SUPERCONDUCTIVITY MODEL WITH AN EXACT SOLUTION

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A system resembling the BCS superconductivity model is considered. The energy levels as well as the ground state energy can be determined exactly, and the variation of the ground state with variation of density can be traced. It is found that the system can be separated into two noninteracting parts (phases) and at low densities (for certain values of the interaction parameter) only one phase is present in the ground state. A gap in the excitation spectrum is encountered in a certain density range; the nature of this gap is, however, different from that in the BSC model. The partition function of the system can be calculated.

IN the well-known paper by Bardeen, Cooper and Schrieffer (BCS) on superconductivity,[1] and also in some of the subsequent papers (see, for example, [2]), a study is made of a model Hamiltonian in the form

$$H = \sum_{\mathbf{k},s} \varepsilon(\mathbf{k}) a^*(\mathbf{k},s) a(\mathbf{k},s) + \frac{1}{V}\sum J(\mathbf{k}_1,\mathbf{k}_2) a^*_{\mathbf{k}_1 1} a^*_{-\mathbf{k}_1,-1} a_{\mathbf{k}_2 1} a_{-\mathbf{k}_2,-1} \qquad (I)$$

in the space of states with fixed number of particles N. The energy of the ground state of the system and the energy of the excited levels are determined in this case asymptotically (for $N \to \infty$, $V \to \infty$, and $N/V \to \rho$). It has been observed that the latter are separated from the ground level by a definite amount (gap in the excitation spectrum). In addition, the partition function is calculated for the Hamiltonian (I) in the BCS paper, and also in [2].

It must be noted, however, that the BCS results have been obtained under the following assumptions: the function $J(\mathbf{k}_1, \mathbf{k}_2)$ describing the interaction differs from zero in a small layer $\lambda_1 < |\mathbf{k}_1| < \lambda_2, \lambda_1 < |\mathbf{k}_2| < \lambda_2$, and the density of the system is such that the boundary of the Fermi sphere corresponding to this density is situated inside the indicated layer. In addition, the variational method used in [1], while apparently leading to the correct answer, is quite difficult to prove rigorously.

We have therefore investigated a certain particular and strongly simplified case of the Hamiltonian (I). In the case considered by us the interaction differs from zero, as before, in a certain momentum region $\lambda_1 < |\mathbf{k}| < \lambda_2$ where it is constant; in addition, and this constitutes the appreciable simplification, the self-energy of the particles is constant in this region $\epsilon(\mathbf{k}) = \text{const}$.

We have succeeded in calculating in explicit form the spectrum of the Hamiltonian and to obtain the dependence of the lower level on the density. In addition, for some ratio of the parameters it turns out that a gap exists in the excitation spectrum. Unfortunately, however, the gap obtained in this manner has an entirely different origin than the gap in the BCS paper. We were unable to obtain in our over-simplified model a gap similar to that arising in the BCS Hamiltonian.

We note also that the Hamiltonian considered in our paper has a formal similarity to the Hamiltonian investigated by Belyaev[3] and describing the levels of sufficiently remote nuclear shells. Of course, in this case we can also employ the method of our article. Incidentally, the cited paper employs methods that are borrowed from papers on superconductivity.

1. DESCRIPTION OF THE MODEL

The Hamiltonian is of the form

$$H = \sum_{\mathbf{k},s} \varepsilon(\mathbf{k}) a^*_{\mathbf{k}s} a_{\mathbf{k}s} - \frac{J}{V} \sum_{\substack{\lambda_1 < |\mathbf{k}_{1,2}| < \lambda_2 \\ \mathbf{k}_1 \neq \mathbf{k}_2}} a^*_{\mathbf{k}_1 1} a^*_{-\mathbf{k}_1,-1} a_{\mathbf{k}_2,-1} a_{-\mathbf{k}_2 1}, \qquad (1)$$

where $a^*_{\mathbf{k}s}$ and $a_{\mathbf{k}s}$ are the operators of creation and annihilation of a particle with momentum $\mathbf{k}$ and spin s. The usual anticommutation relations are satisfied:

$$\{a_{\mathbf{k}s}, a_{\mathbf{k}'s'}\} = \{a^*_{\mathbf{k}s}, a^*_{\mathbf{k}'s'}\} = 0, \quad \{a^*_{\mathbf{k}s}, a_{\mathbf{k}'s'}\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}.$$

The value of s (spin) takes on the two values ±1, while $\mathbf{k}$ runs through a three-dimensional lattice

with a cell volume equal to $1/V$; $J > 0$; $\epsilon(\mathbf{k}) = \epsilon(|\mathbf{k}|)$ is a positive non-diminishing function, $\epsilon(0) = 0$ and, most important, $\epsilon(\mathbf{k})$ is constant when $\lambda_1 < |\mathbf{k}| < \lambda_2$; $\epsilon(\mathbf{k}) = \epsilon_0$. The region of the lattice $\lambda_1 < |\mathbf{k}| < \lambda_2$ will henceforth be denoted by Ω.

2. TRANSFORMATION OF THE HAMILTONIAN

The Hilbert space R in which the operator H acts is represented in the form of a tensor product of two-dimensional space:

$$R = \prod_{\mathbf{k},\, s} \otimes R_{\mathbf{k}s}.$$

$R_{\mathbf{k},\,s}$ is the space of the state of one particle with momentum $\mathbf{k}$ and spin s. We denote by $e_1(\mathbf{k}, s)$ and $e_0(\mathbf{k}, s)$ the basis in $R_{\mathbf{k},\,s}$ corresponding to the fact that the state $(\mathbf{k}, s)$ is filled or empty, respectively. For values $\mathbf{k} \in \Omega$ we form four-dimensional spaces

$$R(\mathbf{k}) = R_{\mathbf{k},\,1} \otimes R_{-\mathbf{k},\,-1}.$$

The basis in $R(\mathbf{k})$ consists of the vectors

$$\begin{aligned} e_{00}(\mathbf{k}) &= e_0(\mathbf{k}, 1) \otimes e_0(-\mathbf{k}, -1), \\ e_{11}(\mathbf{k}) &= e_1(\mathbf{k}, 1) \otimes e_1(-\mathbf{k}, -1), \\ e_{10}(\mathbf{k}) &= e_1(\mathbf{k}, 1) \otimes e_0(-\mathbf{k}, -1), \\ e_{01}(\mathbf{k}) &= e_0(\mathbf{k}, 1) \otimes e_1(-\mathbf{k}, -1). \end{aligned} \tag{2}$$

The meaning of these vectors $e_{ik}(\mathbf{k})$ is as follows: $e_{11}(\mathbf{k})$ is the state in which a pair of particles is present, one with momentum $\mathbf{k}$ and spin 1, and the other with opposite momentum and spin -1, $e_{00}(\mathbf{k})$ is a state where there is no such pair, $e_{10}(\mathbf{k})$ is a state where only the particle with momentum $\mathbf{k}$ and spin 1 is present, and finally, $e_{01}(\mathbf{k})$ is a state where only a particle with momentum $-\mathbf{k}$ and spin -1 is present.

The entire space R is now represented in the form of the tensor product of the spaces

$$R = R_1 \otimes R_2;$$

$$R_1 = \prod_{s,\, \mathbf{k} \overline{\in} \Omega} \otimes R(\mathbf{k}, s), \qquad R_2 = \prod_{\mathbf{k} \in \Omega} \otimes R(\mathbf{k}).$$

The space R_1 will henceforth be called the normal phase and state R_2 the singular phase.

We consider now for $\mathbf{k} \in \Omega$ the operator

$$n(\mathbf{k}) = a_{\mathbf{k}1}^{*} a_{\mathbf{k}1} + a_{-\mathbf{k},\,-1}^{*} a_{-\mathbf{k},\,-1}.$$

It acts in the space $R(\mathbf{k})$ and is specified in the basis (2) by the matrix

$$n(\mathbf{k}) \sim \begin{pmatrix} 00 & 00 \\ 02 & 00 \\ 00 & 10 \\ 00 & 01 \end{pmatrix} = E + 2\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix} = E + 2s_3(\mathbf{k}),$$

where σ_3 is a Pauli matrix:

$$\sigma_3 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

We consider also the operators $a_{\mathbf{k}1}^{*} a_{-\mathbf{k},\,-1}$ and $a_{-\mathbf{k},\,-1} a_{\mathbf{k}1}$, which act in $R(\mathbf{k})$. In the basis (2) they are expressed by the matrices

$$a_{\mathbf{k}1}^{*} a_{-\mathbf{k},\,-1}^{*} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_2 + i\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = s_2(\mathbf{k}) + i s_1(\mathbf{k});$$

$$a_{-\mathbf{k},\,-1} a_{\mathbf{k}1} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_2 - i\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = s_2(\mathbf{k}) - i s_1(\mathbf{k}),$$

$$\sigma_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}.$$

The operator H can be broken up into two components corresponding to the normal and singular phases:

$$H = H_1 + H_2; \quad H_1 = \sum_{s,\, \overline{\Omega}} \varepsilon(\mathbf{k}) a_{\mathbf{k}s}^{*} a_{\mathbf{k}s},$$

$$\begin{aligned} H_2 &= \varepsilon_0 \sum_{s,\, \Omega} a_{\mathbf{k}s}^{*} a_{\mathbf{k}s} - \frac{J}{V} \sum_{\substack{\Omega \\ \mathbf{k}_1 \neq \mathbf{k}_2}} a_{\mathbf{k}_1 1}^{*} a_{-\mathbf{k}_1,\,-1}^{*} a_{-\mathbf{k}_2,\,-1} a_{\mathbf{k}_2 1} \\ &= \varepsilon_0 \sum_{\Omega} n(\mathbf{k}) - \frac{J}{V} \sum_{\substack{\Omega \\ \mathbf{k}_1 \neq \mathbf{k}_2}} [s_2(\mathbf{k}_1) + i s_1(\mathbf{k}_1)][s_2(\mathbf{k}_2) - i s_1(\mathbf{k}_2)] \\ &= \varepsilon_0 |\Omega| E + 2\varepsilon_0 \sum_{\Omega} s_3(\mathbf{k}) - \frac{J}{V}\Big[\Big(\sum_{\Omega} s_2(\mathbf{k})\Big)^2 \\ &\quad + \Big(\sum_{\Omega} s_1(\mathbf{k})\Big)^2 - \sum_{\Omega} s_1^2(\mathbf{k}) - \sum_{\Omega} s_2^2(\mathbf{k})\Big] \\ &= \varepsilon_0 |\Omega| E + 2\varepsilon_0 L_3 - \frac{J}{V}\Big\{[L_1^2 + L_2^2] - \frac{1}{2}\sum_{\Omega} j(\mathbf{k})\Big\}. \end{aligned}$$

The summation over $\mathbf{k}$ in H_1 is carried out here over the region which is complementary to Ω; $|\Omega|$ is the number of lattice points in the region Ω;

$$L_3 = \sum s_3(\mathbf{k}), \qquad L_2 = \sum s_2(\mathbf{k}), \qquad L_1 = \sum s_1(\mathbf{k})$$

(the sums are taken over the region Ω); $j(\mathbf{k})$ is the projection operator in the space $R(\mathbf{k})$ on the subspace subtending the vectors $e_{00}(\mathbf{k})$ and $e_{11}(\mathbf{k})$.

The operator H_1 acts in the normal phase and its spectrum consists of all the possible sums

$$\sum \varepsilon(\mathbf{k}_i), \quad \mathbf{k}_i \overline{\in} \Omega, \quad k_p \neq k_{p'}, \quad p \neq p'.$$

We now determine the spectrum of the operator H_2, which acts in the singular phase.

3. THE SPECTRUM OF H_2

We note that in each of the spaces $R(\mathbf{k})$ we can specify the representation of the group U of the unitary matrix u of second order (det u = 1),[1] namely, in the space subtending the vectors $e_{00}(\mathbf{k})$ and $e_{11}(\mathbf{k})$ (we denote it by $R^{(1)}(\mathbf{k})$), we specify a representation of weight $1/2$, while in the subspace subtending the vectors $e_{01}(\mathbf{k})$ and $e_{10}(\mathbf{k})$ (which we denote by $R^{(0)}(\mathbf{k})$ we specify a double representation of weight 0. It is easy to see that the operators $s_1(\mathbf{k})$, $s_2(\mathbf{k})$, and $s_3(\mathbf{k})$ are infinitesimal operators of this representation.

In the entire space R_2 there acts the representation $u \to T_u$ of the group U, equal to the tensor product of the representations in each $R(\mathbf{k})$. The operators L_1, L_2, and L_3 are infinitesimal operators of the representation $u \to T_u$. The space R_2 can be broken up into a sum of subspaces that are invariant with respect to the representation $u \to T_u$:

$$R_2 = \sum_{n=0}^{|\Omega|} R_n,$$

$$R_n = \sum \prod_{\Omega} \otimes R^{(t(\mathbf{k}))}(\mathbf{k}); \quad t(\mathbf{k}) = 1, 0.$$

In the latter case the summation is over all the sets $t(\mathbf{k})$ such that $\Sigma t(\mathbf{k}) = n$. It is easy to see that the space R_n is the proper space for the operator

$$K = \sum j(\mathbf{k}), \quad \mathbf{k} \in \Omega$$

with eigenvalue equal to n ($0 \le n \le |\Omega|$).

We further expand each R_n into irreducible subspaces $R^{\tau}_{n,l}$ with weight l ($l \le n/2$). The operator

$$L = L_1^2 + L_2^2 + L_3^2$$

in each of these subspaces is a multiple of a unit operator with eigenvalue $l(l+1)$.

Finally, in each of the subspaces $R^{\tau}_{n\,l}$ we choose a canonical basis $\xi^{(n,l,\tau)}_m$ (the basis is made up of the eigenvectors of the operator L_3). The eigenvalues of L_3 are $m = -l, -l+1, \ldots, l$. It follows from all the foregoing that in the basis $\{\xi^{(n,l,\tau)}_m\}$ the operator H_2, which is equal to

$$H_2 = \varepsilon_0|\Omega|E + 2\varepsilon_0 L_3 - \frac{J}{V}\left[L_1^2 + L_2^2 + L_3^2 - L_3^2 - \frac{1}{2}K\right],$$

can be reduced to diagonal form and its eigenvalues are

$$\lambda(n, l, m) = \varepsilon_0|\Omega| + 2\varepsilon_0 m - \frac{J}{V}\left[l(l+1) - m^2 - \frac{1}{2}n\right]. \quad (3)$$

The multiplicity of the eigenvalues is equal to the number of irreducible representations of weight l, contained in the space R_n (of course, a random coincidence of several eigenvalues for different n is possible). Hence, the eigenvalues of the entire operator H are equal to

$$\lambda = \lambda(n, l, m) + \sum_{\Omega} \varepsilon(\mathbf{k}_i), \quad \mathbf{k}_p \neq \mathbf{k}_{p'}, \quad p \neq p'.$$

We now consider the particle-number operator

$$\hat{N} = \sum_{s,\,\mathbf{k}} a_{\mathbf{k}s}^{*} a_{\mathbf{k}s}.$$

Again we break it up into two terms

$$\hat{N} = \hat{N}_1 + \hat{N}_2;$$

$$\hat{N}_1 = \sum_{s,\,\bar{\Omega}} a_{\mathbf{k}s}^{*} a_{\mathbf{k}s}, \qquad \hat{N}_2 = \sum_{s,\,\Omega} a_{\mathbf{k}s}^{*} a_{\mathbf{k}s}.$$

It is obvious that $\hat{N}_2 = |\Omega|E + 2L_3$.

Thus, the eigenvalues of the operator $\hat{N}_2$ are

$$N_2 = |\Omega| + 2m. \quad (4)$$

4. GROUND STATE IN THE SUBSPACE WITH FIXED NUMBER OF PARTICLES

We consider the proper subspace R_N of the operator $\hat{N}$ with eigenvalue equal to N, and investigate the smallest eigenvalue of the operator H in this subspace.[2] We note that

$$R_N = \sum_{N_1+N_2=N} R_{N_1} \otimes R_{N_2},$$

where R_{N_1} is a proper subspace of the operator $\hat{N}_1$ in the space R_1, and R_{N_2} is a proper subspace of the operator $\hat{N}_2$ in the space R_2.

From (4) we get

$$m = (N_2 - |\Omega|)/2.$$

Hence

$$\lambda(m, l, n) \equiv \lambda(N_2, l, n) = \varepsilon_0 N_2 - \frac{J}{V}\left[l(l+1) - \frac{(N_2 - |\Omega|)^2}{4} - \frac{n}{2}\right].$$

It is obvious that

$$\lambda^{(2)}_{min}(N_2) \equiv \min_{l,\,n} \lambda(N_2, l, n) = \varepsilon_0 N_2 - \frac{J}{V}\left[\frac{|\Omega|^2}{4} - \frac{(N_2 - |\Omega|)^2}{4}\right] = \varepsilon_0 N_2 - \frac{J}{4V} N_2[2|\Omega| - N_2].$$

[1] All the concepts and facts employed here from the theory of representations of the group U can be found in the books[4,5].

[2] The operators H and $\hat{N}$ commute and, consequently, the proper subspace of the operator $\hat{N}$ is invariant for H.

The smallest eigenvalue of the operator H_2 in R_2 is

$$\lambda^{(1)}_{min}(N_1) = \sum_{\mathbf{k}_i} \varepsilon(\mathbf{k}_i),$$

where $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{N_1}$ are the N_1 lattice points closest to zero from the region $\overline{\Omega}$. It is obvious that

$$\lambda_{min}(N) = \min_{N_1+N_2=N} (\lambda^{(1)}_{min}(N_1) + \lambda^{(2)}_{min}(N_2)).$$

We now make the transition to the thermodynamic limit

$$V\to\infty, \quad N\to\infty, \quad \frac{N}{V}\to\rho, \quad \frac{N_1}{V}\to\rho_1, \quad \frac{N_2}{V}\to\rho_2$$

($\rho_1 + \rho_2 = \rho$); we have

$$\frac{\lambda^{(1)}_{min}(N_1)}{V} \to \int_{|\mathbf{k}|<\nu(\rho_1)} \varepsilon(\mathbf{k})\,d\mathbf{k} = 4\pi \int_0^{\nu(\rho_1)} \varepsilon(k)k^2dk,$$

$\nu(\rho_1) = (3/4\pi^{-1}\rho_1)^{1/3}$ for $\nu(\rho_1) < \lambda_1$;

$\lambda^{(2)}_{min}(N_2)/V \to \rho_2[\varepsilon_0 - 1/4\, J(2s-\rho_2)]$

(s is the volume of the region Ω). Thus

$$\lambda_{min}(\rho_1,\rho_2) = V\left\{4\pi \int_0^{\nu(\rho_1)} \varepsilon(k)k^2dk + \rho_2\left[\varepsilon_0 - \frac{J}{4}(2s-\rho_2)\right]\right\} + o(V). \tag{5}$$

In order to find

$$\lambda_0(\rho) = \min_{\rho_1+\rho_2=\rho} \lambda_{min}(\rho_1,\rho_2),$$

it is necessary to find the minimum of expression (5) with the conditions:

$$\rho_1+\rho_2=\rho, \quad \rho_1\geqslant 0, \quad \rho_2\geqslant 0, \quad 2s-\rho_2\geqslant 0.$$

5. LOWER STATE FOR DIFFERENT ρ_1

Let us find the minimum

$$\min\frac{\lambda_{min}(\rho_1\rho_2)}{V} = \min\left\{4\pi \int_0^{\nu(\rho_1)} \varepsilon(k)k^2dk + \rho_2\left[\varepsilon_0 + \frac{J}{4}(\rho_2-2s)\right]\right\}$$

under the conditions

$$\rho_1+\rho_2=\rho, \quad \rho_1\geqslant 0, \quad \rho_2\geqslant 0, \quad \rho_2\leqslant 2s.$$

We note that

$$d\lambda/d\rho_1 = \varepsilon(\nu(\rho_1)) - \varepsilon_0 + J\rho_1/2 - J(\rho-s)/2.$$

We now consider two cases:

I. $Js/2 - \varepsilon_0 > 0$, II. $Js/2 - \varepsilon_0 < 0$.

The case $Js/2 - \epsilon_0 > 0$. A. For small ρ

$$J(s-\rho)/2 - \varepsilon_0 > 0. \tag{6}$$

Hence $d\lambda/d\rho_1|_{\rho_1=0} > 0$. Since the derivative $d\lambda/d\rho_1$ increases with increasing ρ_1, it remains positive. This means that the minimum of λ is reached when $\rho_1 = 0$, that is,

$$\lambda_{min} = \rho[\varepsilon_0 + 1/4 J(\rho - 2s)].$$

It follows from this that in the ground state only the singular phase is filled.

B. Let ρ_{cr} be such that

$$J(s-\rho_{cr})/2 - \varepsilon_0 = 0.$$

Then for $\rho > \rho_{cr}$ and $\rho < 2s$ the derivative $d\lambda/d\rho_1|_{\rho_1=0} < 0$, while for $\rho_1 = \rho$ we have

$$\frac{d\lambda}{d\rho_1}\Big|_{\rho_1=\rho} = \frac{Js}{2} - \varepsilon_0 + \varepsilon[\nu(\rho)] > 0.$$

Thus, the minimum is reached at some internal point $\rho_1 = \rho_1(\rho)$ and the energy is equal to

$$\lambda_{min} = V\left\{4\pi \int_0^{\nu(\rho_1)} \varepsilon(k)k^2dk + \rho_2[\varepsilon_0 + 1/4 J(\rho_2 - 2s)]\right\} + o(V).$$

Thus, when $\rho > \rho_{cr}$ and $\rho < 2s$, both the normal and the singular phase are contained in the ground state.

C. If $\rho > 2s$, then $\rho - 2s < \rho_1 < \rho$, from which we get

$$\frac{d\lambda}{d\rho_1}\Big|_{\rho_1=\rho-2s} = \varepsilon[\nu(\rho-2s)] - \varepsilon_0 - \frac{Js}{2} < 0,$$

$$\frac{d\lambda}{d\rho_1}\Big|_{\rho_1=\rho} = \frac{Js}{2} - \varepsilon_0 + \varepsilon[(\nu(\rho)] > 0.$$

We see therefore that if $\rho > 2s$, and

$$\varepsilon[\nu(2\rho - s)] < \varepsilon_0 + Js/2,$$

then the minimum is attained as before inside the gap $\rho - 2s < \rho_1 < \rho$, that is, $\rho_2 < 2s$. This means that both phases are present in the ground state, but the singular phase has not yet reached saturation. Finally, when

$$\varepsilon[\nu(\rho-2s)] - \varepsilon_0 - Js/2 > 0,$$

the minimum is reached at $\rho_1 = \rho - 2s$, and $\rho_2 = 2s$. This means that the singular phase has reached saturation, and further increase in the density is at the expense of the normal phase.

Case $Js/2 - \epsilon_0 < 0$. A. Let again ρ be small. Then

$$J(s-\rho)/2 - \varepsilon_0 < 0.$$

Thus

$$\left.\frac{d\lambda}{d\rho_1}\right|_{\rho_1=0} < 0,$$

$$\left.\frac{d\lambda}{d\rho_1}\right|_{\rho_1=\rho} = \varepsilon[\nu(\rho)] - \varepsilon_0 + \frac{Js}{2} < 0.$$

It follows therefore that the minimum is reached when $\rho_1 = \rho$ and

$$\lambda_{min} = \left[4\pi \int_0^{\nu(\rho)} \varepsilon(k) k^2\, dk\right] V + o(V),$$

that is, in this case only the normal phase is present.

B. Starting with a certain $\rho = \rho_{cr}$,

$$\varepsilon[\nu(\rho)] - \varepsilon_0 + Js/2 > 0,$$

and then the minimum is reached when $0 < \rho_1 < \rho$ and $0 < \rho_2 < 2s$, that is, when both phases are present.

C. Further increase in ρ causes an increase in the densities of both phases, and for certain value of ρ the singular phase reaches saturation and only the density of the normal phase increases.

6. GAP IN THE EXCITATION SPECTRUM

We investigate the behavior of the eigenvalue closest to the ground level. It turns out that in the case I, A, that is, when only the singular phase is filled, there exists in the spectrum a finite gap between the main eigenvalue and the one closest to it.

We write out again the expression for the eigenvalues:

$$\lambda = \sum_{\Omega} \varepsilon(\mathbf{k}_i) + \lambda(N_2, l, n).$$

We have already seen that for sufficiently small ρ we have $N_2 = N$ (in the case I, A) and

$$\lambda_{min} = \varepsilon_0 N + JN^2/2V - JN|\Omega|/2V.$$

Here $n = |\Omega|$ and $l = |\Omega|/2$. How can we change the eigenvalue (without changing the number of particles)? We see from (3) that this can be done in the following manner:

1) It is either necessary to take as before $n = |\Omega|$, $l = |\Omega|/2 - 1/2$, and $N_2 = N$. The increase in the eigenvalue will in this case be equal to

$$\Delta\lambda = J|\Omega|/2V.$$

When $V \to \infty$ we get $\Delta\lambda \to Js/2$, that is, its value remains finite.

2) Alternately, we can put $n = |\Omega| - 1$, $l = |\Omega|/2 - 1/2$, and $N_2 = N$. In this case

$$\Delta\lambda = J|\Omega|/2V - J/2V,$$

which gives $\Delta\lambda \to Js/2$ when $V \to \infty$.

3) Finally, we can transfer one of the particles to the normal phase, that is, put $n = |\Omega|$, $l = |\Omega|/2$, $N_2 = N - 1$, and $N_1 = 1$. This yields for $V \to \infty$

$$\Delta\lambda > J(s - \rho)/2 - \varepsilon_0 > 0.$$

It is clear that any other change of the parameters leads to a larger change in the eigenvalue.

We see, therefore, that for sufficiently small ρ, and when relation (6) is satisfied, we have in the spectrum of the operator H_N a gap equal to

$$\Delta\lambda = J(s - \rho)/2 - \varepsilon_0.$$

For other values of ρ, no gap can arise in our model. Indeed, in all other cases there exists the normal phase, which has, generally speaking, a continuous excitation spectrum. However, if we modify slightly the model, namely, if we assume that

$$\lim_{k \to \lambda_2 + 0} \varepsilon(k) = \varepsilon(\lambda_2 + 0) > \varepsilon_0,$$

that is, we consider the energy $\epsilon(k)$ with a discontinuity on the inner edge of the interaction region, then we find that at the density value at which the radius of the Fermi sphere for the normal phase reaches the internal boundary of the region Ω ($\nu(\rho) = \lambda_1$), a gap appears again in the excitation spectrum, equal to $\epsilon(\lambda_2 + 0) - \epsilon_0$. This circumstance suggests that were we to consider a model in which $\epsilon(k)$ is not constant in the interaction region but varies continuously from some value $\epsilon(\lambda_1)$ to $\epsilon(\lambda_2)$, with $\epsilon(\lambda_2) > \epsilon(\lambda_1)$, then the gap at $\nu(\rho) = \lambda_1$ would again exist. This is essentially the case considered by BCS.

7. CALCULATION OF THE PARTITION FUNCTION

Using the expression for the eigenvalue of the operator H, we can calculate the grand partition function ($\beta = 1/kT$):

$$\Xi = \mathrm{Sp}\exp(-\beta H - \mu\hat{N}) = \sum_{N_1 N_2,\, l,\, m,\, n} \exp\Big(-\beta\Big\{\sum_{\Omega} \varepsilon(\mathbf{k}_1) + \varepsilon_0 N_2 - \frac{J}{V}\Big[l(l+1) - m^2 - \frac{n}{2}\Big]\Big\} - \mu(N_1 + N_2)\Big).$$

We note that $Jn/2V < Js/2$, therefore the term $n/2$ is immaterial for the asymptotic behavior of Ξ.

Thus, we calculate

$$\tilde{\Xi} = \sum_{l,\, m,\, N_1} \exp\Big(-\beta\Big[\sum_{\Omega} \varepsilon(\mathbf{k}_1) + \varepsilon_0(|\Omega| + 2m) - \frac{J}{V}(l(l+1) - m^2)\Big]\Big).$$

$$\cdot \exp[-\mu N_1 - \mu(|\Omega| + 2m)]s(l), \qquad (7)$$

where $s(l)$ is the multiplicity of the representation of weight l in the representation $u \to T_u$. The sum (7) can be written in the form of the product

$$\tilde{\Xi} = \sum_{N_1} \exp\left\{-\beta\left[\sum_{\bar{\Omega}} \varepsilon(k_1)\right] - \mu N_1\right\}$$
$$\times \sum_{l,m} \exp[-(\beta\varepsilon_0 + \mu)(|\Omega| + 2m)]$$
$$\times \exp\left\{\frac{\beta J}{V}[l(l+1) - m^2]\right\} s(l).$$

The first factor represents the usual partition function for an ideal Fermi gas. We obtain

$$\Xi_1 = \prod_{\bar{\Omega}} (1 + e^{-\beta\varepsilon(k)}).$$

It remains to calculate the sum

$$\Xi_2 = \sum_{l,m} \exp\left\{-(\beta\varepsilon_0 + \mu)(|\Omega| + 2m) + \frac{J\beta}{2}[l(l+1) - m^2]\right\} s(l).$$

We first calculate $s(l)$. To this end we note that the character of the representation $u \to T_u$ is (see [5])

$$\chi_{T_u}(\varphi) = \left(2 + 2\cos\frac{\varphi}{2}\right)^{|\Omega|}.$$

The character of the irreducible representation of weight l, as is well known, is expressed in the form

$$\chi_l(\varphi) = \frac{\sin(l + 1/2)\varphi}{\sin(\varphi/2)}.$$

For this we get

$$s(l) = \frac{1}{4\pi}\int_0^{2\pi}\left(2 + 2\cos\frac{\varphi}{2}\right)^{|\Omega|}\frac{\sin(l+1/2)\varphi}{\sin(\varphi/2)}\sin^2\frac{\varphi}{2}\,d\varphi.$$

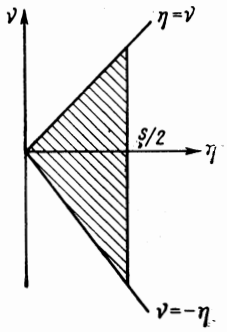
This integral can be evaluated and we obtain asymptotically as $V \to \infty$

$$s(l) \sim \left[\frac{4^s}{(1 - 2\eta/s)^{(s-2\eta)}(1 + 2\eta/s)^{(s+2\eta)}}\right]^V,$$

where $\eta = l/V$.

It is easy to see from this that the sum Ξ_2 coincides asymptotically with the integrals

$$\Xi_2 \sim \exp\{-Vs(\beta\varepsilon_0 + \mu)\}\int_0^{s/2} d\eta \int_{-\eta}^{\eta} d\nu \exp\{-V[2(\beta\varepsilon_0 + \mu)\nu - \beta J\eta^2 + J\nu^2\beta]\}$$
$$\times\left[\frac{4^s}{(1 - 2\eta/s)^{(s-2\eta)}(1 + 2\eta/s)^{(s+2\eta)}}\right]^V.$$



The region of integration is a triangle (see the figure). Two cases are possible:

1) The maximum of the expression

$$F(\eta, \nu) = 2\nu(\beta\varepsilon_0 + \mu) - \beta J\eta^2 + \beta J\nu^2 - (s - 2\eta)\ln(1 - 2\eta/s) - (s + 2\eta)\ln(1 + 2\eta/s)$$

is attained inside the triangle.

2) The maximum is attained outside the triangle. A separate formula is obtained for each case.

Let us consider the equation

$$\ln\left(\frac{s + 2\eta}{s - 2\eta}\right) = \varepsilon_0\beta + \mu \quad (\mu > 0).$$

It has, as can be readily verified, a unique solution η_{cr} on the segment $[0, s/2]$.

Assume now that

$$\partial F/\partial\eta = 2J\beta\eta - 2\ln(s + 2\eta) - 2\ln(s - 2\eta) = 0$$

or

$$\eta = \frac{1}{J\beta}\ln\left(\frac{s + 2\eta}{s - 2\eta}\right). \qquad (8)$$

Equation (8) has on the segment $[0, s/2]$ not more than one solution for $\bar{\eta}$.

Two cases are possible:

1) $\bar{\eta} > \eta_{cr}$. In this case the maximum of $F(\eta, \nu)$ is reached inside the triangle and is equal to

$$\max F(\eta, \nu) = \frac{(\beta\varepsilon_0 + \mu)^2}{J\beta} + \beta J\bar{\eta} - (s - 2\bar{\eta})\ln\left(1 - \frac{2}{s}\bar{\eta}\right) - (s + 2\bar{\eta})\ln\left(1 + \frac{2\bar{\eta}}{s}\right);$$

2) $\bar{\eta}$ either does not exist or else $\bar{\eta} < \eta_{cr}$. In this case the maximum is reached on the boundary of the triangle at the point $\nu = \eta = \eta_{cr}$ and is equal to

$$\max F(\eta, \nu) = (\varepsilon_0\beta + \mu)2\eta_{cr} - (s - 2\eta_{cr})\ln\left(1 - \frac{2\eta_{cr}}{s}\right) - (s + 2\eta_{cr})\ln\left(1 + \frac{2\eta_{cr}}{s}\right).$$

We thus arrive at two different expressions for the free energy for cases 1) and 2):

$$f(\beta, \mu) = f_{id}(\beta, \mu) - (\beta\varepsilon_0 + \mu)s + \max F(\eta, \nu),$$

where f_{id} is the free energy of the ideal Fermi gas (normal phase).

It is easy to verify that this result agrees with the general expression obtained by BCS[1] and in [2].

[1] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, Novyĭ metod v teorii sverkhprovodimosti (A New Method in the Theory of Superconductivity), Gostekhizdat, 1958.

[3] S. T. Belyaev, JETP **39**, 1387 (1960), Soviet Phys. JETP **12**, 968 (1961).

[4] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, Predstavleniya gruppy vrashcheniĭ i gruppy Lorentsa (Representation of the Rotation and Lorentz Groups), AN SSSR, 1958.

[5] G. Ya. Lyubarskiĭ, Teoriya grupp i ee primenenie v fizike (Group Theory and Its Use in Physics), Fizmatgiz, 1958.

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