ANGULAR DISTRIBUTION OF BREMSSTRAHLUNG RADIATION AND THE LANDAU-POMERANCHUK EFFECT

N. P. KALASHNIKOV and M. I. RYAZANOV

Moscow Institute of Engineering Physics

Submitted to JETP editor October 14, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 791–794 (March, 1966)

The effect of multiple scattering on the direction and frequency distribution of bremsstrahlung radiation is considered. It is shown that, in the range in which the Landau-Pomeranchuk effect is encountered, multiple scattering decreases the radiation intensity at small angles, but does not affect the large-angle radiation. This agrees with the results of a qualitative treatment.[8] Integration of the formula derived leads to the results obtained by Landau, Pomeranchuk, and Migdal.[1, 2]

1. In 1953, Landau and Pomeranchuk[1] pointed out the inapplicability at very high energies of the Bethe-Heitler formula for bremsstrahlung, the reason being the effect of multiple scattering on the radiation length of the quantum. The quantum theory of this effect was given by Migdal;[2] however, in [2], as in later papers by other authors, only the energy spectrum of bremsstrahlung was studied. The study of the energy spectrum was preferred because the form of the energy spectrum depends on multiple scattering only through the effective radiation length and is not at all dependent on the direction of the initial particle's momentum, and, consequently, does not depend on multiple scattering prior to the emission of a quantum. On the other hand, the angular distribution depends on the multiple scattering of the particle before the time of emission and, therefore, on the path traversed by the particle in the substance.

The difficulty in calculating the photon frequency and direction distribution lies in the fact that it is impossible to separate beforehand the nontrivial effect of the influence of multiple scattering on the effective length from the trivial effect of multiple scattering before emission, which results in directional dispersion of the initial momentum of the particle. Gol'dman's study of the angular distribution of the quanta[3] was limited in that the expression was formally derived from the classical theory in the form of a certain integral of the Fourier transform of the distribution function, and the integration was not carried out explicitly.

2. It is convenient to carry out the study of the bremsstrahlung angular distribution by describing multiple scattering in the language of quantum mechanics, that is, by characterizing the state of a spinor particle in a material by the wave function[4]

$$\Psi(\mathbf{r}, t) = \exp(i\mathbf{p}_0\mathbf{r} - iE_0t)\,(1 - i(2E_0)^{-1})\,(\boldsymbol{\alpha}\nabla)\,u_0 \times \exp\Big\{\sum_a \Phi(\mathbf{p}_0, \mathbf{r}-\mathbf{R}_a)\Big\}. \tag{2.1}$$

As was shown in [4], the quantity $\Phi(\mathbf{p}_0, \mathbf{r}-\mathbf{R}_a)$ is expressible in terms of the amplitude for single-center scattering (the retention of terms of order E^{-1} is connected with the specific nature of the bremsstrahlung problem[5]). By inserting the wave function of Eq. (2.1) into the matrix element and averaging the probability of bremsstrahlung over the positions of the atoms by the method developed in [4], it is not difficult to obtain an expression for the bremsstrahlung probability per unit path length ($h = c = 1$):

$$dW = d^3p_2 d^3k\,\delta(E_1 - E_2 - \omega) \int d^3l\,W_0(\mathbf{p}_2, \mathbf{k}, \mathbf{l}) \int_0^L dx_0 \int d^3R \times \exp[i(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k} + \mathbf{l})R] \times \exp\Big\{-n_0R_\perp^2\langle q_\perp^2\rangle L + in_0R_x\langle q_\perp^2\rangle \times\Big[\frac{L - x_0 - |\mathbf{R}_x|/2}{2p_2} - \frac{x_0 - |\mathbf{R}_x|/2}{2p_1} + \frac{|\mathbf{R}_x|\omega}{4E_1^2}\Big]\Big\}. \tag{2.2}$$

Here $\omega \ll E$ and the notation

$$\langle q_\perp^2\rangle = \int q_\perp^2 |U_0(\mathbf{q}_\perp)|^2 d^2q_\perp$$

has been used. The number of atoms per unit volume of material is n_0, L is the thickness of the slab of material, and $U_0(\mathbf{q})$ is the Fourier transform of the potential due to an individual atom. The function $W_0(\mathbf{p}_2, \mathbf{k}, \mathbf{l})$ is connected with the Bethe-Heitler radiation probability by the formula

$$dW_{\mathrm{BH}} = d^3p_2 d^3k \int d^3l \delta(E_1 - E_2 - \omega) \times \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k} + \mathbf{l}) W_0(\mathbf{p}_2, \mathbf{k}, \mathbf{l}). \tag{2.3}$$

In the low-density limit ($n_0 \to 0$), Eq. (2.2) tends to Eq. (2.3). To obtain the angular and frequency distribution of the radiated photons, Eq. (2.2) must be integrated over p_2. In doing this, it is convenient to change to the variable $\mathbf{p}_2' = \mathbf{p}_2 + \boldsymbol{\vartheta}(L - x_0 - \frac{1}{2}|R_x|)$, the vector $\boldsymbol{\vartheta}$ being defined as the change in the particle momentum as a result of multiple scattering along the distance $L - x_0 - \frac{1}{2}|R_x|$, that is, as a vector satisfying the conditions

$$\langle \boldsymbol{\vartheta}^2(x) \rangle = (E_s / E)^2 (x / L_{\mathrm{rad}}), \qquad \langle 2\mathbf{p}_2 \boldsymbol{\vartheta}(x) \rangle = -(E_s / E)^2 (x / L_{\mathrm{rad}}); \tag{2.4}$$

$(E_s^2/E^2 L_{\mathrm{rad}})$ is the rms multiple-scattering angle per unit path length. From (2.4) it follows that $(p_2')^2 = (p_2)^2$.

In the integration below, it is assumed that the effective radiation length is small in comparison with the total path length of the particle in the material, $(E^2/\omega m^2) \ll L$. In this case, it is possible to neglect boundary effects and to extend the integration to the whole x axis. Taking this into account, the distribution per unit path length of the probability for radiation of photons with frequency ω by a fast particle, in the direction of the element of solid angle $d\Omega_{\mathbf{k}}$, can be put in the form

$$\frac{dW}{d\omega\, d\Omega_{\mathbf{k}}} = L^{-1} \int_0^L dx \int d\Omega_{\mathbf{p}} W(x, \boldsymbol{\vartheta}_{\mathbf{p}}) W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}; \omega, \boldsymbol{\vartheta}_{\mathbf{k}}), \tag{2.5}$$

where $W(x, \boldsymbol{\vartheta}_{\mathbf{p}})$ is the usual probability for the deflection of the particle into the angle $\boldsymbol{\vartheta}_{\mathbf{p}}$ on the path x because of multiple scattering. $W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}, \omega, \boldsymbol{\vartheta}_{\mathbf{k}})$ is the probability that a particle which has been deflected into the angle $\boldsymbol{\vartheta}_{\mathbf{p}}$ by multiple scattering will emit a quantum with frequency ω at an angle $\boldsymbol{\vartheta}_{\mathbf{k}}$ to the original direction:

$$W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}, \omega, \boldsymbol{\vartheta}_{\mathbf{k}}) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\xi Z^2 e^6 \ln M \cdot \omega \times \exp\left[is\xi + i\frac{n_0 \langle q_\perp^2 \rangle}{4E_1^2} \omega\xi|\xi| \right] \times \left[\frac{\omega m^2}{2E_1^2} + \frac{\omega}{2}(\boldsymbol{\vartheta}_{\mathbf{k}} - \boldsymbol{\vartheta}_{\mathbf{p}})^2 - s \right]^{-2}$$
$$= (2\pi) b^{-1} Z^2 e^6 \ln M \,\mathrm{Im}\, \gamma \exp(\gamma^2/4) D_{-1}(\gamma). \tag{2.6}$$

Here $D_{-1}(\gamma)$ is a parabolic-cylinder function[6] and

$$\gamma \equiv -\frac{1}{\sqrt{2ib}} \left\{ \frac{\omega m^2}{2E_1^2} + \frac{\omega}{2}(\boldsymbol{\vartheta}_{\mathbf{k}} - \boldsymbol{\vartheta}_{\mathbf{p}})^2 \right\}, \qquad b \equiv n_0 \omega \frac{\langle q_\perp^2 \rangle}{4E_1};$$
$$M^{-1} = (m\omega / 2E^2)^2 + e^4 Z^{2/3}.$$

Formulae (2.5) and (2.6) give the exact expression for the bremsstrahlung angular and frequency distribution. We will show that these formulae yield, in limiting cases, results which coincide with those of other authors.

In the low density limit, $\omega \gg (E_s^2 E^2/m^4 L_{\mathrm{rad}})$, ($\gamma \gg 1$, that is, when the influence of multiple scattering is not yet important), $W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}; \omega, \boldsymbol{\vartheta}_{\mathbf{k}})$ coincides with the Bethe-Heitler cross section, as is easily seen by making use of the asymptotic formula[6]

$$D_{-1}(y) \approx y^{-1} \exp(-y^2/4);$$

the integration of Eq. (2.5) for this case was carried out in [7].

In the other limiting case, $\omega \ll (E_s^2 E^2/m^4 L_{\mathrm{rad}})$, for large values of ϑ_k ($\vartheta^2 \gg [E_s^2/E^2 \omega L_{\mathrm{rad}}]^{1/2}$), we have $\gamma \gg 1$ and again $W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}; \omega, \boldsymbol{\vartheta}_{\mathbf{k}})$ coincides with the Bethe-Heitler cross section. For small angles ($\vartheta^2 \ll [E_s^2/E^2 \omega L_{\mathrm{rad}}]^{1/2}$), γ will be much less than unity and, in this case, multiple scattering in the effective region of bremsstrahlung changes the form of W_γ substantially:

$$W_\gamma(\mathbf{p} + \boldsymbol{\vartheta}_{\mathbf{p}}; \omega, \boldsymbol{\vartheta}_{\mathbf{k}}) = 8Z^2 e^6 \ln M \times \sqrt{\frac{\pi^3\omega}{2}} \frac{[(m/E)^2 + (\boldsymbol{\vartheta}_{\mathbf{k}} - \boldsymbol{\vartheta}_{\mathbf{p}})^2]}{[E_s^2/E^2 L_{\mathrm{rad}}]^{3/2}}. \tag{2.7}$$

This formula is valid if

$$[(m/E)^2 + (\boldsymbol{\vartheta}_{\mathbf{k}} - \boldsymbol{\vartheta}_{\mathbf{p}})^2] \ll (E_s^2 / 4E^2 L_{\mathrm{rad}} \omega). \tag{2.8}$$

Thus, multiple scattering suppresses bremsstrahlung when the quantum is emitted at small angles and does not change the intensity of radiation when emission occurs at large angles. This leads to an increase of the effective angles for bremsstrahlung.

Analysis of formula (2.6) shows that the maximum of the angular distribution corresponds to angles

$$\vartheta_{max}^2 \sim 4(E_s/E)(\omega L_{\mathrm{rad}})^{-1/2}. \tag{2.9}$$

This expression supports the estimate of the broadening of the bremsstrahlung angular distribution in the presence of multiple scattering, which was obtained from the qualitative study by Galitskiĭ and Gurevich.[8]

If Eq. (2.5) is also integrated over the angles of emission of the photon, the energy spectrum of the radiation is obtained:

$$\frac{d\sigma}{d\omega} = e^2\alpha^2 \ln M \cdot (2\pi)^2(-2)\,\mathrm{Im}\,\frac{1}{\sqrt{2ib}} \times \exp\left(-i\frac{a^2}{8b}\right) D_{-1}\left(-i\frac{a}{\sqrt{2ib}}\right),$$

$$a = (m^2\omega / 2E^2). \qquad (2.10)$$

In the low-density limit, Eq. (2.10) gives the Bethe-Heitler spectrum, while for $\omega \ll (E_s^2 E^2/m^4 L_{rad})$ it tends to the expression obtained by Landau, Pomeranchuk, and Migdal.[1, 2]

In conclusion the authors take the opportunity to thank V. M. Galitskiĭ for useful discussions.

[1] L. D. Landau and I. Ya. Pomeranchuk, DAN SSSR **92**, 535, 735 (1953).

[2] A. B. Migdal, DAN SSSR **96**, 49 (1954); JETP **32**, 633 (1957), Soviet Phys. JETP **5**, 527 (1957).

[3] I. I. Gol'dman, JETP **38**, 1866 (1960), Soviet Phys. JETP **11**, 1341 (1960).

[4] N. P. Kalashnikov and M. I. Ryazanov, JETP **47**, 1055 (1964), Soviet Phys. JETP **20**, 707 (1965).

[5] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics) Fizmatgiz, 1959.

[6] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products) Fizmatgiz, 1962.

[7] A. Sirlin, Phys. Rev. **101**, 1219 (1956).

[8] V. M. Galitsky and I. I. Gurevich, Nuovo Cimento **32**, 396 (1964).

Translated by G. H. Renninger
98