QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN A DISPERSIVE MEDIUM

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Submitted to JETP editor June 15, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 915-925 (April, 1966)

The free macroscopic electromagnetic field in a homogeneous, transparent, nonmagnetic, dispersive, and anisotropic medium is quantized taking into account all types of waves excited in the medium. The Green's function and the retarded Green's function of the electromagnetic field are obtained. The method is used to determine the radiation from an impurity atom in an anisotropic dispersive medium. Selection rules for the emission of longitudinal quanta are derived. The probability for long-wave emission of transverse and longitudinal quanta is calculated for an impurity molecule located in a homogeneous isotropic plasma with spatial dispersion.

In problems of crystal optics and plasma physics as well as in the investigation of electromagnetic phenomena, it often occurs that only the long-wave part of the spectrum is important. Then the electromagnetic field can be regarded as macroscopic. In such problems the quantum approach may be very useful, as it is based on the well developed methods of quantum electrodynamics. In order to translate the methods of quantum electrodynamics into the above-mentioned region, we must first of all quantize the macroscopic electromagnetic field in the medium. The phenomenological approach of taking account of the medium through the introduction of the dielectric tensor is often sufficient. In the absence of dispersion and absorption such a quantization of the field has been carried out earlier for isotropic [1-3] and anisotropic media.[4] Some problems of the phenomenological quantum electrodynamics in a transparent dispersive isotropic medium have been considered in [5-7]. In absorptive media the quantum approach is based on the use of Green's functions.[8]

Below we shall quantize the free macroscopic electromagnetic field in a transparent, nonmagnetic, dispersive, and anisotropic medium. By free macroscopic field we always understand a radiation field which can be excited in the medium and exists thereafter independently of its sources (variable charges and currents). In the Schrödinger representation the operator of the interaction of an arbitrary quantum system with the free macroscopic field is determined by the laws of quantum mechanics. The interaction with other fields is accounted for in the usual way. Hence the result of the quantization given below allows us to solve the electrodynamical problems in a transparent dispersive medium by the methods of quantum electrodynamics. This circumstance is of paramount importance for problems in which the classical description loses its power.[9]

For the sake of generality we have considered media with frequency and spatial dispersion whose dielectric tensor $\epsilon_{\alpha\beta}(\omega, \mathbf{k})$ is known. In contrast to nondispersive media, we encounter here a number of new features, which justify a detailed consideration of the field quantization in a dispersive medium. These features are essentially connected with the fact that the number of wave types in a dispersive medium is greater.[10] We use a classification for these which arises in natural fashion from the formal solution of the field equations for a medium with given $\epsilon_{\alpha\beta}(\omega, \mathbf{k})$. The generalization to other cases is obvious.

As an illustration we consider the long wave radiation of an impurity atom in an anisotropic dispersive medium. In the transition from a given excited level, the impurity atom may with definite probability emit quanta of different types, which have the same energy but different polarizations and dispersion laws. This circumstance leads to a spreading of the spectral line owing to the increase in the probability of an atomic transition from the excited level. Among the emitted quanta there are also longitudinal quanta, which do not occur in the absence of dispersion. The formula for the radiation and the selection rules for the emission of longitudinal quanta have their specific character. For example, magnetic dipole transitions are forbidden, and the $0 \to 0$ quadrupole transition is allowed. Analogous remarks apply

also to the radiation from an impurity atom in an isotropic medium with spatial dispersion. In an anisotropic medium the longitudinal quanta may have a distinguished direction of propagation,[10] which has a noticeable effect on their emission probability. In particular, longitudinal quanta which propagate only along a distinguished axis or in a plane are not emitted by individual impurity atoms in an infinite anisotropic medium. Such longitudinal quanta can be emitted, for example, in a wave-guide or a resonator filled with an anisotropic dielectric if the axis of the waveguide (resonator) coincides with the distinguished direction of the wave vector of the longitudinal quantum.

The resulting formulas for the radiation of an impurity atom in an anisotropic medium allow for a new approach to the explanation of the polarization of the light of luminescent crystals and also of the solid state lasers[4]. In particular, these formulas give the possibility, using the correspondence principle, to determine the angular distribution and the frequency spectrum of the radiation of classical (not quantal) objects in the anisotropic medium.

1. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

As usual, we assume that the free electromagnetic field is enclosed in a volume V of the form of a right parallelepiped whose linear dimensions are large compared with the characteristic wave lengths of the problem under consideration. We choose a gauge in which the scalar potential is identically zero. The vector potential is written as a superposition of plane waves:

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}\lambda}[a(\mathbf{k},\lambda;t)e^{i\mathbf{kx}}+a^*(\mathbf{k},\lambda;t)e^{-i\mathbf{kx}}]\,\mathbf{l}^{\mathbf{k}\lambda}$$

$$+\frac{1}{\sqrt{V}}\sum_{\mathbf{k}\nu}[a^l(\mathbf{k},\nu;t)e^{i\mathbf{kx}}+a^{l*}(\mathbf{k},\nu;t)e^{-i\mathbf{kx}}]\frac{\mathbf{k}}{k}, \qquad (1)$$

where $\mathbf{k}$ runs through a discrete sequence of values determined by the expansion of the free field in a triple Fourier series, and the amplitudes $a(\mathbf{k},\lambda;t)$ and $a^l(\mathbf{k},\nu;t)$ depend on the time t through the factors $\exp(-i\omega_{\mathbf{k}\lambda}t)$ and $\exp(-i\omega_{\mathbf{k}\nu}t)$, respectively.

For each value of the wave vector $\mathbf{k}$ in (1) the frequencies $\omega_{\mathbf{k}\nu}$ ($\nu = 1,2,\ldots$) of the purely longitudinal waves as functions of $\mathbf{k}$ are determined by the solution of the equation

$$|\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\nu},\mathbf{k})|=0, \qquad (2)$$

which follows from the equation for the longitudinal waves

$$\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\nu},\mathbf{k})k_\beta=0. \qquad (3)$$

The frequencies $\omega_{\mathbf{k}\lambda}$ ($\lambda = 1,2,\ldots$) and the directions of the polarization vectors $\mathbf{l}^{\mathbf{k}\lambda}$ of the ordinary waves[1)] are found by solving the system of algebraic equations

$$[c^2(k_\alpha k_\beta-k^2\delta_{\alpha\beta})+\omega_{\mathbf{k}\lambda}^2\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\lambda},\mathbf{k})]l_\beta^{\mathbf{k}\lambda}=0. \qquad (4)$$

Setting the determinant of the system (4) equal to zero, we obtain the known Fresnel equation

$$|c^2(k_\alpha k_\beta-k^2\delta_{\alpha\beta})+\omega_{\mathbf{k}\lambda}^2\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\lambda},\mathbf{k})|=0, \qquad (5)$$

which for each fixed value of the wave vector $\mathbf{k}$ determines the frequency $\omega_{\mathbf{k}\lambda}$ as a function of $\mathbf{k}$:

$$\omega_{\mathbf{k}\lambda}=\omega_{\mathbf{k}\lambda}(k,\mathbf{k}/k). \qquad (6)$$

The index λ runs through the integer values $\lambda = 1,2,\ldots$, labeling the different roots of the Fresnel equation (5). For each value of the frequency (6) Eq. (4) determines the direction of the unit vector of polarization $\mathbf{l}^{\mathbf{k}\lambda}$ in the form

$$\mathbf{l}^{\mathbf{k}\lambda}=\left(\boldsymbol{\tau}^{\mathbf{k}\lambda}-\mathbf{k}\frac{k_\alpha\tau_\beta^{\mathbf{k}\lambda}\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\lambda},\mathbf{k})}{k_{\alpha'}k_{\beta'}\varepsilon_{\alpha'\beta'}(\omega_{\mathbf{k}\lambda},\mathbf{k})}\right)\frac{1}{g_{\mathbf{k}\lambda}}, \qquad (7)$$

where

$$g_{\mathbf{k}\lambda}^2=1+k^2\left(\frac{k_\alpha\tau_\beta^{\mathbf{k}\lambda}\varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\lambda},\mathbf{k})}{k_{\alpha'}k_{\beta'}\varepsilon_{\alpha'\beta'}(\omega_{\mathbf{k}\lambda},\mathbf{k})}\right)^2,$$

and $\boldsymbol{\tau}^{\mathbf{k}\lambda}$ is a unit vector perpendicular to $\mathbf{k}$. The transverse vector $\boldsymbol{\tau}^{\mathbf{k}\lambda}$ is easily determined by going to a coordinate system with a Z axis directed along $\mathbf{k}$. In this coordinate system the vector $\boldsymbol{\tau}^{\mathbf{k}\lambda}$ is directed along one of the two principal axes of the two-dimensional reciprocal dielectric tensor $\epsilon_{\alpha\beta}^{-1}(\omega_{\mathbf{k}\lambda},\mathbf{k})$, where the indices α and β run through the two values 1 and 2 corresponding to the projections on the X and Y axes. By a coordinate transformation we find the vector $\boldsymbol{\tau}^{\mathbf{k}\lambda}$ in an arbitrary reference system.

In the chosen gauge of the potentials, the Maxwell equations together with the relations (4) and (7) lead to the following general relations for the polarization vectors of ordinary waves:

$$\varepsilon_{\alpha\beta}k_\alpha l_\beta^{\mathbf{k}\lambda}=0, \qquad (8)$$

$$\varepsilon_{\alpha\beta}l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda'}=\varepsilon_{\alpha\beta}l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda}\delta_{\lambda\lambda'}, \qquad (9)$$

$$\frac{\partial(\omega_{\mathbf{k}\lambda}^2\varepsilon_{\alpha\beta})}{\partial\omega_{\mathbf{k}\lambda}}l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda}=\varepsilon_{\alpha\beta}l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda}\frac{2\omega_{\mathbf{k}\lambda}^2}{k}\frac{\partial k}{\partial\omega_{\mathbf{k}\lambda}}, \qquad (10)$$

where $\omega_{\mathbf{k}\lambda}$, as a function of $\mathbf{k}$ is determined

[1)] For brevity we call ordinary those plane waves for which the dispersion law and the polarization vector have the forms (6) and (7), respectively. In the case of a uniaxial crystal these waves are further subdivided into ordinary and extraordinary waves.

from the Fresnel equation (5), and the unit polarization vector $l^{\mathbf{k}\lambda}$ refers to given $\mathbf{k}$ and $\omega_{\mathbf{k}\lambda}$. In all formulas (8) to (10) the partial derivatives are taken for fixed direction $\mathbf{k}$, and the dielectric permittivity tensor $\epsilon_{\alpha\beta}$ depends on the frequency $\omega_{\mathbf{k}\lambda}$ and on the wave vector $\mathbf{k}$:

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\omega_{\mathbf{k}\lambda}, \mathbf{k}).$$

In an arbitrary gauge of the potentials, Eq. (4) and the relations (7) to (10) are satisfied by a unit vector directed along the electric vector of a monochromatic plane wave. Purely longitudinal waves do not, of course, satisfy equations (7), (9), and (10), since for these equation (4) turns into (3) with the necessary condition (2).

It is known that in a transparent medium with frequency and spatial dispersion the energy of a packet of monochromatic waves with wave vector $\mathbf{k}$ and a narrow distribution of frequencies near the main frequency ω is given by[11]

$$\frac{1}{8\pi}\int\left[\frac{\partial(\omega\varepsilon_{\alpha\beta})}{\partial\omega}\overline{E_\alpha(\omega)E_\beta(\omega)}+\overline{H_\alpha(\omega)H_\beta(\omega)}\right]dV, \quad (11)$$

where the bar denotes the time average over the period $2\pi/\omega$, and the electric and magnetic field intensities $\mathbf{E}(\omega)$ and $\mathbf{H}(\omega)$ have a harmonic time dependence with a slowly varying amplitude.

The expression (11) does not depend on the small width of the distribution of frequencies near the main frequency ω and therefore remains valid in the limit of an infinitely narrow packet. This means that the energy of a monochromatic wave in a transparent dispersive anisotropic medium is also given by (11). Hence the total energy of the electromagnetic field in a transparent dispersive anisotropic medium can be written as a sum of the energies of the individual plane waves. Then the quantization of the field is carried out in the usual manner.[4] As a result the Hamiltonian H_γ of the free macroscopic electromagnetic field in a homogeneous, transparent, nonmagnetic, dispersive, and anisotropic medium and the operator of the vector potential $\mathbf{A}(\mathbf{x})$ in the Schrödinger representation take the form

$$H_\gamma = \sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}\lambda}(c_{\mathbf{k}\lambda}^{+}c_{\mathbf{k}\lambda}+{}^1/_2) + \sum_{\mathbf{k}\nu}\hbar\omega_{\mathbf{k}\nu}(c_{\mathbf{k}\nu}^{+}c_{\mathbf{k}\nu}+{}^1/_2), \quad (12)$$

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k}\lambda}\left(\frac{2\pi\hbar c^2 k}{\omega_{\mathbf{k}\lambda}^2\varepsilon_{\alpha\beta}l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda}V}\frac{\partial\omega_{\mathbf{k}\lambda}}{\partial k}\right)^{1/2} \times(c_{\mathbf{k}\lambda}e^{i\mathbf{k}\mathbf{x}}+c_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\mathbf{x}})\mathbf{l}^{\mathbf{k}\lambda} + \sum_{\mathbf{k}\nu}\left(\frac{4\pi\hbar c^2k^2}{V\omega_{\mathbf{k}\nu}^2k_\alpha k_\beta\,\partial\varepsilon_{\alpha\beta}/\partial\omega_{\mathbf{k}\nu}}\right)^{1/2} \times(c_{\mathbf{k}\nu}e^{i\mathbf{k}\mathbf{x}}+c_{\mathbf{k}\nu}^{+}e^{-i\mathbf{k}\mathbf{x}})\frac{\mathbf{k}}{k}, \quad (13)$$

where $c_{\mathbf{k}\lambda}$ and $c_{\mathbf{k}\lambda}^{+}$ are the absorption and creation operators for a quantum with wave vector $\mathbf{k}$, polarization $l^{\mathbf{k}\lambda}$ and energy $\hbar\omega_{\mathbf{k}\lambda}$, where the quantities $\omega_{\mathbf{k}\lambda}$ and $l^{\mathbf{k}\lambda}$ are for each fixed $\mathbf{k}$ determined from equations (5) and (4). Finally, $c_{\mathbf{k}\nu}$ and $c_{\mathbf{k}\nu}^{+}$ are the absorption and creation operators for a longitudinal quantum with wave vector $\mathbf{k}$ and energy $\hbar\omega_{\mathbf{k}\nu}$, where the frequencies $\omega_{\mathbf{k}\nu}(\nu = 1,2,\ldots)$ for given $\mathbf{k}$ are found from equation (2). The creation and absorption operators satisfy the following commutation relations:

$$c_{\mathbf{k}\lambda}c_{\mathbf{k}'\lambda'}^{+} - c_{\mathbf{k}'\lambda'}^{+}c_{\mathbf{k}\lambda} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'},$$

$$c_{\mathbf{k}\nu}c_{\mathbf{k}'\nu'}^{+} - c_{\mathbf{k}'\nu'}^{+}c_{\mathbf{k}\nu} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\nu\nu'}.$$

The first and second terms in (12) and (13) refer to the ordinary and purely longitudinal waves, respectively. In the absence of dispersion the purely longitudinal waves vanish, and we have the relation

$$\partial\omega_{\mathbf{k}\lambda}/\partial k = \omega_{\mathbf{k}\lambda}/k,$$

so that formulas (12) and (13) coincide with those obtained earlier for a nondispersive medium.[4] Formulas (12) and (13) are valid also for homogeneous isotropic media with frequency and spatial dispersion. The generalization of our results to gyrotropic media presents no difficulties.

In many applications it is important to know the Green's function

$$D_{\alpha\beta}(\mathbf{x}-\mathbf{x}', t-t') = -i\langle T(A_\alpha(\mathbf{x}, t)A_\beta(\mathbf{x}', t'))\rangle \quad (14)$$

and the retarded Green's function

$$D_{\alpha\beta}^{R}(\mathbf{x}-\mathbf{x}', t-t') = \begin{cases} -i\langle A_\alpha(\mathbf{x}, t)A_\beta(\mathbf{x}', t') - A_\beta(\mathbf{x}', t')A_\alpha(\mathbf{x}, t)\rangle, & t' < t, \\ 0, & t' > t \end{cases} \quad (15)$$

of the free electromagnetic field in an anisotropic dispersive medium. Here T denotes the T product[12] of Heisenberg operators of the vector potential $\mathbf{A}(\mathbf{x}, t)$, and the brackets denote the quantum mechanical average over the ground state of the field.

Expanding the T product in (14) with the help of (13) and (10), we obtain

$$D_{\alpha\beta}(\mathbf{x}, t) = \frac{1}{(2\pi)^4}\int D_{\alpha\beta}(\mathbf{k}, \omega)e^{i(\mathbf{k}\mathbf{x}-\omega t)}\,d\mathbf{k}\,d\omega, \quad (16)$$

$$D_{\alpha\beta}(\mathbf{k}, \omega) = \sum_\lambda\frac{4\pi\hbar c^2 l_\alpha^{\mathbf{k}\lambda}l_\beta^{\mathbf{k}\lambda}}{\omega^2\varepsilon_{\alpha'\beta'}(\omega, \mathbf{k})l_{\alpha'}^{\mathbf{k}\lambda}l_{\beta'}^{\mathbf{k}\lambda} - c^2k^2 + c^2(\mathbf{k}\mathbf{l}^{\mathbf{k}\lambda})^2 + i\delta} + \frac{4\pi\hbar c^2k_\alpha k_\beta}{\omega^2\varepsilon_{\alpha'\beta'}(\omega, \mathbf{k})k_{\alpha'}k_{\beta'} + i\delta}, \quad (17)$$

where the unit polarization vector $l^{\mathbf{k}\lambda}$ is for each

fixed value of $\mathbf{k}$ defined by (7), and the infinitesimal term $i\delta$ ($\delta > 0$) in the denominators of (17) indicates the way in which the poles must be bypassed in the integration over ω. The contribution from the pole $\omega = 0$ in the second term of (17) must be discarded, i.e., this pole has no relation to the free macroscopic field.

For an anisotropic medium without spatial dispersion, formula (17) coincides with the expression obtained by another method in [8]. We note also that the quantity $\omega_{\mathbf{k}\lambda}$ which enters in $l^{\mathbf{k}\lambda}$ in (17) can be formally replaced by ω. This does not alter the Green's function (16). After this replacement the components of the vector $l_\alpha{}^{\mathbf{k}\lambda}$ will be proportional to the cofactors $A_{\lambda\alpha}$ of the elements of the λ-th row of the determinant

$$|\omega^2\varepsilon_{\alpha\beta}(\omega, \mathbf{k}) - c^2k^2\delta_{\alpha\beta} + c^2k_\alpha k_\beta|,$$

and the expression (17) is rewritten in the form

$$D_{\alpha\beta}(\mathbf{k}, \omega) = 4\pi\hbar c^2 A_{\alpha\beta} / |\omega^2\varepsilon_{\alpha\beta}(\omega, \mathbf{k}) - c^2k^2\delta_{\alpha\beta} + c^2k_\alpha k_\beta|.$$

The first and second terms in (17) refer to the ordinary and purely longitudinal waves, respectively, where the second term vanishes if the purely longitudinal waves are directed along a distinguished axis or in a plane. Hence such purely longitudinal waves are not emitted in an infinite anisotropic medium. The second term in (17) is different from zero only if the directions of the wave vectors $\mathbf{k}$ of the purely longitudinal waves lie in a solid angle of finite dimensions. For example, this is the case in an anisotropic medium with the dielectric constant [10]

$$\varepsilon_{\alpha\beta}(\omega, \kappa) = C^\alpha\delta_{\alpha\beta}\varepsilon(\omega, \kappa), \qquad (18)$$

where the C^α ($\alpha = 1,2,3$) are constants. The directions of the wave vectors of the purely longitudinal waves are in the general case determined from (2) and (3).

Analogously, the Fourier transform of the retarded Green's function (15) is written in the form (17) without the imaginary term $i\delta$ in the denominator. The poles are in this case circumvented in the usual way by displacing the integration contour into the upper half-plane of the complex variable ω. The analytic properties of the Green's function remain the same as for an isotropic medium.

In solving specific problems in electrodynamics it is convenient to go to the interaction representation, in which the operators of the interacting fields depend on the time in the same way as the free field operators, and the presence of the interaction of the fields is reflected only in the change of the wave function of the entire quantum mechanical system. Thus formulas (12), (13), and (17) permit the use of a graph technique in a transparent dispersive medium in analogy to the graph technique in vacuum.[13]

2. EMISSION BY AN ATOM IN A DISPERSIVE MEDIUM

To solve the problem of long-wave radiation by an impurity atom in a dispersive medium we use the known S matrix formalism.[13] Simple calculations lead to the following value for the probability $dW_{\mathbf{k}\lambda}$ for the emission per unit time of a quantum $\hbar\omega_{\mathbf{k}\lambda}$ with wave vector $\mathbf{k}$ in the solid angle $d\Omega$ and polarization $l^{\mathbf{k}\lambda}$ from an impurity atom located in a transparent dispersive anisotropic medium:

$$dW_{\mathbf{k}\lambda} = \frac{|\mathbf{l}^{\mathbf{k}\lambda}\mathbf{M}_{12}|^2 k^3\, d\Omega}{2\pi\hbar\omega_{21}^2\varepsilon_{\alpha\beta}(\omega_{21}, \mathbf{k})\, l_\alpha^{\mathbf{k}\lambda} l_\beta^{\mathbf{k}\lambda}}. \qquad (19)$$

Here $\mathbf{M}_{12}$ is the matrix element for the transition of the atom from an excited state characterized by the complete set of quantum numbers n_2 and energy $\mathscr{E}_{n_2}$ to the lower state n_1, $\mathscr{E}_{n_1}$:

$$M_{\alpha 12} = \omega_{21}d_\alpha + c[\boldsymbol{\mu}\mathbf{k}]_\alpha - i\omega_{21}Q_{\alpha\beta}k_\beta / 2, \qquad (20)^*$$

where $\mathbf{d}$, $\boldsymbol{\mu}$, and $Q_{\alpha\beta}$ are the corresponding matrix elements of the dipole, magnetic dipole, and quadrupole moments of the atom, respectively, and ω_{21} is the frequency of the atomic transition

$$\omega_{21} = (\mathscr{E}_{n_2} - \mathscr{E}_{n_1}) / \hbar. \qquad (21)$$

For a given direction $\mathbf{k}$ the absolute value of the wave vector in (19) is expressed through ω_{21} via the energy conservation law:

$$\omega_{\mathbf{k}\lambda}(k, \mathbf{k}/k) = \omega_{21}, \qquad (22)$$

where from all roots of (22) one must take only such values k which together with $\omega_{\mathbf{k}\lambda} = \omega_{21}$ satisfy the Fresnel equation (5).

We emphasize that in (20) the trace of the quadrupole tensor of the impurity atom,

$$Q_{\alpha\beta} = e\int \psi_{n_1}^+(\mathbf{x})\, x_\alpha x_\beta\, \psi_{n_2}(\mathbf{x})\, dV, \qquad (23)$$

is different from zero, where e is the electron charge, and $\psi_{n_1}(\mathbf{x})$ and $\psi_{n_2}(\mathbf{x})$ are the electron wave functions normalized to unity. This is connected with the fact that the polarization vector $l^{\mathbf{k}\lambda}$ of the ordinary waves in an anisotropic medium (7) has in general a longitudinal component.

The probability for emission of ordinary waves

*$[\mu\mathbf{k}] \equiv \mu \times \mathbf{k}$.

into the solid angle $d\Omega$ (19) has the same form as in the absence of dispersion.[4] In problems of the emission and absorption of ordinary waves in a transparent dispersive anisotropic medium one can therefore ignore the dispersion in the intermediate calculations and regard the dielectric constant in the final result as dependent on the wave vector and the frequency of the atomic transition. An analogous conclusion was arrived at earlier in [13,14] for media without spatial dispersion. However, in a medium with frequency and spatial dispersion the number of wave types increases, so that the index λ must now assume several values instead of just two, corresponding to all the roots of the Fresnel equation for given $\mathbf{k}$.

In the absence of spatial dispersion the Fresnel equation (5) has no multiple roots. For a given frequency of the atomic transition (21) only one ordinary wave is therefore emitted whose frequency $\omega_{\mathbf{k}\lambda}$ satisfies (22). If spatial dispersion is present the ranges of variation of the functions (6) with different λ may partly overlap, so that for some frequencies of the atomic transition (21) the equation (22) will be satisfied by several functions $\omega_{\mathbf{k}\lambda}$ with different values $\mathbf{k}$ and λ. This means that the atom can with definite probability emit quanta of different types having the same energy and differing among themselves in their polarizations and the dispersion law (6). Hence the probability for radiation from an impurity atom in an anisotropic medium with spatial dispersion is in general increased in correspondence with the increase of the statistical weight of the final states of the system.

In the problem of the radiation of longitudinal quanta the dispersion of the dielectric constant is important at all stages of the calculation. Using the standard S matrix formalism, we find for the probability W^l of the emission of a longitudinal quantum per unit time by an impurity atom in a transparent dispersive anisotropic medium

$$W^l = \frac{1}{\pi\hbar\omega_{21}^2}\int \frac{|\mathbf{k}\mathbf{M}_{12}|^2 k^2 d\Omega}{|\partial\omega_{\mathbf{k}\nu}/\partial\mathbf{k}|\; k_\alpha k_\beta\, \partial\varepsilon_{\alpha\beta}(\omega_{21}, \mathbf{k})/\partial\omega_{21}}, \quad (24)$$

where the numerical value of k for given direction $\mathbf{k}$ is determined from the energy conservation law

$$\omega_{\mathbf{k}\nu}(k, \mathbf{k}/k) = \omega_{21}, \quad (25)$$

and the frequency of the longitudinal quantum $\omega_{\mathbf{k}\nu}(\mathrm{k}, \mathbf{k}/\mathrm{k})$ is determined by (2). From all the roots of (25) one must choose only those values k which together with $\omega_{\mathbf{k}\nu} = \omega_{21}$ satisfy (2).

The region of integration over the angles of the vector $\mathbf{k}$ in (24) is determined from (2) and (3). For example, if the vector $\mathbf{k}$ lies in a plane or is directed along a straight line, the volume V will remain uncompensated in the denominator of (24). In this case the probability for the emission of a longitudinal quantum in an infinite medium $V \to \infty$ vanishes in accordance with what we have said earlier. However, if the radiation of the atom takes place in a wave guide or a resonator with anisotropic dielectric filling, the probability for the emission of a longitudinal quantum is different from zero if the axis of the wave guide or the resonator coincides with the distinguished direction of the wave vector $\mathbf{k}$ of the longitudinal quantum. Formula (24) holds also for an isotropic medium with frequency and spatial dispersion.

In the general case the matrix element $\mathbf{M}_{12}$ in (24) must be written in the form

$$\mathbf{M}_{12} = \frac{ie}{2m}\int \psi_{n_1}^{+}(\mathbf{x})\, e^{-i\mathbf{k}\mathbf{x}}(2\mathbf{p} - i\mathbf{k})\,\psi_{n_2}(\mathbf{x})\, dV,$$

where m and $\mathbf{p}$ are the mass and the momentum operator of the radiating electron, respectively. For long-wave emission of a longitudinal quantum the expansion (20) is valid.

It follows from (24) and (20) that there are no magnetic dipole transitions with emission of a long-wave quantum, and for quadrupole transitions the transition $0 \to 0$ is allowed. The remaining selection rules are easily established in each specific case, starting from the form of the corresponding matrix elements (20) (cf., for example[10]).

It should be noted that in contrast to the isotropic medium, the division into dipole, magnetic-dipole, and quadrupole radiation in the anisotropic medium is of formal character, connected with the expansion of the matrix element (20) in the small parameter a/λ, where a is the radius of the radiating atom and λ the wave length of the emitted wave. Since the angular momentum is in general not conserved in an anisotropic medium, the radiated field cannot be expanded in multipoles with definite angular momentum and projection. As a consequence, only a quantum with definite energy, wave vector, and polarization is emitted in dipole as well as quadrupole transitions of an impurity atom in an anisotropic medium. Owing to the smallness of the coupling constant with the radiation field one usually introduces selection rules with respect to quantum numbers characterizing the state of the radiating electrons in the impurity atom. These quantum numbers may also contain the angular momentum and its projection. One must only keep in mind that the angular momentum may not be conserved in the long-wave radiation

in a transparent anisotropic medium (in contrast to the energy).

For definiteness we apply formulas (19) and (24) to the long-wave ($kr_D \ll 1$) radiation from an impurity molecule in an isotropic plasma with dielectric constant:[11]

$$\varepsilon_{\alpha\beta}(\omega, \mathbf{k}) = \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\varepsilon^{tr}(\omega, k) + \frac{k_\alpha k_\beta}{k^2}\varepsilon^{l}(\omega, k),$$

$$\varepsilon^{tr}(\omega, k) = 1 - \frac{\omega_L{}^2}{\omega^2}(1 + k^2 r_D{}^2),$$

$$\varepsilon^{l}(\omega, k) = 1 - \frac{\omega_L{}^2}{\omega^2}(1 + 3k^2 r_D{}^2),$$

where r_D is the Debye radius and ω_L is the Langmuir frequency of the electrons of the plasma. Let us consider the case where the frequency ω_{21} of the molecular transition is close to the Langmuir frequency, $(\omega_{21} - \omega_L)/\omega_L \ll 1$, where $\omega_{21} > \omega_L$. For $\omega_{21} < \omega_L$ no longitudinal waves are emitted. As a result the probabilities for dipole emission per unit time of transverse, $W_d{}^{tr}$, and longitudinal quanta, $W_d{}^{l}$, are given by the formulas

$$W_d{}^{tr} = (n_{\omega_{21}} + 1) W_d{}^0 (1 - \omega_L{}^2 / \omega_{21}{}^2)^{1/2}, \tag{26}$$

$$W_d{}^{l} = (n_{\omega_{21}} + 1) W_d{}^0 \left(1 - \frac{\omega_L{}^2}{\omega_{21}{}^2}\right)^{1/2} \frac{1}{6\sqrt{3}} \left(\frac{mc^2}{\varkappa T}\right)^{3/2},$$

$$1 \gg \frac{\varkappa T}{mc^2} \gg \frac{4\pi e^2 N_e{}^{1/3}}{mc^2}, \tag{27}$$

where $W_d{}^0$ is the dipole transition probability in vacuum, κ is the Boltzmann constant, T is the plasma temperature, and N_e is the electron density in the plasma.

In writing (26) and (27) we have assumed that before the radiation of the impurity molecule the electromagnetic field of the plasma was in thermodynamical equilibrium with the charged particles of the plasma. Therefore the number of quanta n_ω with frequency ω and definite direction of the polarization vector and wave vector is

$$n_\omega = [\exp(\hbar\omega / \varkappa T) - 1]^{-1}.$$

The quantity $n_{\omega_{21}} + 1$ in (26) and (27) takes account of the spontaneous and induced emission of the impurity molecule.

It is seen from (26) and (27) that the probability for the emission (absorption) of longitudinal waves is much larger than for transverse waves in the frequency region indicated. This remark holds not only for the long-wave radiation by an impurity molecule in a plasma. According to the correspondence principle, the radiation of a classical charged system characterized by a variable dipole moment $\mathbf{d}(t)$ is also described by (26) and (27) with the matrix element $\mathbf{d}$ replaced by $\mathbf{d}(t)/\sqrt{2}$. This means, for example, that the radiation of transverse waves from a linear antenna in the ionosphere in the indicated frequency region is strongly weakened through power losses to the radiation of longitudinal waves whose radiation intensity is larger by a factor $(1/6\sqrt{3})(mc^2/\kappa T)^{3/2}$.

3. RADIATION OF A CLASSICAL CHARGED SYSTEM IN AN ANISOTROPIC MEDIUM

Above we have obtained formulas for the radiation from an impurity atom in an anisotropic dispersive medium. However, if the correspondence principle is applied one can also easily determine the radiation law for classical (unquantized) objects. We note in this connection that the direct integration of the Maxwell equations in an anisotropic medium is mathematically much more complicated even in the dipole case and has not yet been carried out.

For simplicity we write down the formulas for the radiation in a uniaxial nondispersive crystal. The direction of the polarization vectors of the ordinary and extraordinary waves and the angular distribution of the atomic multipole radiation with wave vector $\mathbf{k}$ lying in a solid angle $d\Omega = 2\pi \sin\theta\, d\theta$ were given in [4]. Here θ is the angle between the vector $\mathbf{k}$ and the optical axis of the crystal.[2)] The Poynting vector of the extraordinary wave lies in the same plane as the wave vector $\mathbf{k}$ and the optical axis of the crystal, spanning the angle θ' with the latter, where

$$\cos^2\theta = \frac{\varepsilon_\perp{}^2 \cos^2\theta'}{\varepsilon_\parallel{}^2 + (\varepsilon_\perp{}^2 - \varepsilon_\parallel{}^2)\cos^2\theta'}, \tag{28}$$

with $\epsilon_\perp$ and $\epsilon_\parallel$ the dielectric constant of the uniaxial crystal in the transverse and longitudinal directions with respect to the optical axis, respectively.

Using the correspondence principle and (28), we find for the energy $d\mathscr{E}_{n\omega}$ radiated with frequency ω into the solid angle $d\Omega' = 2\pi \sin\theta' d\theta'$ in the form of an extraordinary wave

$$d\mathscr{E}_{n\omega} = \frac{|\ddot{\mathbf{d}}(\omega)|^2}{12\pi^2 c^3} \frac{\varepsilon_\parallel{}^2 \varepsilon_\perp \, d\Omega' \, d\omega}{[\varepsilon_\parallel + (\varepsilon_\perp - \varepsilon_\parallel)\cos^2\theta']^{5/2}}, \tag{29}$$

where $\ddot{\mathbf{d}}(\omega)$ is the Fourier transform of the second derivative of the dipole moment $\mathbf{d}(t)$ of the classical charged system located in a uniaxial crystal. The refraction of the rays as the radiation leaves the crystal is calculated with the laws

[2)] We take this opportunity to note that there is a printing error in formulas (15) to (17) of [4]. The first term $\varepsilon_\parallel$ in the square bracket in the denominator must be replaced by $\varepsilon_\perp$.

of geometrical optics. For a random orientation of the dipole moment the frequency distribution of the radiation of ordinary waves is the same as in an isotropic medium with dielectric constant $\epsilon_\perp$.

Expression (29) is averaged over the direction of the dipole moment. If the electric dipole moment is directed along the optical axis, then formula (29) is multiplied by $3\sin^2\theta'$, and there is no radiation of ordinary waves.

For magnetic dipole radiation from a classical charged system with random orientation of the magnetic moment $\mu(t)$ one must make the replacement

$$\ddot{\mathbf{d}}(\omega) \to \ddot{\boldsymbol{\mu}}(\omega),$$

in formula (29), and the exponent $5/2$ of the square bracket in the denominator must be changed to $3/2$. In this case the radiation of ordinary waves is the same as in an isotropic medium with dielectric constant $\epsilon_\perp$. If the magnetic moment is directed along the optical axis, no extraordinary waves will be emitted, and the energy $d\mathscr{E}_{\mathbf{n}\omega}$ of the radiation with frequency ω into the solid angle $d\Omega'$ in the form of ordinary waves is

$$d\mathscr{E}_{\mathbf{n}\omega} = \frac{|\ddot{\boldsymbol{\mu}}(\omega)|^2 \varepsilon_\perp^{3/2} \sin^2\theta'}{4\pi^2 c^3} d\Omega'\, d\omega.$$

The quadrupole radiation of a classical charged system is calculated in an analogous fashion. For example, the energy $d\mathscr{E}_{\mathbf{n}\omega}$ radiated with frequency ω into the solid angle $d\Omega'$ in the form of ordinary waves is

$$d\mathscr{E}_{\mathbf{n}\omega} = \frac{|\dddot{Q}_{\alpha\beta}(\omega)|^2 \varepsilon_\parallel^2 \varepsilon_\perp}{240\pi^2 c^5} \times \frac{\varepsilon_\parallel^2 + (3\varepsilon_\perp - \varepsilon_\parallel)(\varepsilon_\perp - \varepsilon_\parallel)\cos^2\theta' - 2(\varepsilon_\perp - \varepsilon_\parallel)^2 \cos^4\theta'}{[\varepsilon_\parallel + (\varepsilon_\perp - \varepsilon_\parallel)\cos^2\theta']^{7/2}} d\Omega' d\omega, \quad (30)$$

where $\dddot{Q}_{\alpha\beta}(\omega)$ is the Fourier transform of the third derivative of the quadrupole moment $Q_{\alpha\beta}(t)$ of the classical charged system. Expression (30) is averaged over the directions of the axes of the quadrupole tensor.

The authors are very grateful to V. M. Galitskiĭ, V. I. Kogan, and M. M. Ryazanov for a discussion of the results of this paper.

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Translated by R. Lipperheide
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