s are defined by relations (6a). Expressions (A.20)–(A.22) are valid for both positive and negative values of l.

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