THE GENERAL SOLUTION OF THE SIMULTANEOUS SYSTEM OF EINSTEIN'S AND NEWTON'S EQUATIONS

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All the solutions of Einstein's equations which in a suitably chosen coordinate system also satisfy the equations of Newtonian theory are found. The general solution of the problem consists of two series of particular solutions. Every solution of the first series represents the gravitational field of an isolated particle falling freely in an external, variable, homogeneous gravitational field. Such a solution may be interpreted either as the limiting case of the two-body problem, if the mass of one of the two points is infinite and located at infinity, or as a Schwarzschild field of only one mass point considered from an accelerated reference system. The solutions of the other series coincide essentially with Kasner's solution. The Schwarzschild solution can be derived directly from the corresponding solution of the Newtonian theory because the Schwarzschild field can be described by a solution of the simultaneous system of Einstein's and Newton's equations. The problem of the physical interpretation of the coordinate system and the related problem of the uniqueness of the relativistic corrections to the formulas of Newtonian theory, in particular the uniqueness of the Kustaanheimo-Lehti effect, are considered in the case of central symmetry as applications of the obtained solution. It is shown that Petrov's classification of gravitational fields has no analog in Newtonian theory.

1. INTRODUCTION

It is well known that the Schwarzschild solution can be derived by elementary methods, on the basis of the equivalence principle and Newtonian gravitational theory, without solving Einstein's equations.[1-7] There are many more or less satisfactory explanations of this fact. We shall show that the Schwarzschild field can equally well be considered as the solution of Einstein's equations, or as the solution of Newton's equations; in the light of this the above property of the Schwarzschild solution turns out to be trivial.

The solution of this problem makes it possible to cast light on many similar problems. We shall consider in more detail the problem of the uniqueness of the relativistic corrections to the formulas of Newtonian gravitational theory. These corrections are obtained by comparing the Newtonian formulas with the corresponding relativistic ones. However, the form of the formulas depends, on the one hand, on the nature of the gravitational field, and, on the other, on the reference system in which the gravitational field is considered. Since many relativistic reference systems may coincide in the Newtonian limit, it remains in general uncertain which relativistic reference system should be compared with a given nonrelativistic reference system. As a result of this there appears in the calculation of relativistic corrections a peculiar arbitrariness to which Rao[8,9] drew attention. He showed that the relativistic corrections depend on the choice of the coordinate conditions. This ambiguity is generally removed by the introduction of some general principle for choosing the coordinate conditions. For example, Fock[10] requires the introduction of the property of harmonicity of the relativistic reference system. However such a solution of the problem is not the only possible solution. We shall show that in certain particular cases it is advantageous to proceed differently.

As another particular problem we shall consider the Petrov classification of gravitational fields from the point of view of Newtonian theory.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

We will start with a coordinate system for which the metric form is[1)]

$$ds^2 = c^2dt^2 - \gamma_{rs}d\xi^r d\xi^s, \tag{2.1}$$

[1)]Latin indices take on the values 1, 2, and 3, Greek indices—the values 0, 1, 2, and 3.

where for $c \rightarrow \infty$ the coefficients γ_{ik} have finite limiting values. In this coordinate system Einstein's equations for a vacuum can be written in the form

$$R_{00} = 0, R_{i0} = 0, \quad P_{ik} + c^{-2}Q_{ik} = 0, \tag{2.2}$$

where P_{ik} is the three-dimensional Ricci tensor set up for γ_{ik}, and Q_{ik} is a certain quantity, an exact expression for which has been given in [11]. In the limit $c \rightarrow \infty$ and with the additional condition $R_{0ijk} \rightarrow 0$ Eqs. (2.2) go over into Newton's equations of the gravitational field:[11]

$$R_{00} = 0, \ R_{0ijk} = 0, \ P_{ik} = 0. \tag{2.3}$$

Our problem is the solution of the simultaneous system of (2.2) and (2.3).

By virtue of the last equation of the system (2.3) the three-dimensional space of coordinates ξ^i is Euclidean at each instant of time. One can, therefore, introduce a Cartesian rectangular coordinate system x^i in such a manner that the relative velocity $v^i(x^k, t)$ of the space of points ξ^i = const relative to the space of points x^i = const will satisfy the condition of being irrotational [11][2)]

$$v^i{}_{,k} - v^k{}_{,i} = 0.$$

Then v^i is the gradient of the scalar potential $\varphi(x^i, t)$:

$$v^i = \varphi_{,i}. \tag{2.4}$$

The Newtonian potential Φ is expressed in terms of φ according to the formula:[12]

$$\Phi = \varphi_{,0} + {}^1/_2\varphi_{,s}\varphi_{,s}. \tag{2.5}$$

In the system of coordinates x^i, t the metric form (2.1) takes on the form:

$$ds^2 = (c^2 - \varphi_{,s}\varphi_{,s})\,dt^2 + 2\varphi_{,s}dx^s dt - dx^s dx^s, \tag{2.6}$$

and the system (2.2) and (2.3) reduces to the following system:

$$\varphi_{,ss0} + \varphi_{,r}\varphi_{,ssr} + \varphi_{,rs}\varphi_{,rs} = 0, \tag{2.7}$$

$$\varphi_{,ik0} + \varphi_{,s}\varphi_{,sik} + \varphi_{,ss}\varphi_{,ik} = 0. \tag{2.8}$$

Our problem consists thus of finding the function $\varphi(x^i, t)$ from the system of equations (2.7) and (2.8).

3. ONE PARTICULAR SOLUTION

The system (2.7) and (2.8) has a particular solution:

$$\varphi(x^i, t) = -2\eta\,(2GMr)^{1/2}, \ r^2 = x^s x^s, \ \eta = \pm 1, \tag{3.1}$$

2)A comma before a subscript indicates ordinary differentiation, a point and a comma covariant differentiation.

to which there corresponds according to (2.5) the Newtonian potential

$$\Phi = GMr^{-1}, \tag{3.2}$$

i.e., the potential of the gravitational field of the point mass M, and according to formula (2.6) the metric form

$$ds^2 = c^2(1 - r_0 r^{-1})\,dt^2 - 2\eta c r_0^{1/2} r^{-1/2} dr dt$$
$$- dr^2 - r^2(d\theta^2 + \sin^2\theta d\vartheta^2) \tag{3.3}$$

(we have written it in the polar coordinates r, θ, ϑ), where

$$r_0 = 2GMc^{-2}. \tag{3.4}$$

The transformation

$$ct = c\tau + 2\eta\,(rr_0)^{1/2}$$
$$+ r_0 \ln|\,(r^{1/2} - \eta r_0^{1/2})\,(r^{1/2} + \eta r_0^{1/2})^{-1}| \tag{3.5}$$

transforms the linear element (3.3) into the linear element of Schwarzschild. We see that the function φ being a solution of the simultaneous system of Einstein's and Newton's equations characterizes the gravitational field of a point mass both in the relativistic and in the nonrelativistic theory.

The factor η in Eq. (3.3) is of no appreciable importance. Finkelstein[13] expressed the opinion that for $\eta = +1$ the metric form (3.3) describes the gravitational field of the usual material particle, whereas $\eta = -1$ describes the field of the "antiparticle." Actually the transition in formula (3.3) from $\eta = +1$ to $\eta = -1$ indicates a transition to a new time coordinate t′

$$ct = ct' + 4(rr_0)^{1/2}$$
$$+ 2r_0 \ln|\,(r^{1/2} - r_0^{1/2})\,(r^{1/2} + r_0^{1/2})^{-1}|, \tag{3.6}$$

i.e., a change in the allowance for simultaneity in the space. The fact that the Newtonian potential is independent of η is also evidence that the function describes the same gravitational field for both values of η.

The scalar φ is specified in a strictly Euclidean three-dimensional space in a system of polar coordinates r, θ, and ϑ. In this space one can consider the relativistic laws of mechanics and electrodynamics with the space-time metric (3.3) which characterizes the field φ in its relativistic aspect, and the corresponding nonrelativistic laws with the gravitational potential (3.2) which characterizes the field φ from the point of view of Newton's gravitational theory. The situation can be interpreted as if we had introduced into the framework of the relativistic theory of gravitation without distortion a Newtonian gravitational field, and produced thereby the conditions for a direct comparison of nonrelativistic and relativistic formulas.

The polar coordinates r, θ, and ϑ in (3.3) are the same as those used in Newtonian mechanics. Consequently, the relativistic corrections to the Newtonian formulas which contain only the variables r, θ, and ϑ are obtained uniquely. Such corrections are the secular shift of the perihelion of Mercury and the bending of a light ray (as an effect of the anisotropy of motion of the light signal).

Rao[9] gave an expression for the secular shift of the perihelion of Mercury as a function of coordinate conditions and concluded from it that in the general theory of relativity one can obtain for the magnitude of the shift of the perihelion of Mercury an arbitrary theoretical value, and that for this reason the general theory of relativity does not "explain" this effect and merely permits the selection of certain privileged reference systems from the point of view of the terrestrial observer. However, as we have seen, in the case considered the space part of the reference system will be naturally fixed independently of the phenomenon under consideration, so that the effect of the shift of the perihelion of Mercury will be calculated as a unique theoretical prediction, and in agreement with the observations.

If the central mass M is not at the origin $x^i = 0$, but at some point $x^1 = \xi$, $x^2 = 0$, and $x^3 = 0$ on the positive x^1 axis, then all the cited formulas remain valid if r is determined by the formula

$$r^2 = (x^1 - \xi)^2 + (x^2)^2 + (x^3)^2.$$

The Newtonian gravitational force acting on a unit weight at the fixed point x^i is

$$F_i = \Phi_{,i} = -GMr^{-2}[(x^1 - \xi)r^{-1},\ x^2r^{-1},\ x^3r^{-1}].$$

Let the center of attraction move along the positive x^1 axis to infinity with the mass M increasing in such a way that the ratio M/ξ^2 remains finite and tends to a definite limit m. Then $r/\xi \to 1$ and in the limit the force F_i will be of constant magnitude $F = Gm$ directed along the positive x^1 axis. With the aid of the considered passage to the limit one obtains from a centrally-symmetric gravitational field a homogeneous gravitational field produced by a material point of infinite mass located at infinity. This homogeneous field can be derived from the Newtonian potential

$$\Phi = Gmx + {}^1/_2 G^2m^2t^2$$

(we have written x instead of x^1), which is in turn derived according to formula (2.5) from the scalar

$$\varphi = Gmtx, \tag{3.7}$$

which satisfies the system (2.7) and (2.8). Thus the homogeneous field (3.7) is both Newtonian and relativistic, and according to (2.6) it can be described with the aid of the metric form

$$ds^2 = c^2(1 - G^2m^2t^2c^{-2})dt^2 + 2Gmtdxdt$$
$$- dx^2 - dy^2 - dz^2. \tag{3.8}$$

Thus the form (3.8) derived from the Schwarzschild equation by a passage to the limit describes the gravitational field of a point infinite mass located at infinity. However the form (3.8) can also be obtained from the linear element of Minkowski

$$ds^2 = c^2dt^2 - d\bar{x}^2 - dy^2 - dz^2 \tag{3.9}$$

by means of the transformation

$$\bar{x} = x - {}^1/_2 GMt^2, \tag{3.10}$$

i.e., by means of a transition from an inertial reference system to an accelerated system. Now the form (3.8) describes the inertial field of the special theory of relativity considered from an accelerated reference system. The form (3.8) admits therefore two equally justified physical interpretations. This is an expression of the equivalence principle of the general theory of relativity. In particular, it follows from the possibility of transforming the form (3.8) to the form (3.9) that the homogeneous gravitational field of an infinite point mass located at infinity can be described with the aid of the metric form of Minkowski (3.9).

4. THE KUSTAANHEIMO-LEHTI EFFECT

The scalar φ given by formula (3.1) does not depend on the time. Unlike in the case of a spatial coordinate system, insertion of the Newtonian field into the framework of relativity theory determines the relativistic time coordinate t ambiguously, accurate to the transformation (3.6). Consequently the relativistic corrections to the Newtonian formulas containing the time are determined in general ambiguously. There are however exceptions. As an example let us consider the Kustaanheimo-Lehti effect,[14] viz., the lengthening of the sidereal period of motion of a planet. In our coordinate system this effect can be calculated as follows.

For the metric (3.3) the differential equation which determines the orbit of the particle moving in the plane $\theta = {}^1/_2\pi$ is of the form[3)]

$$u'' + u = \alpha p^{-1} + {}^3/_2 r_0 u^2, \tag{4.1}$$

where $u = 1/r$, p — constant, $\alpha = 1$ in the case of a material point and $\alpha = 0$ in the case of a photon.

[3)]A prime denotes differentiation with respect to ϑ, and a dot over a letter – differentiation with respect to t.

The general solution of Eq. (4.1) calculated accurate to first-order terms in r_0 is

$$u = (\alpha + e\cos\vartheta)p^{-1}\{1 + {}^1/_2 r_0 p^{-1}[{}^5/_2\alpha - e\cos\vartheta + ({}^1/_2\alpha^2 + 2e^2 + 3\alpha e\vartheta\sin\vartheta)(\alpha + e\cos\vartheta)^{-1}]\}. \quad (4.2)$$

It has been assumed here that at the perihelion $\vartheta = 0$; e is the constant of integration. In addition the integral of

$$(d\vartheta / ds)^2 = {}^1/_2 r_0 p u^4 \quad (4.3)$$

exists. Eliminating ds from (3.3) and (4.3) and taking into account the relation $\dot{u} = u'\dot{\vartheta}$, we obtain an equation for $\dot{\vartheta}$:

$$\dot{\vartheta}^2 - GMpu^4 + {}^1/_2 p r_0 u^2[(1 + u'^2 u^{-2})\dot{\vartheta}^2 - 2\eta(2GMu)^{1/2}u'\dot{\vartheta} + 2GMu^3) = 0, \quad (4.4)$$

from which it follows, assuming $\alpha = 1$ and using formula (4.2), that

$$dt = d\vartheta p^{3/2}(GM)^{-1/2}(1 + e\cos\vartheta)^{-2}\{1 + {}^1/_2 r_0 p^{-1} \times[-{}^7/_2 + {}^1/_2 e^2 + 4e\cos\vartheta - (1 + 4e^2 + 6e\vartheta\sin\vartheta) \times(1 + e\cos\vartheta)^{-1} + \eta\sqrt{2}e\sin\vartheta(1 + e\cos\vartheta)^{1/2}]\}. \quad (4.5)$$

Assume that we have for $\vartheta = 0$ the initial values: $r = q$ and $\dot{\vartheta} = \mu$. Substituting these values in (4.2) and (4.5), we obtain two equations for e and p from which we find

1) for $r_0 = 0$ (nonrelativistic motion)

$$GMp_0 = \mu^2 q^4, \qquad q(1 + e_0) = p_0;$$

2) for $r_0 \neq 0$ (relativistic motion)

$$p = p_0[1 + {}^1/_2 r_0 q^{-1}(3 + e_0)],$$
$$e = e_0\{1 + {}^1/_4 r_0 q^{-1}[6 + 2e_0 + 5(1 + e_0)^{-1}]\}. \quad (4.6)$$

If we substitute (4.6) in (4.5), we obtain

$$dt = d\vartheta p_0^{3/2}(GM)^{-1/2}(1 + e_0\cos\vartheta)^{-2} \times\{1 + r_0 q^{-1}[-1 + (2e_0\cos\vartheta - 4) \times (1 + e_0)^{-1} + (5 - e_0)(1 + e_0\cos\vartheta)^{-1} - 3e_0\vartheta\sin\vartheta(1 + e_0)^{-1}(1 + e_0\cos\vartheta)^{-1} + \eta e_0\sin\vartheta(1 + e_0\cos\vartheta)^{1/2}2^{-1/2}(1 + e_0)^{-1}]\}. \quad (4.7)$$

Integration from 0 to 2π yields the sidereal period of the planet

$$P = P_0\{1 + {}^3/_2 r_0 q^{-1}[-1 - e_0 + 2(1 - e_0)^{-1} - (1 - e_0)^{3/2}(1 + e_0)^{-3/2}]\}, \quad (4.8)$$

where

$$P_0 = 2\pi a_0^{3/2}(GM)^{-1/2}, \qquad a_0 = p_0(1 - e_0^2)^{-1}. \quad (4.9)$$

P_0 is the period of the planet moving for the given initial conditions according to the laws of Newtonian mechanics, P is the corresponding relativistic period of motion of the planet with the same initial conditions. The difference between the two periods is in general negligible, and in the case $e_0 = 0$ there is altogether no difference. However, when e_0 is close to unity, then in the right-hand part of Eq. (4.8) the term containing $1 - e_0$ in the denominator will be large compared with the remaining terms and we obtain the approximate expression

$$P = P_0[1 + 3r_0 q^{-1}(1 - e_0)^{-1}] = P_0(1 + 6GMa_0c^{-2}q^{-2}). \quad (4.10)$$

This formula expresses the effect of the lengthening of the sidereal period of a comet with an orbit having a large eccentricity. The effect is calculated uniquely. Formula (4.10) coincides with formula (33) of Kustaanheimo-Lehti if one assumes in the latter that $\gamma = 0$. For $\gamma = 1$ the Kustaanheimo-Lehti formula yields a somewhat larger value for the effect.

5. THE GENERAL SOLUTION

Let us return to the system (2.7) and (2.8) which we rewrite in the form

$$\Phi_{,ik} = \varphi_{,si}\varphi_{,sk} + \varphi_{,ss}\varphi_{,ik}, \qquad \Phi_{,ss} = 0. \quad (5.1)$$

From these equations one can readily derive the relation

$$\Phi_{,ik} = \text{the cofactor of the element } \varphi_{,ik} \text{ in } \det(\varphi_{,ik}), \quad (5.2)$$

whence, if we set $\det(\varphi_{,ik}) = A$, it follows that

$$\varphi_{,si}\Phi_{,sk} = A\delta_{ik}. \quad (5.3)$$

If we differentiate the first Eq. (5.1) with respect to t, substitute in place of the third derivatives of the type $\varphi_{,ik0}$ their expressions from Eqs. (2.7) and (2.8), and then take into account Eq. (5.1) and the relation

$$\varphi_{,rs}\varphi_{,rs} - \varphi_{,ss}{}^2 = 0, \quad (5.4)$$

which follows from it, we obtain the equation

$$\Phi_{,ik0} + \varphi_{,s}\Phi_{,sik} + 2\varphi_{,ss}\Phi_{,ik} = 0. \quad (5.5)$$

Differentiating with respect to x^j and then interchanging j and k, we obtain with account of (5.3)

$$A_{,k}\delta_{ij} - A_{,j}\delta_{ik} + 2(\varphi_{,ssj}\Phi_{,ik} - \varphi_{,ssk}\Phi_{,ij}) = 0. \quad (5.6)$$

Summation over i and j yields

$$A_{,h} + \varphi_{,ssr}\Phi_{,rh} = 0. \quad (5.7)$$

Now one must differentiate two cases: $\varphi_{,ssi} \neq 0$ and $\varphi_{,ssi} = 0$.

1. $\varphi_{,ssi} \neq 0$. In this case we have also $A_{,i} \neq 0$ if we do not consider the case of a homogeneous gravitational field which we have already consid-

ered above. Actually, if $A_{,i} = 0$, then it follows from (5.6) that

$$\Phi_{,ik} = k_i \varphi_{,ssk}, \tag{5.8}$$

and hence $k_i\varphi_{,ssk} = k_k\varphi_{,ssi} = 0$, from which it follows that $k_i = k\varphi_{,ssi}$, so that we have the relation $\Phi_{,ik} = k\varphi_{,ssi}\varphi_{,ssk}$, or on the basis of (5.1) $k\varphi_{,ssp}\varphi_{,rrp} = 0$. Since, according to the assumption $\varphi_{,ssi} \neq 0$, $k = 0$, and consequently according to (5.8) $\Phi_{,ik} = 0$, i.e., the gravitational field is homogeneous.

Multiplying (5.6) by $\varphi_{,ssi}$ and summing over i, we find with account of (5.7) that

$$A_{,k}\varphi_{,ssj} - A_{,j}\varphi_{,ssk} = 0,$$

whence it follows that

$$\varphi_{,ssi} = \lambda A_{,i}. \tag{5.9}$$

If we substitute in (5.6) instead of $\varphi_{,ssi}$ expression (5.9), we obtain an equation from which it follows that

$$\delta_{ik} - 2\lambda\Phi_{,ik} = \varkappa A_{,i}A_{,k}, \tag{5.10}$$

$$\varkappa A_{,s}A_{,s} = 3. \tag{5.11}$$

In order to determine the coefficient λ, we set up a determinant, taking into account (5.10):

$$8\lambda^3 \det(\Phi_{,ik}) = \det(\delta_{ik} - \varkappa A_{,i}A_{,k}).$$

By virtue of (5.2) we have $\det(\Phi_{,ik}) = A^2$. Consequently, bearing in mind (5.11), we obtain the equation $4\lambda^3A^2 + 1 = 0$, from which it follows that

$$\lambda = -(2A)^{-2/3}. \tag{5.12}$$

We multiply now (5.10) and (5.7) by $\varphi_{,kj}$, sum over k, and then eliminate from the obtained two equations $A_{,s}\varphi_{,sj}$. As a consequence of (5.3), (5.9), and (5.12), we obtain the relation

$$\varphi_{,ij} = -2^{-2/3}A^{1/3}(2\delta_{ij} - \varkappa A_{,i}A_{,j}), \tag{5.13}$$

from which it follows that

$$\varphi_{,ss} = -3\cdot 2^{-2/3}A^{1/3}. \tag{5.14}$$

In order to obtain an equation for the determination of A, we eliminate φ from (5.13) by differentiation and interchange:

$$A_{,k}\delta_{ij} - A_{,j}\delta_{ik} = {}^3/_2A[\varkappa(A_{,ik}A_{,j} - A_{,ij}A_{,k}) + A_{,i}(\varkappa_{,k}A_{,j} - \varkappa_{,j}A_{,k})]. \tag{5.15}$$

Multiplying by $A_{,i}$ and summing over i with allowance for the fact that on the basis of (5.11) we have

$$\varkappa_{,i} = -{}^2/_3\varkappa^2A_{,s}A_{,si}, \tag{5.16}$$

we see that

$$\varkappa_{,k}A_{,j} - \varkappa_{,j}A_{,k} = 0. \tag{5.17}$$

By virtue of this the equality (5.15) takes on the form

$$A_{,k}(\delta_{ij} + {}^3/_2\varkappa AA_{,ij}) - A_{,j}(\delta_{ik} + {}^3/_2\varkappa AA_{,ik}) = 0,$$

and from this it follows that

$$\delta_{ik} + {}^3/_2\varkappa AA_{,ik} = \mu A_{,i}A_{,k}, \tag{5.18}$$

where μ is some scalar.

The condition for the integrability of (5.18) is

$$(\mu + {}^3/_2\varkappa)(A_{,k}\delta_{ij} - A_{,j}\delta_{ik}) + {}^3/_2A(\varkappa_{,k}\delta_{ij} - \varkappa_{,j}\delta_{ik}) = {}^3/_2\varkappa AA_{,i}(\mu_{,j}A_{,k} - \mu_{,k}A_{,j}). \tag{5.19}$$

Multiplying by $A_{,i}$ and summing over i, we find with account of (5.17) and (5.11) that

$$\mu_{,j}A_{,k} - \mu_{,k}A_{,j} = 0,$$

as a result of which (5.19) is brought to the form

$$(\mu + {}^3/_2\varkappa)A_{,k} + {}^3/_2A\varkappa_{,k} = 0. \tag{5.20}$$

On the other hand, multiplying (5.18) by $A_{,i}$, summing over i and taking into account (5.16) and (5.11), we obtain

$$(\mu - {}^1/_3\varkappa)A_{,k} + {}^3/_4A\varkappa_{,k} = 0. \tag{5.21}$$

Comparison of (5.20) and (5.21) shows that

$$\mu = {}^{13}/_6\varkappa \tag{5.22}$$

and

$$\varkappa_{,i}\varkappa^{-1} = -22A_{,i}(9A)^{-1}$$

or

$$\varkappa = k(t)A^{-22/9}. \tag{5.23}$$

With the aid of the obtained values of κ and μ Eq. (5.18) can be expressed in the form

$$^{27}/_8k(A^{-4/9})_{,ik} = \delta_{ik},$$

whence it follows with the additional condition (5.11) that

$$^{27}/_4kA^{-4/9} = r^2, \quad r^2 = (x^s - a^s)(x^s - a^s), \quad a^i = a^i(t). \tag{5.24}$$

Knowing A, we obtain from (5.13) an equation for φ:

$$\varphi_{,ik} = -\eta(2GM)^{1/2}r^{-3/2} \times[\delta_{ik} - {}^3/_2(x^i - a^i)(x^k - a^k)r^{-2}], \tag{5.25}$$

where we have used the notation

$$GM = 2^{-1/3}(^{27}/_4k)^{3/2}, \qquad \eta = \pm 1,$$

G is the gravitational constant.

The general solution of Eq. (5.25) is of the form

$$\varphi = -2\eta(2GMr)^{1/2} + b_s(t)x^s + b(t). \tag{5.26}$$

But this expression satisfies the system (2.7) and (2.8) if and only if

$$\dot{M} = 0, \quad b_i = \dot{a}^i.$$

Consequently, the general solution of the system (2.7) and (2.8) is

$$\varphi = -2\eta(2GMr)^{1/2} + \dot{a}^s x^s + b(t), \qquad (5.27)$$

where M is a constant, and b(t) and $a^i(t)$ are arbitrary functions of t.

The field (5.27) has the Newtonian potential

$$\Phi = GMr^{-1} + \ddot{a}^s x^s + {}^1/_2 \dot{a}^s \dot{a}^s + \dot{b}, \qquad (5.28)$$

and the metric form which describes this field is

$$ds^2 = [c^2 - 2GMr^{-1} + 2\eta(2GM)^{1/2} r^{-3/2} \dot{a}^s (x^s - a^s) - \dot{a}^s \dot{a}^s] dt^2$$
$$-2[\eta(2GM)^{1/2} r^{-3/2}(x^s - a^s) - \dot{a}^s] dx^s dt - dx^s dx^s. \quad (5.29)$$

From the expression (5.28) for the Newtonian potential it is seen that the scalar φ describes the gravitational field of a point mass M together with the external variable homogeneous field in which the mass M falls freely. If $\ddot{a}^i$ = const, i.e., the homogeneous field is constant, then it can be considered as the gravitational field of an infinite point mass located at infinity. With such an interpretation of the solution (5.27) it represents a limiting case of the solution of the two-body problem. However, the form (5.29) can be derived from the form (3.3) by the transformation $x^i \to x^i - a^i$. Therefore the field (5.27) can also be interpreted as the Schwarzschild field of one point mass considered from an accelerated reference system. We note that the possibility of transforming the form (5.29) into the form (3.3) indicates that the Schwarzschild solution is consistent with a distribution of infinite masses at infinity.

2. $\varphi_{,ssi} = 0$. If $\varphi_{,ss} = 0$ then by virtue of (5.4) $\varphi_{,ik} = 0$ too, since φ and consequently also Φ are linear functions of the coordinates x^i, and φ describes thus a homogeneous gravitational field. Setting this already considered case aside, one can assume that $\varphi_{,ss} \neq 0$. Then one obtains from (2.7) and (5.4)

$$\varphi_{,ss} = t^{-1}. \qquad (5.30)$$

Differentiation of (5.4) yields

$$\varphi_{,rs}\varphi_{,rsi} = 0.$$

Differentiating again with respect to x^i and summing over i, we obtain the equality

$$\varphi_{,rsi}\varphi_{,rsi} = 0,$$

whence it follows that $\varphi_{,ikj} = 0$, or

$$(t\varphi)_{,ikj} = 0. \qquad (5.31)$$

To this we add Eq. (2.8) which can be written in the form

$$(t\varphi)_{,ik0} = 0. \qquad (5.32)$$

From Eqs. (5.31) and (5.32) it follows that

$$\varphi = {}^1/_2 t^{-1} a_{rs} x^r x^s + b_s(t) x^s + b(t), \qquad (5.33)$$

where $b_i(t)$ and b(t) are arbitrary functions and a_{ik} are constants which satisfy as a result of relations (5.30) and (5.4) the following conditions:

$$a_{ik} = a_{ki}, \quad a_{ss} = 1, \quad a_{rs}a_{rs} = 1. \qquad (5.34)$$

The function (5.33) is the general solution of the system (2.7) and (2.8) with the condition $\varphi_{,ssi} = 0$.

The Newtonian potential of the field (5.33) is of the form

$$\Phi = {}^1/_2 t^{-2} A_{rs} x^r x^s + (\dot{b}_s + t^{-1} a_{ps} b_p) x^s$$
$$+ \dot{b} + {}^1/_2 b_s b_s, \qquad (5.35)$$

where A_{ik} is the cofactor of the element a_{ik} in the det (a_{ik}). The metric form describing the field (5.33) is:

$$ds^2 = (c^2 - t^{-2} a_{pr} a_{ps} x^r x^s - 2t^{-1} a_{ps} b_p x^s - b_s b_s) dt^2$$
$$+ 2(t^{-1} a_{ps} x^p + b_s) dx^s dt - dx^s dx^s. \qquad (5.36)$$

One possible choice of the constants a_{ik} is:

$$a_{22} = p_2, \; a_{33} = p_3, \; a_{12} = a_{23} = a_{31} = 0,$$

with

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \qquad (5.37)$$

Koppel showed that in this case and for $b_i = b = 0$ the transformation

$$x^i = y_i t^{p_i}$$

reduces the metric form (5.36) to the form

$$ds^2 = c^2 dt^2 - t^{2p_1} dy_1^2 - t^{2p_2} dy_2^2 - t^{2p_3} dy_3^2. \quad (5.38)$$

This is Kasner's[15] solution which is of such importance in the cosmological investigations of E. Lifshitz and Khalatnikov.[16] The nature of the singularity of the solution (5.38) for t = 0 is seen from the expression for the Newtonian potential of the field (5.38):

$$\Phi = {}^1/_2 t^{-2} [p_2 p_3 (x^1)^2 + p_3 p_1 (x^2)^2 + p_1 p_2 (x^3)^2]. \quad (5.39)$$

The general case (5.35) does not differ considerably from the special case (5.39). In the identity

$$A_{ss}^2 - A_{rs} A_{rs} \equiv 2a a_{ss}, \qquad a = \det(a_{ik})$$

we have by virtue of (5.34) $A_{ss} = 0$ and $a_{ss} = 1$, so that there remains the equality

$$2a = -A_{rs} A_{rs}.$$

If follows hence that $a \leq 0$. If a = 0, then $A_{ik} = 0$

too, and the field (5.35) is homogeneous. One can therefore assume that $a < 0$.

The real symmetric matrix (a_{ik}) has three mutually orthogonal unit eigenvectors $\gamma_i = \{\gamma_i^k\}$ which belong to the roots p_i of the characteristic equation

$$\det(a_{ik} - p\delta_{ik}) = 0$$

or

$$p^3 - p^2 - a = 0.$$

It is seen from the latter equation that the numbers p_i satisfy condition (5.37). In the coordinate system y^i related to the coordinate system x^i by the orthogonal transformation

$$x^i = \gamma_s^i(y^s - c^s), \qquad c^k = tb_s\gamma_k^s p_k^{-1},$$

the scalar φ is of the form

$$\varphi = {}^1/_2 t^{-1}[p_1(y^1)^2 + p_2(y^2)^2 + p_3(y^3)^2] + \bar{b},$$

and the potential Φ is expressed by the formula

$$\Phi = {}^1/_2 t^{-2}[p_2p_3(y^1)^2 + p_3p_1(y^2)^2 + p_1p_2(y^3)^2] + \dot{\bar{b}}. \quad (5.40)$$

6. A. Z. PETROV'S CLASSIFICATION

In order to carry out a Petrov classification of gravitational fields one can use the theory of quadratic forms in complex three-dimensional plane space R_3.[17] We propose the following practical method based on the isomorphism of the Lorentz group and the group of orthogonal transformations in the complex space R_3.

Let $e^\mu = \{e^\mu\}$ be mutually orthogonal unit vectors, such that we have

$$g^{\rho\sigma}e_\rho^\mu e_\sigma^\nu = \eta_{\mu\nu}, \qquad \eta_{\mu 0} = \delta_{\mu 0}, \qquad \eta_{ik} = -\delta_{ik}. \quad (6.1)$$

Transition to another orthogonal frame of reference f^μ is accomplished by means of the Lorentz transformation

$$e^\mu = b_\sigma^\mu f^\sigma, \quad (6.2)$$

$$\eta_{\rho\sigma}b_\mu^\rho b_\nu^\sigma = \eta_{\rho\sigma}b_\rho^\mu b_\sigma^\nu = \eta_{\mu\nu}, \qquad \det(b_\mu^\nu) = 1,$$
$$b_0^0 > 0. \quad (6.3)$$

Then the complex quadratic forms

$$E^j = \varepsilon_{jrs}e^r e^s + i(e^0e^j - e^je^0)$$

are transformed into the forms

$$F^j = \varepsilon_{jrs}f^r f^s + i(f^0f^j - f^jf^0)$$

with the aid of the complex orthogonal transformation

$$E^j = B_s^j F^s, \quad (6.4)$$

$$B_k^j = b_0^0 b_k^j - b_k^0 b_0^j - i\varepsilon_{jrs}b_0^r b_k^s$$
$$= {}^1/_2\varepsilon_{kpq}\varepsilon_{jrs}b_p^r b_q^s + i\varepsilon_{krs}b_r^0 b_s^j, \quad (6.5)$$

$$B_j^s B_k^s = \delta_{jk}, \qquad \det(B_j^k) = 1. \quad (6.6)$$

Here we have used the notation

$$e^\mu e^\nu - e^\nu e^\mu = \{e_\alpha^\mu e_\beta^\nu - e_\alpha^\nu e_\beta^\mu\},$$

where ϵ_{ijk} is antisymmetric with respect to each pair of indices and $\epsilon_{123} = 1$.

Let $R_{(\mu\nu\kappa\lambda)}$ be the components of the curvature tensor in the frame of reference e^μ. We shall use the notation

$$R_{(0i0k)} = M_{ik}, \qquad R_{(0ijk)} = N_{is}\varepsilon_{jks},$$
$$R_{(ij0k)} = \varepsilon_{ijs}K_{sk}, \qquad R_{(ijhk)} = \varepsilon_{ijr}\varepsilon_{hks}L_{rs}.$$

By virtue of the symmetry properties and in the case when the curvature tensor satisfies Einstein's equations $R_{\mu\nu} = 0$ or $\eta_{\rho\sigma}R_{(\rho\mu\nu\sigma)} = 0$ the following relations hold

$$M_{ik} = M_{ki}, \qquad L_{ik} = L_{ki}, \qquad N_{ik} = N_{ki} = K_{ik},$$
$$M_{ik} + L_{ki} = 0, \qquad M_{ss} = N_{ss} = 0.$$

By virtue of these relations we have the equality

$$R_{(\rho\sigma\mu\nu)}e^\rho e^\sigma e^\mu e^\nu = -\mathrm{Re}(T),$$
$$T = T_{rs}E^rE^s, \qquad T_{jk} = M_{jk} + iN_{jk}. \quad (6.7)$$

According to (6.7) the problem of bringing the curvature tensor into canonical form with the aid of the Lorentz transformation reduces to the problem of bringing the complex quadratic form T to canonical form with the aid of an orthogonal transformation. The matrix

$$\begin{pmatrix} M & N \\ K & L \end{pmatrix} = \begin{pmatrix} M & N \\ N & -M \end{pmatrix} \quad (6.8)$$

then takes on one of Petrov's canonical forms.[17]

In order to bring the form T to canonical form with the aid of an orthogonal transformation, one must solve the characteristic equation

$$\det(T_{ik} - \lambda\delta_{ik}) = 0$$

and determine the eigenvectors $B_k = \{B_k^j\}$ from the system

$$(T_{sj} - \lambda\delta_{sj})B^s = 0.$$

By virtue of the relation $T_{ss} = 0$ the roots of the characteristic equation satisfy the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \quad (6.9)$$

Case I. There exist three mutually orthogonal unit eigenvectors B_k. The form T is transformed with the aid of the orthogonal transformation

$$E^j = B_s^j F^s \quad (6.10)$$

into the canonical form

$$T = \lambda_1 F^1 F^1 + \lambda_2 F^2 F^2 + \lambda_3 F^3 F^3. \qquad (6.11)$$

Hence we determine the canonical values of the elements of the matrices M and N. The Lorentz transformation corresponding to the transformation (6.10) can be found by solving the equations (6.5) and (6.3) for b_μ^ν:

$$b_0{}^0 = \lambda^{1/2}, \quad b_k{}^j = \lambda^{-1/2}\Lambda_k{}^j,$$

$$b_0{}^k = {}^1/{}_2\lambda^{-1/2}\varepsilon_{kpq}\mu_s{}^p\lambda_s{}^q, \qquad b_k{}^0 = {}^1/{}_2\lambda^{-1/2}\varepsilon_{kpq}\lambda_p{}^s\mu_q{}^s, \quad (6.12)$$

where $\lambda = \det(\lambda_i^k)$, Λ_i^k is the cofactor of the element λ_k^i in $\det(\lambda_k^i)$ and

$$B_k{}^j = \lambda_k{}^j + i\mu_k{}^j.$$

Case II. One of the roots λ_i, for instance $\lambda_2 (=\lambda_3)$ is a multiple root and there exist two eigenvectors $\bar{B}_1$ and $\bar{B}_2$ which satisfy the conditions

$$\bar{B}_1{}^s\bar{B}_2{}^s = 0, \qquad \bar{B}_2{}^s\bar{B}_2{}^s = 0, \qquad \bar{B}_1{}^s\bar{B}_1{}^s = 1.$$

The third vector $\bar{B}_3$ can be determined from the system

$$(T_{sj} - \lambda_2\delta_{sj})\bar{B}_3{}^s = -2i\bar{B}_2{}^j.$$

After appropriate normalization it satisfies the conditions

$$\bar{B}_1{}^s\bar{B}_3{}^s = 0, \qquad \bar{B}_3{}^s\bar{B}_3{}^s = 0, \qquad \bar{B}_2{}^s\bar{B}_3{}^s = -{}^1/{}_2 i.$$

Now the quantities

$$B_1{}^k = \bar{B}_1{}^k, \quad B_2{}^k = -(i\bar{B}_3{}^k + \bar{B}_2{}^k), \qquad B_3{}^k = \bar{B}_3{}^k + i\bar{B}_2{}^k$$

are elements of an orthogonal matrix and the transformation (6.10) brings the form T into the canonical form

$$T = \lambda_1 F^1F^1 + (\lambda_2 + 1)F^2F^2$$
$$+ (\lambda_2 - 1)F^3F^3 + i(F^2F^3 + F^3F^2). \qquad (6.13)$$

Case III. $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and there exists only one eigenvector $\bar{B}_3$ which satisfies the condition

$$\bar{B}_3{}^s\bar{B}_3{}^s = 0.$$

The vectors $\bar{B}_1$ and $\bar{B}_2$ will be determined from the system

$$T_{sj}\bar{B}_2{}^s = -2i\bar{B}_3{}^j, \qquad T_{sj}\bar{B}_1{}^s = -\bar{B}_2{}^j,$$

normalizing them in such a way that

$$\bar{B}_2{}^s\bar{B}_2{}^s = 1, \qquad \bar{B}_1{}^s\bar{B}_1{}^s = 0, \qquad \bar{B}_2{}^s\bar{B}_3{}^s = 0,$$
$$\bar{B}_1{}^s\bar{B}_2{}^s = 0, \qquad \bar{B}_1{}^s\bar{B}_3{}^s = -{}^1/{}_2 i.$$

The quantities

$$B_1{}^k = -(\bar{B}_1{}^k + i\bar{B}_3{}^k), \quad B_2{}^k = \bar{B}_2{}^k, \qquad B_3{}^k = \bar{B}_3{}^k + i\bar{B}_1{}^k$$

are elements of an orthogonal matrix and the transformation (6.10) brings the form T into the canonical form

$$T = F^1F^2 + F^2F^1 - i(F^2F^3 + F^3F^2). \qquad (6.14)$$

In the case of the metric (2.6) there exists in each world point an orthonormalized frame of reference

$$e^0 = \{c, 0, 0, 0\}, \qquad e^i = \{-\varphi_{,i}, \delta_1{}^i, \delta_2{}^i, \delta_3{}^i\},$$

with respect to which the curvature tensor has the components

$$R_{(0i0k)} = c^{-2}(\varphi_{,si}\varphi_{,sk} - \varphi_{,ss}\varphi_{,ik}), \qquad R_{(0ijk)} = 0.$$

Consequently, according to (5.1),

$$M_{ik} = c^{-2}\Phi_{,ik}, \qquad N_{ik} = 0,$$

so that

$$T = c^{-2}\Phi_{,rs}E^rE^s. \qquad (6.15)$$

Since the gravitational field φ can be considered as a Newtonian field, it can be seen from (6.15) that from the point of view of the Newtonian theory of gravitation the Petrov classification of gravitational fields is on par with the classification of gravitational fields according to the canonical forms of the matrix $(\Phi_{,ik})$. This result can be considered to be general, since the curvature tensor corresponds in Newtonian theory to the orthogonal tensor $\Phi_{,ik}$.[11] However, the real symmetric matrix $(\Phi_{,ik})$ has always three mutually orthogonal unit eigenvectors, so that cases II and III actually drop out, and there is in Newtonian theory no real classification of gravitational fields according to the canonical forms of the matrix $(\Phi_{,ik})$. Petrov's classification of gravitational fields has a purely relativistic significance. It does not, however, follow from the above statements that for $c \to \infty$ the relativistic gravitational fields of the type of cases II and III cannot have a Newtonian limit. Example 2 in [11] proves the opposite.

The gravitational fields (5.29) and (5.36) refer thus to case I. For example, taking into account for the metric (5.29) formulas (5.28), (5.14), and (5.24), we obtain the characteristic equation

$$\lambda^3 - 3(GM)^2r^{-6}c^{-4}\lambda - 2(GM)^3r^{-9}c^{-6} = 0,$$

whence we find

$$\lambda_1 = 2GMr^{-3}c^{-2}, \qquad \lambda_2 = \lambda_3 = -GMr^3c^{-2}.$$

There exist three eigenvectors:

$$B_1{}^i = (x^i - a^i)r^{-1}, \qquad B_2{}^i = \varepsilon_{i3s}(x^s - a^s)\rho^{-1},$$

$$B_3{}^i = r\rho^{-1}[\delta^{i3} - (x^i - a^i)(x^3 - a^3)r^{-2}],$$

$$\rho^2 = (x^1 - a^1)^2 + (x^2 - a^2)^2,$$

and the form T has the canonical form:

$$T = GMr^{-3}c^{-2}(2F^1F^1 - F^2F^2 - F^3F^3).$$

Since the B_k^i are real, i.e., $\mu_k^i = 0$, and consequently $\lambda_k^i = B_k^i$ and $\lambda = 1$, it follows from (6.12) that

$$b_0{}^0 = 1, \quad b_0{}^k = b_k{}^0 = 0, \quad b_k{}^i = B_k{}^i.$$

The curvature tensor is brought into canonical form with the aid of the real three-dimensional orthogonal transformation

$$e^i = B_s{}^i f^s, \qquad e^0 = f^0.$$

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