ON THE THEORY OF MULTIPLE SCATTERING

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Submitted to JETP editor, November 11, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 1081-1083 (April, 1966)

An expression for the scattering cross section of relativistic electrons into fairly large angles is obtained. It takes account of the finite size of the nucleus and the effect of multiple scattering for the case when the correction to the single-scattering formula is relatively small.

THE angular distribution of the large-angle scattering of fast electrons in matter can be calculated on the basis of the well developed theory of multiple scattering.[1-3] However, this involves numerical computations which become more and more complex as the scattering angle increases. Below we shall use the Green's function method developed by Fradkin and Milekhin[4-5] (cf. also the work of Kalashnikov and Ryazanov[6]). This allows us to obtain the final result in an explicit form, with logarithmic accuracy.

For the following considerations it is important that for $\sigma nL \ll 1$, where σ is the cross section for large angle scattering, n is the density of scattering centers, and L is the target thickness, single scattering into a large angle is much more probable than double or multiple scattering. The inequality just mentioned is usually fulfilled owing to the sharp anisotropy of the scattering of relativistic electrons (we are here considering energies of tens of MeV) and corresponds to the fact that the observed cross section differs little from the "tail" of the single scattering cross section. We shall therefore describe the behavior of the electron by a Green's function in which the interaction of the electron with the medium comes about through the exchange of one hard "photon" and an arbitrary number of soft ones. If we denote the initial and final momenta by p and k, respectively, this Green's function can be written for a Dirac electron[5]

$$G(p,k) = -ie(i\gamma_\nu k_\nu + m)\gamma_\mu(i\gamma_\nu p_\nu + m) \times \int d^4x e^{i(p-k)x} A_\mu(x) \int_0^\infty ds_1 \int_0^\infty ds_2 \exp\Big\{-is_1(p^2+m^2) - is_2(k^2+m^2) - 2ie\int_0^{s_1} dt_1\, p_\nu a_\nu(x-2pt_1) - 2ie\int_0^{s_2} dt_2\, k_\nu a_\nu(x+2kt_2)\Big\}. \quad (1)$$

Here A_μ is the field of that atom which is responsible for the main part of the scattering with large momentum transfers, and a_μ is the field of the remaining atoms: since the momentum transfer is much larger than the reciprocal dimensions of the atom, we must take for A_μ only the field of the nucleus. In the following we shall consider only such target thicknesses L that $\sigma nL \ll 1$ as well as the condition $na^2L \gg 1$ are fulfilled, where a is a characteristic dimension of the atom.

In formula (1), the principal scattering is included in Born approximation, which is valid only for light nuclei; it is therefore reasonable to generalize the expression $\gamma_\mu A_\mu$ by replacing it by the complete invariant electron-nucleus scattering amplitude. In the following we restrict ourselves to electron energies E which do not exceed 50 to 100 MeV; then only the nuclear charge distribution is important and we may take $i\gamma_4\Phi$ instead of $\gamma_\mu A_\mu$. The Fourier transform of Φ is written in the form

$$|e\tilde{\Phi}(\mathbf{q})|^2 = (4\pi Ze^2/\mathbf{q}^2)^2 F(|\mathbf{q}|R), \quad (2)$$

where F is the nuclear electric form factor and R is a characteristic dimension of the nucleus. In Born approximation the quantity F is equal to the square modulus of the Fourier transform of the nuclear charge density distribution. Inelastic processes can be neglected for $E \lesssim 50$ MeV[7] (a simple method for taking inelasticity into account by changing the form of F is presented in the work of Cooper and Rainwater[8]). In the potentials a_μ one may take the fourth component $a_4 = i\varphi$ as the only nonvanishing component.

Multiplying (1) on the left by $\bar{u}_f(\mathbf{k})(i\gamma_\nu k_\nu - m)$ and on the right by $(i\gamma_\nu p_\nu - m)u_i(\mathbf{p})$ and taking the limit $i\gamma_\nu k_\nu \to i\gamma_\nu p_\nu \to m$, we find an expression for the scattering matrix element ($\bar{u}_f$ and u_i are the spinors of the final and initial electron

states). Making the usual transition from the matrix element to the differential cross section and averaging over the spin orientations of the electron, we obtain

$$d\sigma = \mathbf{p}^2(1 - v^2 \sin^2\theta/2) I d\Omega / 4\pi^2 v^2, \tag{3}$$

$$I = \int\int d^3x\, d^3y \exp[i(\mathbf{p}-\mathbf{k})(\mathbf{x}-\mathbf{y})]\, e^2\Phi(\mathbf{x})\Phi^*(\mathbf{y}) \times \exp\left\{2ieE\int_0^\infty ds[\varphi(\mathbf{x}-2\mathbf{p}s)+\varphi(\mathbf{x}+2\mathbf{k}s) - \varphi(\mathbf{y}-2\mathbf{p}s)-\varphi(\mathbf{y}+2\mathbf{k}s)]\right\}. \tag{4}$$

Here θ is the scattering angle of the electron, E is its energy, and v is the velocity in units of the light velocity.

The potentials φ are composed of the potentials of the separate scattering centers

$$\varphi(\mathbf{x}) = \sum_{i=1}^{N} V(|\mathbf{x}-\mathbf{x}_i|)$$

and the cross section depends on the position of each atom. Naturally, we are interested in an averaged expression. We shall assume that the scatterers are distributed randomly, i.e., all coordinates within the boundaries of the target, a plate of thickness L, are equally probable for each $\mathbf{x}_i$. Then the averaging can be carried out in the asymptotic limit ($N \to \infty$) and we obtain for the average of the last exponential in (4)

$$\exp\left\{n\int d^3z\left[\exp\left(2ieE\int_0^\infty ds[V(\mathbf{x}-2\mathbf{p}s-\mathbf{z}) - V(\mathbf{y}-2\mathbf{p}s-\mathbf{z}) + V(\mathbf{x}+2\mathbf{k}s-\mathbf{z}) - V(\mathbf{y}+2\mathbf{k}s-\mathbf{z})]\right) - 1\right]\right\}. \tag{5}$$

It is easy to see that at the points where the exponent of the inner exponential in (5) is not close to zero, the integral over $\mathbf{z}$ has a negative real part which is equal in order of magnitude to na^2L, where a is an atomic dimension, and thus much larger than unity. Therefore the main contribution to $\langle I\rangle$ comes from the region of integration where the inner exponential is almost canceled by unity. This, in turn, allows us to restrict ourselves to the first nonvanishing terms of the expansion of this exponential. Then it turns out that the first order terms do not contain L and can be discarded, neglecting quantities of order na^3. Computing the second order terms with the same accuracy, we obtain for (5)

$$\exp\{-\alpha[(L+x_1+y_1)(\mathbf{x}-\mathbf{y})^2_{\perp\mathbf{p}} + (L-x_1-y_1)(\mathbf{x}-\mathbf{y})^2_{\perp\mathbf{k}}/\cos\theta]\}, \tag{6}$$

where

$$\alpha = \frac{\pi n e^2}{v^2}\int_0^\infty du\int_0^\infty dv\int_0^\infty dw\, w \times \frac{d}{dw}V(\sqrt{u^2+w^2})\frac{d}{dw}V(\sqrt{v^2+w^2}) \tag{7}$$

(the momentum p and the 1 axis are perpendicular to the target plate).

Substituting (6) and (7) in the expression for $\langle I\rangle$ and going to momentum space, we see that the characteristic momentum transfer corresponding to the combined effect of all scatterings with small momentum transfers is equal to $(\alpha L)^{1/2}$ in order of magnitude. In accordance with our basic assumptions, this means that the following inequality must be satisfied:

$$\alpha L \ll (\mathbf{p}-\mathbf{k})^2 = (\Delta p)^2. \tag{8}$$

Taking this into account, we can, in the expression for $\langle I\rangle$, expand the function $\tilde{\Phi}(\mathbf{k}-\mathbf{p}+\mathbf{q})$ under the integral sign in powers of $\mathbf{q}$, so that all necessary integrations can be carried out in elementary fashion. Averaging, moreover, over all possible positions of the principal scatterer, we obtain for the cross section

$$d\sigma = d\sigma_{Mott} F(\Delta pR)\left\{1 + \frac{\alpha L}{(\Delta p)^2}\frac{1+\cos\theta}{2\cos\theta}[8(1+3\cos\theta) - (5+9\cos\theta)\Delta pRF'/F + (1+\cos\theta)(\Delta pR)^2F''/F]\right\}. \tag{9}$$

The second term in the curly bracket gives a correction due to the multiple scattering. It remains to give an explicit expression for the constant α. In general the integral (7) is logarithmically divergent at small distances if one takes a screened atomic potential for V. We must, however, recognize that α describes the scattering with momentum transfer q_{max}, which is smaller than Δp. Since q_{max} enters in the result as an argument of a logarithm, we can replace it by Δp with logarithmic accuracy. Then we obtain from (7)

$$\alpha = (\pi n (Ze^2)^2/v^2)\ln(a\Delta p). \tag{10}$$

Formulas (9) and (10) give the expression for the cross section. When the scattering angle is small ($\theta \ll 1$) and the nucleus is regarded as point-like ($F \equiv 1$), these two formulas go over, with logarithmic accuracy, into the known result of Moliére.[9]

The author is grateful to E. L. Feĭnberg for a discussion of this work.

[1] S. A Goudsmit and J. L. Saunderson, Phys. Rev. 57, 24 (1940) and 58, 36 (1940).

[2] M. G. Wang and E. Guth, Phys. Rev. 84, 1092 (1951).

[3] Yu. N. Gnedin and A. Z. Dolginov, JETP **45**, 1136 (1963) and **48**, 548 (1965), Soviet Phys. JETP **18**, 784 (1964) and **21**, 364 (1965).

[4] E. S. Fradkin, Dissertation, ITEF, 1960.

[5] G. A. Milekhin and E. S. Fradkin, JETP **45**, 1926 (1963), Soviet Phys. JETP **18**, 1323 (1964).

[6] N. P. Kalashnikov and M. I. Ryazanov, JETP **47**, 1055 (1964), Soviet Phys. JETP **20**, 707 (1965).

[7] M. E. Rose, Phys. Rev. **73**, 279 (1948).

[8] L. N. Cooper and J. Rainwater, Phys. Rev. **97**, 492 (1955).

[9] G. Moliére, Z. Naturforschg. **3a**, 78 (1948) and **10a**, 177 (1955).

Translated by R. Lipperheide
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