ELECTROMAGNETIC PAIR PRODUCTION

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The method of invariant tensor integration is used to calculate the cross section for electromagnetic pair creation. The cross section for electromagnetic pair creation in the collision of a photon with a charged particle is calculated taking into account the recoil and the contribution of dispersion diagrams. The cross section for the annihilation of a pair of particles into two pairs of charged particles is determined exactly. The integration of the Compton tensor of fourth rank over the final states of the fermion pair is carried out.

I. INTRODUCTION

PHENOMENA such as the emission of bremsstrahlung in the collision of charged particles and the pair creation in the collision of photons (or charged particles) with charged particles are being studied extensively at present, in order to verify the validity of quantum electrodynamics at small distances, to investigate the electromagnetic structure of particles, etc. (see for example[1,2]). It is evident that for this purpose the theoretical cross sections must be known with sufficiently high accuracy. It is known, on the other hand, that the theory of such processes leads to considerable computational difficulties. The precise cross section is known only for the case of pair production by photons in the field of a nucleus (Bethe-Heitler formula). For other cases only approximate expressions for the cross section for certain limiting situations are known.

A number of earlier papers[3-6] have dealt with the bremsstrahlung in the collision of two charged particles and the emission of photons in the two-body annihilation of particle pairs. This was done by using a method whose main idea was to integrate the separate parts of diagrams exploiting their relativistic, gauge, and charge-conjugation invariance. This method makes it possible to find the total cross sections (integrated over the states of the created particles) without having to do lengthy calculations of differential cross sections. The same method can be applied to calculate the cross sections for electromagnetic pair creation. In the following we shall treat all charged particles as distinguishable, since for identical particles the interference between direct and exchange diagrams leads to considerable complications. If the emitted particles are in a narrow cone about the direction of incidence the correction for such interference terms turns out to be very small in practice.

In Sec. 2 we calculate the cross section for the creation of pairs of fermions with spin 1/2 or scalar particles in the collision of a photon with a charged particle. We find exact expressions for the differential cross section with respect to the invariant mass of the pair of charged particles. The cross sections are discussed from the point of view of the study of the form factors of the particles and the verification of the validity of quantum electrodynamics at small distances. In Sec. 3 we consider the annihilation of a pair of particles into two pairs of charged particles. We obtain an exact value for the differential cross section with respect to the invariant masses of the created pairs and discuss these cross sections; we also obtain approximate expressions for the total cross sections. In Sec. 4 we carry out the integration of the Compton tensor of fourth rank over the final states of the fermion pair.

2. PAIR CREATION IN A PHOTON-FERMION COLLISION

Consider, to lowest order of perturbation theory, the creation of a pair in the collision of a photon with a fermion of spin 1/2: $\gamma + A \rightarrow A + B + \overline{B}$. It turns out that we can consider the creation of scalar particles and of spin 1/2 fermions together. In the case of a pair of fermions the process can be represented by four diagrams (Fig. 1).

The total cross section, integrated over all final states of the created particles (all particles participating in the reaction are assumed unpolarized) can be calculated as in [3,4] 1) noting that the required expressions can be obtained from the equations of [3,4] by the substitution

$$k \to -k, \quad -p_2^+ \to p_2. \tag{2.1}$$

The cross section appears in the form

$$d\sigma = d\sigma_a + d\sigma_b, \tag{2.2}$$

where $d\sigma_a$ is the contribution from the diagrams labelled a in Fig. 1 and $d\sigma_b$ that of the diagrams labelled b; the interference term between diagrams a and b vanishes after integration of the final states as in [3-5].

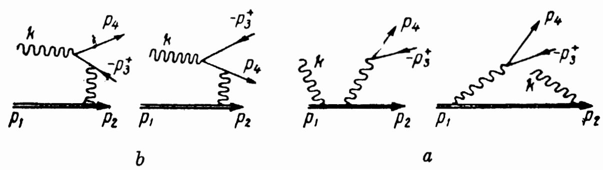

FIG. 1

In the following we shall carry out the integration over the final states of the particle A (in [3,4] this required integration over the final states of the photon). It is convenient to use covariant variables:

$$\frac{d^3p_2}{E_2} = \frac{d\Delta^2\, d\Lambda^2\, d\varphi}{4\varkappa_1}. \tag{2.3}$$

The integration over the azimuthal angle φ can be done trivially and then the cross section for pair creation can be represented in the following form (the substitution (2.1) is most conveniently carried out by using Eqs. (3.2), (3.4) of [5] and (2.10) and (2.21) of [4]).

The contribution of the b diagrams is

$$\frac{d^2\sigma_b}{d\Delta^2\, d\Lambda^2} = \frac{\alpha^3\beta_0}{2\varkappa_1^2}\frac{1}{\Lambda^4}\Big\{(2m^2+\Lambda^2)\Big[c_1 + \frac{c_2\Lambda^2 + c_3}{(k\Lambda)^2}\Big] + \frac{2\varkappa_1\varkappa_2\Lambda^2}{(k\Lambda)^2}\Big[c_1 + \frac{c_3 + c_4\Lambda^2}{(k\Lambda)^2}\Big]\Big\}, \tag{2.4}$$

where for fermions

$$c_1^F = 1 - L_1, \quad c_2^F = \Delta^2(1 - L_1/2),$$
$$c_3^F = \mu^2(\Delta^2 c_1^F + 2\mu^2 L_1),$$
$$c_4^F = 2\Delta^2 - (2\mu^2 + \Delta^2/2)L_1, \tag{2.5}$$

and for scalar particles

$$c_1^S = -1/2, \quad c_2^S = 1/2(\mu^2 L_1 - \Delta^2/2),$$
$$c_3^S = -1/2 c_3^F, \quad c_4^S = -1/2 c_4^F. \tag{2.6}$$

1)In this section we use the notation of the earlier papers[3-5].

Here

$$\varkappa_1 = -\varkappa = (kp_1),$$
$$\varkappa_2 = -\varkappa' = (kp_2) = 1/2(\Lambda^2 - \Delta^2) + \varkappa_1,$$
$$\Delta = p_3 + p_4^+, \quad \Lambda = p_1 - p_2, \quad \beta_0 = \left(\frac{\Delta^2 - 4\mu^2}{\Delta^2}\right)^{1/2},$$
$$L_1 = \frac{1}{\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}, \tag{2.7}$$

μ is the mass of one of the particles of the pair, and m the mass of A. The contribution from the a diagrams is

$$\frac{d^2\sigma_a}{d\Delta^2\, d\Lambda^2} = \frac{\alpha^3\beta_0}{8\varkappa_1^2\Delta^2}\Big[D_1 - D_2\frac{\beta_0^2}{3}\Big]Z, \tag{2.8}$$

where

$$Z = (\Delta^2 + 2m^2)\Big[m^2\Big(\frac{1}{\varkappa_1^2} + \frac{1}{\varkappa_2^2}\Big) + 2\Big(\frac{1}{\varkappa_1} - \frac{1}{\varkappa_2}\Big) + \frac{\Delta^2 - 2m^2}{\varkappa_1\varkappa_2}\Big] + 2\Big(\frac{\varkappa_1}{\varkappa_2} + \frac{\varkappa_2}{\varkappa_1}\Big) \tag{2.9}$$

and for point fermions

$$D_1^{0F} = 1, \quad D_2^{0F} = 1, \tag{2.10a}$$

and for scalar point particles

$$D_1^{0S} = 0, \quad D_2^{0S} = -1/2. \tag{2.10b}$$

Equation (2.8) is valid also for the creation of pairs of extended particles, but in that case D_1 and D_2 are known combinations of the form factors [4].

The variables Δ^2 and Λ^2 vary within the following limits:

$$4\mu^2 \leqslant \Delta^2 \leqslant (\sqrt{s^2} - m)^2, \quad s^2 = (p_1 + k)^2 = 2\varkappa_1 + m^2;$$
$$x^2_{\substack{max\\min}} = \frac{\varkappa_1(2\varkappa_1 + \Delta^2)}{s^2} - \Delta^2 \pm \frac{\varkappa_1}{s^2}[(2\varkappa_1 - \Delta^2)^2 - 4m^2\Delta^2]^{1/2} \tag{2.11}$$

(see Fig. 2; we use the variable $x^2 = -\Lambda^2$).

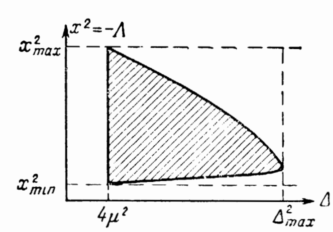

FIG. 2

By inserting the expressions (2.4) and (2.8) in (2.2) we find an exact expression for the cross section for the creation of a pair on a point fermion.

Another important case is the creation of a pair of point particles on particles with a known electromagnetic structure (the creation of electron-positron or muon pairs on nucleons or nuclei). This structure is easily taken into account for the contribution from the b diagrams by introducing appropriate form factors. It turns out that in this case one can write down a general formula for a particle A with an arbitrary spin[5-7]:

$$\frac{d^2\sigma_b}{d\Delta^2\,d\Lambda^2}=\frac{\alpha^3}{8\pi\varkappa_1^2}\frac{1}{\Lambda^4}\Big\{\frac{\Lambda^2}{2}D_1{}^A(2a_1-\Lambda^2 a_2)$$
$$+D_2{}^A\Big[a_1\Big(2m^2-\frac{\Lambda^2}{2}\Big)+\frac{\Lambda^2}{2}(a_1+\Lambda^2 a_2)\Big(\frac{\varkappa_1+\varkappa_2}{\varkappa_1-\varkappa_2}\Big)^2\Big]\Big\}, \tag{2.12}$$

$$a_1=4\pi\beta_0\Big[c_1+\frac{c_3+c_2\Lambda^2}{(k\Lambda)^2}\Big],\quad a_2=4\pi\beta_0\frac{c_4-c_2}{(k\Lambda)^2}, \tag{2.13}$$

where the c_i are given by (2.5) and (2.6), and D_1^A and D_2^A are functions of the corresponding form factors. For example, if A is a nucleon

$$D_1{}^A=|F_1+gF_2|^2,\quad D_2{}^A=|F_1|^2-\frac{\Lambda^2g^2}{4m^2}|F_2|^2; \tag{2.14}$$

where F_1 and F_2 are the electromagnetic form factors for the nucleon. Similar relations for a scalar particle (e.g. the He^4 nucleus) and a vector particle (e.g. the deuteron) are given in[8].

It is much more difficult to allow for the structure of the A particle in the contributions from the diagrams in Fig. 1a. It seems that one of the most convenient ways of tackling this problem is to introduce inelastic form factors. We have considered this question in[8] where we have given an expression for the contribution of the a diagrams to the cross section in terms of inelastic form factors. No information is at present available about these form factors, and to find them it is probably necessary to construct dynamical models. We hope to return to this problem in the future.

It is particularly interesting to consider the ratio between the contributions from the a and b diagrams (assuming A to be a point fermion).[2)]

1. If $\kappa_1 \gg m^2, \mu^2$ we find:

(a) In the strip $x^2_{min} \le x^2 \le m^2$ always $d\sigma_b \gg d\sigma_a$ and this strip gives the main contribution to the total cross section of the reaction. For large Δ^2 the ratio of the cross sections has, in this strip, the form

$$\frac{d\sigma_b}{d\sigma_a}\approx\frac{\varkappa_1}{m^2}L_1. \tag{2.15}$$

(b) In the region $4\mu^2 \le \Delta^2 \le \mu\sqrt{\kappa_1}$, $2\kappa_1 - 2m\sqrt{\kappa_1} \le x^2 \le x_b^2$

$$\frac{d\sigma_a}{d\sigma_b}\sim\frac{\varkappa_1^2}{\varkappa_2\Delta^2L_1} \tag{2.16}$$

with $m^2 \le \kappa^2 \le m\sqrt{\kappa_1}$, i.e., in this region $d\sigma_a \gg d\sigma_b$. The relative increase of the contribution $d\sigma_a$ is due to the smallness of the denominator of the propagator for the fermion A and the smallness of Δ^2.

(c) On the line $\Delta^2 = x^2$, subject to $4\mu^2 \le \Delta^2 \le \mu\sqrt{\kappa_1}$ we have

$$\frac{d\sigma_b}{d\sigma}\approx\frac{\varkappa_1^2}{\Delta^4}L_1\gg 1, \tag{2.17}$$

and near the intersection of this line with the upper limit of the region:

$$\frac{d\sigma_a}{d\sigma_b}\approx\frac{\varkappa_1}{m^2}\frac{1}{L_1}\gg 1 \tag{2.18}$$

In the rest of the region in general (apart from factors of the order of the logarithm L_1), $d\sigma_a \sim d\sigma_b$.

2. If $\kappa_1 \sim \mu^2$, $\kappa_1 \ll m^2$ ($\mu \ll m$) then $d\sigma_b \gg d\sigma_a$ for almost all values of the variables (with the possible exception of the region 1(b)).

Consider the region in which the b diagrams dominate. To study the electromagnetic structure of nucleons (or nuclei) one should then study the cross section for large momentum transfer Λ^2 while the quantity Δ^2 can remain arbitrary. For the study of the validity of quantum electrodynamics at small distances we must look at events in which the virtual line of the fermion belonging to the pair carries a large momentum. In general this momentum is not directly related to the quantity Δ^2. However, if the pair creation takes place at a large angle to the direction of the incident photon, and the angle between the components of the pair is large, a large momentum of the virtual state corresponds to a large Δ^2. For a small scattering angle of the pair there is no such correspondence.

Thus, subject to certain conditions (assuming that the contributions from the a diagrams can be calculated at least in the approximation of point particles), the calculated cross sections can be used both for the study of the electromagnetic structure of nucleons (nuclei) and for verifying the validity of quantum electrodynamics at small distances.

Another point of considerable interest is an exact result for the electrodynamic cross section (differing from the well known Bethe-Heitler formula by taking into account the recoil of A and the contribution from the a diagrams). By carrying out the integrations with respect to Λ^2 we obtain a differential cross section with respect to the invariant mass of the emitted pair:

$$\frac{d\sigma_b}{d\Delta^2}=\frac{\alpha^3\beta_0}{2\varkappa_1^2}\Big\{\frac{4(L_3-L_2)}{\Delta^4}\Big[2\varkappa_1\Big(\varkappa_1-\frac{\Delta^2}{2}\Big)c_1+2m^2c_2$$

[2)]The ratio of the a and b contributions is also discussed in[9] (for the completely differential cross section with five differentials) and in [10] (for a cross section with two differentials but assuming $\kappa_1/m^2 \ll 1$).

$$+\left(1+\frac{4(m^2-\varkappa_1)}{\Delta^2}+\frac{8\varkappa_1^2}{\Delta^4}\right)c_3\Big]-c_1L_2+[(2\varkappa_1-\Delta^2)^2$$
$$-4m^2\Delta^2]^{1/2}\left[\frac{8\varkappa_1}{\Delta^4}c_1+\frac{2}{\Delta^2\varkappa_1}\left(1+\frac{2m^2}{\Delta^2}\right)\left(c_2-\frac{c_4}{3}\right)+\frac{4c_3}{3\Delta^4\varkappa_1}\right.$$
$$\left.\times\left(1+\frac{2(m^2-2\varkappa_1)}{\Delta^2}+34\frac{\varkappa_1^2}{\Delta^4}\right)+\frac{4c_4}{3\Delta^4}\left(\frac{4\varkappa_1}{\Delta^2}-1\right)\right]\Big\}, \qquad (2.19)$$

$$\frac{d\sigma_a}{d\Delta^2}=\frac{\alpha^3\beta_0}{8\varkappa_1^2\Delta^2}\left[D_1-D_2\frac{\beta_0^2}{3}\right]$$
$$\times\Big\{2L_4\left[2\varkappa_1-(2\varkappa_1+2m^2-\Delta^2)\frac{2m^2+\Delta^2}{\varkappa_1}\right]$$
$$+\frac{4[(2\varkappa_1-\Delta^2)^2-4\Delta^2m^2]^{1/2}}{\varkappa_1}$$
$$\times\left[2m^2+\Delta^2+\frac{\varkappa_1^2(2\varkappa_1+2m^2-\Delta^2)}{2(2\varkappa_1+m^2)^2}\right]\Big\}, \qquad (2.20)$$

where

$$L_2=\ln\frac{x^2_{max}}{x^2_{min}}, \quad L_3=\ln\frac{\Delta^2+x^2_{max}}{\Delta^2+x^2_{min}},$$
$$L_4=\ln\frac{2\varkappa_1-\Delta^2-x^2_{min}}{2\varkappa_1-\Delta^2-x^2_{max}}. \qquad (2.21)$$

By inserting the results (2.19) and (2.20) in (2.2) we find an exact expression for the differential cross section with respect to the invariant mass of the pair for the creation of a pair of spin-1/2 fermions or of scalar particles on a fermion. Evidently we can in the same way easily obtain the cross sections for the production of a pair of charged particles on scalar or vector particles (see (2.8) and (2.12)).

The exact expression for the total cross section of the reaction is quite lengthy, and we therefore consider here only the limiting case in which $\kappa_1 \gg m^2, \mu^2$. We then find for the creation of a pair of fermions

$$\sigma^F=\sigma_b{}^F=\alpha r_0^2\left(\frac{28}{9}\ln\frac{2\varkappa_1}{m\mu}-\frac{218}{27}\right), \qquad (2.22)$$

and for the creation of a pair of scalar particles

$$\sigma^S=\sigma_b{}^S=\alpha r_0^2\left(\frac{4}{9}\ln\frac{2\varkappa_1}{m\mu}-\frac{26}{27}\right). \qquad (2.23)$$

These results for the total cross section have the following important properties: (1) the whole cross section comes from the b diagrams, apart from terms of the order m^2/κ_1 and $m\mu/\kappa_1$; (2) the resulting expressions agree (in the rest system of A) with the cross section for the creation of such a pair in a Coulomb field[11]. Both these findings are connected with the fact that the main contribution to the total cross section comes from the region of small momentum transfer Λ^2:

$$x^2_{min}\approx m^2\Delta^4/4\varkappa_1^2 \qquad (2.24)$$

and in this sense the b diagrams can be called "peripheral." At small Λ^2 the cross section $d\sigma_b$ rises sharply (because of the presence of the factor $1/\Lambda^4$) while $d\sigma_a$ varies slowly and remains small. For this reason the contribution $d\sigma_b$ to the integrated cross section turns out to be dominant. The proximity of the pole in Λ^2 allows one to calculate the main contribution to the total cross section σ_b by means of the pole approximation (Weizsäcker-Williams method)[12]; the range of integration in the pole approximation is the interval $x^2_{min} \le x^2 \le m^2$. For the a diagrams for which $\Delta^2 \ge 4\mu^2$, there is no such "pole region," so that for their contribution the Weizsäcker-Williams method is not applicable. In addition it is clear that for small Λ^2 the recoil is negligible and this leads to the result that regardless of the ratio of the masses m and μ the total cross section is the same as that for pair creation in a Coulomb field.

3. ANNIHILATION OF A PAIR OF CHARGED PARTICLES INTO TWO PAIRS

The method used for investigating the annihilation of a pair with the emission of a photon and the creation of a particle pair by a photon is also very useful for the investigation of reactions in which the photon is virtual. This includes the annihilation of a particle pair into two pairs of charged particles, and the creation of a pair of particles upon collision of two charged particles. To lowest order of perturbation theory such a reaction is represented by six diagrams (Fig. 3; we assume for definiteness that all particles are distinguishable fermions of spin 1/2). We consider here the annihilation of a pair into two pairs. The calculation of the cross section for this process turns out to be considerably simpler than that of the cross section for the creation of a pair in a collision be-

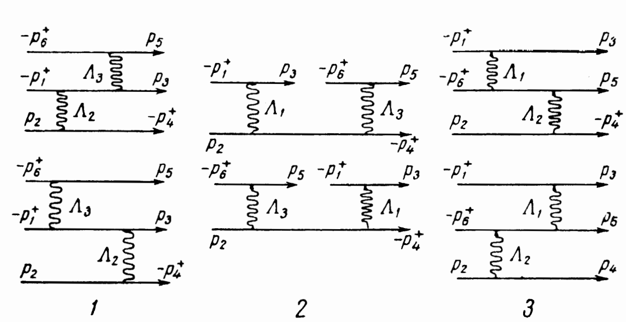

FIG. 3

tween two particles. The reason is that the method we use is based on integrating the contribution of separate fermion lines, which is particularly convenient for annihilation processes.

We represent the matrix element of the process in the form

$$M = A\Big[\frac{(\bar v_4\gamma_\nu u_2)(\bar u_5\gamma_\alpha v_6)(\bar u_3 L_1^{\alpha\nu}v_1)}{\Lambda_2^2\Lambda_3^2} + \frac{(\bar v_4 L_2^{\beta\mu}u_2)(\bar u_3\gamma_\mu v_1)(\bar u_5\gamma_\beta v_6)}{\Lambda_1^2\Lambda_3^2} + \frac{(\bar v_4\gamma_\alpha u_2)(\bar u_3\gamma_\mu v_1)(\bar u_5 L_3^{\mu\alpha}v_6)}{\Lambda_1^2\Lambda_2^2}\Big]. \quad (3.1)$$

Here

$$A = -ie^4(2\pi)^{-5}m_1m_2m_3(E_1E_2E_3E_4E_5E_6)^{-1/2}; \quad (3.2)$$

$$L_1^{\alpha\nu} = \gamma^\alpha\frac{-p_1^+ + \Lambda_2 + m_1}{\varkappa_1}\gamma^\nu + \gamma^\nu\frac{p_3 - \Lambda_2 + m_1}{\varkappa_2}\gamma^\alpha,$$

$$L_2^{\beta\mu} = L_1^{\beta\mu}(1-2); \quad L_3^{\mu\alpha} = L_1^{\mu\alpha}(1-3), \quad (3.3)$$

where (1—2) indicates the substitution $-p_1^+ \to p_2$, $p_3 \to -p_4^+$, $m_1 \to m_2$, and (1—3) stands for $-p_1^+ \to -p_6^+$, $p_3 \to p_5$, $m_1 \to m_3$; finally

$$\Lambda_1 = p_1^+ + p_3, \quad \Lambda_2 = p_2 + p_4^+, \quad \Lambda_3 = p_5 + p_6^+,$$

$$\Lambda_2 = \Lambda_1 + \Lambda_3, \quad \varkappa_1 = \Lambda_2^2 - 2(p_1^+\Lambda_2),$$

$$\varkappa_2 = \Lambda_2^2 - 2(p_3\Lambda_2). \quad (3.4)$$

Averaging over the spins of the initial particles and summing over the final spins we find

$$\bar S_i S_f|M|^2 = -\frac{|A|^2}{4}\Big[\frac{T_1}{\Lambda_3^4\Lambda_2^4} + \frac{T_2}{\Lambda_1^4\Lambda_3^4} + \frac{T_3}{\Lambda_1^4\Lambda_2^4} + \frac{2T_4}{\Lambda_1^2\Lambda_2^2\Lambda_3^4} + \frac{2T_5}{\Lambda_1^2\Lambda_2^4\Lambda_3^2} + \frac{2T_6}{\Lambda_1^4\Lambda_2^2\Lambda_3^2}\Big]. \quad (3.5)$$

Here T_1, T_2, and T_3 are the contributions from diagrams 1, 2 and 3 respectively (Fig. 3); T_4, T_5, and T_6 are interference terms. We are interested in the cross section in which the integration over the final states of the created particles has been carried out. In this integration the contributions from the interference terms vanish. Indeed, we may write these contributions in the form

$$\int d^4\Lambda_1\int\Big[\frac{T_4}{\Lambda_1^2\Lambda_2^2\Lambda_3^4} + \frac{T_5}{\Lambda_1^2\Lambda_2^4\Lambda_3^2} + \frac{T_6}{\Lambda_1^4\Lambda_2^2\Lambda_3^2}\Big] \times\delta(\Lambda_2-\Lambda_1-p_5-p_6^+)\,\delta(\Lambda_1-p_1^+-p_3) \times\frac{d^3p_1^+}{E_1^+}\frac{d^3p_3}{E_3}\frac{d^3p_5}{E_5}\frac{d^3p_6^+}{E_6^+}. \quad (3.6)$$

The quantities T_4, T_5 and T_6 contain the interference tensor $K^{\nu\nu'\alpha}$ [3], which is antisymmetric in the momenta of the pair of final particles. In that case the integral which contains a δ function must evidently vanish (Eq. 2.39) of[3]).

It is thus sufficient to consider only the contributions T_1, T_2, and T_3. We write T_1 in the form

$$T_1 = M_1^{\alpha\nu\alpha'\nu'}J_{2\nu\nu'}J_{3\alpha\alpha'}/m_1^2m_2^2m_3^2, \quad (3.7)$$

where

$$M_1^{\alpha\nu\alpha'\nu'}/m_1^2 = \mathrm{Sp}[L_1^{\alpha\nu}\Lambda_-(p_1^+)\bar L_1^{\alpha'\nu'}\Lambda_+(p_3)] \quad (3.8)$$

is the Compton tensor, which is proportional to the cross section of the Compton scattering of a polarized heavy "proton" of mass Λ_2^2 into a polarized heavy "photon" of mass Λ_3^2, and

$$J_{2\nu\nu'}/m_2^2 = \mathrm{Sp}[\gamma_\nu\Lambda_+(p_2)\gamma_{\nu'}\Lambda_-(p_4^+)] \quad (3.9)$$

is the current tensor.

The integration over the final momenta p_5 and p_6^+ can be done as in[3]:

$$\int\frac{d^3p_6^+}{E_6}\frac{d^3p_5}{E_5}\delta(\Lambda_3-p_5-p_6^+)J_{3\alpha\alpha'} = C_3\Big(g_{\alpha\alpha'} - \frac{\Lambda_{3\alpha}\Lambda_{3\alpha'}}{\Lambda_3^2}\Big), \quad (3.10)$$

$$C_3 = \frac{2\pi}{3}(\Lambda_3^2 + 2m_3^2)\Big(\frac{\Lambda_3^2 - 4m_3^2}{\Lambda_3^2}\Big)^{1/2}. \quad (3.11)$$

From the gauge invariance of the Compton tensor we have

$$M_1^{\alpha\nu\alpha'\nu'}\Lambda_{3\alpha} = M_1^{\alpha\nu\alpha'\nu'}\Lambda_{3\alpha'} = M_1^{\alpha\nu\alpha'\nu'}\Lambda_{2\nu} = M_1^{\alpha\nu\alpha'\nu'}\Lambda_{2\nu'} = 0, \quad (3.12)$$

and the only contribution comes from the contraction $M_1^{\alpha\nu\alpha'\nu'}g_{\alpha\alpha'} \equiv M_1^{\nu\nu'}$.

The integral over the term containing T_1 can be written in the form (following a method used in[4])

$$B_{\nu\nu'} = \int d^4\Lambda_1\frac{C_3}{\Lambda_2^4\Lambda_3^4}\int M_1^{\nu\nu'}\delta(\Lambda_1-p_1^+-p_3)\frac{d^3p_1^+}{E_1}\frac{d^3p_3}{E_3} = \Big(g^{\nu\nu'} - \frac{\Lambda_2^\nu\Lambda_2^{\nu'}}{\Lambda_2^2}\Big)\int d^4\Lambda_1\frac{C_3}{\Lambda_2^4\Lambda_3^4}f_1. \quad (3.13)$$

To find the function f_1 it is sufficient to contract both sides of (3.13) with the tensor $g_{\nu\nu'}$:

$$f_1 = \frac{1}{3}\int M_1^{\nu\nu'}g_{\nu\nu'}\delta(\Lambda_1-p_1^+-p_3)\frac{d^3p_1^+}{E_1}\frac{d^3p_3}{E_3}. \quad (3.14)$$

For the case in which both "photons" are heavy, the contracted constant tensor takes the form

$$M_1 = M_1^{\nu\nu'}g_{\nu\nu'} = 2\varkappa_1^{-2}\{\varkappa_1^2 + 2\varkappa_1(m_1^2 - (\Lambda_2\Lambda_3)) + 2m_1^2(2m_1^2 + \Lambda_2^2 + \Lambda_3^2) + \Lambda_2^2\Lambda_3^2\} + \text{terms } (p_1^+ \leftrightarrow p_3) + 8\varkappa_1^{-1}\varkappa_2^{-1}\{2m_1^4 + m_1^2(\Lambda_2\Lambda_3) + {}^1\!/_2\Lambda_1^2(\Lambda_2^2 + \Lambda_3^2)\}. \quad (3.15)$$

The calculation of the invariant integral of the contracted constant tensor is easiest carried out in the center-of-mass system of the particles p_1^+ and p_3; we finally obtain:

$$f_1 = \frac{8\pi}{3}\beta_1\Big\{1 + \frac{1}{b^2 - a^2}[4m_1^4 + 2m_1^2(\Lambda_2^2 + \Lambda_3^2) + \Lambda_2^2\Lambda_3^2]$$

$$+\frac{1}{ab}\ln\left(\frac{b+a}{b-a}\right)\left[2m_1^2(m_1^2+(\Lambda_2\Lambda_3))-(\Lambda_2\Lambda_3)^2\right.$$
$$\left.\left.-\frac{\Lambda_1^2}{2}(\Lambda_2^2+\Lambda_3^2)\right]\right\}, \qquad (3.16)$$

where

$\beta_1=[\Lambda_1^2-4m_1^2)/\Lambda_1^2]^{1/2}$, $b=(\Lambda_2\Lambda_3)$,
$a=\beta_1\xi$, $\xi=[(\Lambda_2\Lambda_3)^2-\Lambda_2^2\Lambda_3^2]^{1/2}=[(\Lambda_1\Lambda_2)^2-\Lambda_1^2\Lambda_2^2]^{1/2}$.

It is convenient to write also the volume element $d^4\Lambda_1$ in the form

$$d^4\Lambda_1=(\xi/4\Lambda_2^2)\,d\Lambda_3^2d\Lambda_1^2d\cos\vartheta d\varphi, \qquad (3.17)$$

where ϑ and φ are the angles of the vector $\mathbf{\Lambda}_1$ in the system in which $\mathbf{\Lambda}_2=0$; the angle integration then becomes trivial. We finally find

$$\frac{d^2\sigma_1}{d\Lambda_1^2\,d\Lambda_3^2}=-\frac{6\alpha^4}{(2\pi)^4|F|}\frac{C_3}{\Lambda_3^4}\frac{C_2}{\Lambda_2^4}\frac{f_1\xi}{\Lambda_2^2\beta_2},$$
$$C_2=C_3(3\to2),\quad \beta_2=\beta_1(1\to2). \qquad (3.18)$$

The contribution from T_3 can be calculated in a similar manner so that

$$\frac{d^2\sigma_3}{d\Lambda_1^2\,d\Lambda_3^2}=\frac{d^2\sigma_1}{d\Lambda_1^2\,d\Lambda_3^2}(1\leftrightarrow3). \qquad (3.19)$$

In the calculation of the contribution from the T_2 term the integration can at once be carried out over both current tensors:

$$d\sigma_2=-\frac{4\alpha^4}{(2\pi)^4|F|}\int d^4\Lambda_1\frac{C_3}{\Lambda_3^4}\frac{C_1}{\Lambda_1^4}M_2,$$
$$M_2=M_1(1\leftrightarrow2),\quad C_1=C_3(3\leftrightarrow1). \qquad (3.20)$$

We integrate the contracted Compton tensor over the angles and obtain

$$f_2=\frac{1}{6}\beta_2\int M_2d(\cos\vartheta)\,d\varphi=f_1(1\leftrightarrow2). \qquad (3.21)$$

Finally the contribution to the cross section takes the form

$$\frac{d^2\sigma_2}{d\Lambda_3^2\,d\Lambda_1^2}=-\frac{6\alpha^4}{(2\pi)^4|F|}\frac{C_3}{\Lambda_3^4}\frac{C_1}{\Lambda_1^4}\frac{f_2\xi}{\Lambda_2^2\beta_2}. \qquad (3.22)$$

We have thus obtained an exact expression for the differential cross section with respect to Λ_1^2 and Λ_2^2 in the form

$$\frac{d^2\sigma}{d\Lambda_1^2\,d\Lambda_3^2}=\frac{d^2\sigma_1}{d\Lambda_1^2\,d\Lambda_3^2}+\frac{d^2\sigma_2}{d\Lambda_1^2\,d\Lambda_3^2}+\frac{d^2\sigma_3}{d\Lambda_1^2\,d\Lambda_3^2}, \qquad (3.23)$$

in which the terms are given by (3.18), (3.19), and (3.22).

The range of variation of the variables Λ_1^2 and Λ_3^2 is determined by the conditions

$$4m_1^2\leqslant\Lambda_1^2\leqslant(\sqrt{\Lambda_2^2}-2m_3)^2,$$
$$4m_3^2\leqslant\Lambda_3^2\leqslant(\sqrt{\Lambda_2^2}-\sqrt{\Lambda_1^2})^2. \qquad (3.24)$$

These are shown in Fig. 4. The lower straight line

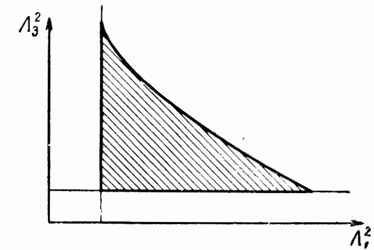

FIG. 4

represents the threshold for pair production (the particles in each pair have equal momentum). The upper boundary corresponds to the case in which the components of each pair go off in opposite directions carrying the same momentum in their c.m.s.

The cross section $d\sigma_1(d\sigma_3)$ contains the function C_3/Λ_3^4 (C_1/Λ_1^4). The properties of this function are investigated in detail in [5], where it is shown that as a function of Λ_3^2 (Λ_1^2) it has a peak near the lower limit. This peak has a clear physical meaning: in the region of the peak the two components of the pair are emitted in the same direction, and the invariant mass of the pair is small, so that we can speak of a front which is converted into a particle pair. The cross section $d\sigma_2$ contains the two functions C_1/Λ_1^4 and C_3/Λ_3^4; it has therefore a double peak (near the apex of the right angle in the domain of variation of the parameters (see Fig. 4)). As shown in [5], these peaks are high, but very narrow so that they do not give logarithmic contributions to the total cross section.

The determination of the exact integrated cross section is a very complicated task. We shall carry out only an estimate of the main contributions to the integrated cross section.

In the case in which $\Lambda_2^2\gg m_1^2, m_2^2, m_3^2$ the main contribution to the integrated cross section contains the cube of a logarithm:

$$\sigma=\frac{10}{27}\frac{\alpha^4}{\pi|F|}\ln^3\frac{\Lambda_2^2}{4m^2}. \qquad (3.25)$$

In the case

$$\sigma_1=\sigma_3=\frac{1}{27}\frac{\alpha^4}{\pi|F|}\ln^3\frac{\Lambda_2^2}{4m^2}\qquad \sigma_2=\frac{8}{27}\frac{\alpha^4}{\pi|F|}\ln^3\frac{\Lambda_2^2}{4m^2}. \qquad (3.26)$$

It is much more difficult to obtain the integral cross section to the next order (the square of the logarithm). We consider only two limiting cases:

(1) $m_2^2\ll m_1^2m_3^2/\Lambda_2^2$, $\Lambda_2^2\gg m_1^2, m_2^2, m_3^2$, $m_1>m_2$. In this case the integrated cross section is

$$\sigma=(4\alpha^4/9\pi|F|)\,[{}^1/_6 l_1^3+{}^1/_{12}l_2^3+{}^1/_4 l_1l_3^2-{}^1/_4 l_1^2l_3$$
$$+\ln2\cdot l_2^2-{}^{21}/_4 l_1l_3-({}^{13}/_{12}-{}^1/_2\ln2)(l_1^2+l_3^2)], \qquad (3.27)$$

where

$$l_1 = \ln\frac{\Lambda_2^2}{4m_1^2},\quad l_2 = \ln\frac{\Lambda_2^4}{16m_1^2m_3^2},\quad l_3 = \frac{\Lambda_2^2}{4m_3^2}. \quad (3.28)$$

(2) $4m_2^2 \sim \Lambda_2^2$, $m_1 > m_3$

Then

$$\sigma = (4\alpha^4/9\pi|F|)\,[{}^1/_4 l_1^3 + {}^3/_8 l_1 l_3^2 - {}^3/_8 l_1^2 l_3 - {}^{15}/_8 l_1 l_3 + {}^3/_4 \ln 2\cdot(l_1^2 + l_3^2)]. \quad (3.29)$$

Note that in this case σ_2 does not contain a cubic logarithmic term as the result of the smallness of the fermion propagator in the contribution from diagram 2 in Fig. 3.

We remark that in both these cases the contributions from the cubic and quadratic logarithmic terms are, over most of the range, of the same order of magnitude and of opposite sign. The approximation based on the cubic logarithm only may therefore be rather inaccurate except in the extreme asymptotic region.

4. INTEGRATION OF THE CROSS SECTION FOR PAIR PRODUCTION IN COLLISIONS OVER THE FINAL STATES OF THE PAIR

For the case in which all three particles involved are fermions the diagrams of the lowest order of perturbation theory are shown in Fig. 3, and here Eqs. (3.1) to (3.5) are applicable if we make the substitutions

$$-p_1^+ \to p_1,\quad -p_4^+ \to p_4;$$
$$\Lambda_1 \to -\Delta_1,\quad \Lambda_2 \to \Delta_2,\quad \Lambda_3 \to \Delta_3. \quad (4.1)$$

In the process of integrating over the final states of the pair (p_5, p_6^+) there are the following differences from the case of the annihilation of one pair into two pairs (Sec. 3): (1) the interference term between diagrams 1 and 2 does not vanish and this considerably complicates the calculation; (2) one has to integrate over final states the Compton tensor of fourth rank $M_3^{\sigma\rho\sigma\rho'}$ which appears in the expression T_3. The present section is concerned with this question.

The sought tensor

$$B^{\sigma\rho\sigma'\rho'} = \int M_3^{\sigma\rho\sigma'\rho'}\delta(p_5 + p_6^+ - \Delta_1 - \Delta_2)\frac{d^3p_5}{E_5}\frac{d^3p_6^+}{E_6} \quad (4.2)$$

depends only on the vectors Δ_1 and Δ_2. Using the fact that

$$M_3^{\sigma\rho\sigma'\rho'} = M_3^{\sigma'\rho'\sigma\rho} \quad (4.3)$$

and also the fact that this tensor will be multiplied eventually by current tensors for which we know from gauge invariance that

$$J_1^{\mu\mu'}\Delta_{1\mu} = J_1^{\mu\mu'}\Delta_{1\mu'} = J_2^{\mu\mu'}\Delta_{2\mu} = J_2^{\mu\mu'}\Delta_{2\mu'} = 0, \quad (4.4)$$

the only non-vanishing contribution comes from the following tensor:

$$\bar{B}^{\sigma\rho\sigma'\rho'} = 2\pi\beta_3\Big[d_1 g^{\sigma\rho}g^{\sigma'\rho'} + d_2 g^{\rho\rho'}g^{\sigma\sigma'} + d_3 g^{\rho\sigma'}g^{\rho'\sigma} + \frac{d_4}{D}g^{\sigma\sigma'}\Delta_1^\rho\Delta_1^{\rho'} + \frac{d_5}{D}g^{\rho\rho'}\Delta_2^\sigma\Delta_2^{\sigma'} + \frac{d_6}{a_3 D}(g^{\rho\sigma}\Delta_1^{\rho'}\Delta_2^{\sigma'} + g^{\rho'\sigma'}\Delta_1^\rho\Delta_1^\sigma) + \frac{d_7}{a_3 D}(g^{\rho\sigma'}\Delta_1^{\rho'}\Delta_2^\sigma + g^{\rho'\sigma}\Delta_1^\rho\Delta_2^{\sigma'}) + \frac{d_8}{D^2}\Delta_1^\rho\Delta_1^{\rho'}\Delta_2^\sigma\Delta_2^{\sigma'}\Big], \quad (4.5)$$

where

$$a_1 = \Delta_1^2,\quad a_2 = \Delta_2^2,\quad a_3 = (\Delta_1\Delta_2),$$
$$D = a_3^2 - a_1a_2,\quad \beta_3 = [(\Delta_3^2 - 4m_3^2)/\Delta_3^2]^{1/2}. \quad (4.6)$$

We note here that the tensor $B^{\sigma\rho\sigma'\rho'}$ (allowing for the restriction (4.3)) contains 27 terms d_n of which only eight are independent; for these we may choose, for example, d_1 to d_8.

The tensors (3.10) and (4.5) are sufficient to express any cross section for the production of spin 1/2 particles in the collision of two charged particles or of a photon and a charged particle, after integration over the final states of the pair.

The coefficients d_i ($i = 1, \ldots, 8$) can be determined if we contract the tensor $B^{\sigma\rho\sigma'\rho'}$ in eight independent gauge invariant ways (so as to eliminate all terms with d_n for $n > 8$) and compute the corresponding integrals (4.2). In this way we obtained a set of eight equations for the coefficient d_i. Their solution is:

$$8d_1 = 16 + g_1 + 8g_2 - \frac{4\Delta_3^2(6a_3 - m_3^2)}{D} + \left[g_3 + 16a_3^2\left(1 - \frac{a_3\Delta_3^2}{D}\right) + \frac{a_1a_2}{2}g_1\right]\frac{L}{a_3},$$

$$8d_2 = g_1 + 8g_2 + \frac{4\Delta_3^2(\Delta_3^2 - 2a_3 + m_3^2)}{D} + \left[-g_3 - 16m_3^4 + \frac{4a_3^2\Delta_3^2(\Delta_3^2 + 2m_3^2)}{D} + \frac{a_1a_2}{2}g_1\right]\frac{L}{a_3},$$

$$8d_3 = g_1 - 8g_2 + \frac{4\Delta_3^2(2a_3 + m_3^2)}{D} + \left[g_3 + \frac{a_1a_2}{2}g_1\right]\frac{L}{a_3},$$

$$d_4 = a_2d_2 + \frac{a_2}{2D}\left[2\Delta_3^2a_2 + \frac{3a_1\Delta_3^4a_2}{D} + 2a_1\Delta_3^2g_2 + \left(g_4 + \frac{a_1D}{2}g_1\right)\frac{a_2}{a_3}L\right],\quad d_5 = d_4(a_1 \leftrightarrow a_2),$$

$$d_6 = -a_3^2d_1 + \frac{a_1a_2a_3}{2D}\left\{4\Delta_3^2 - \frac{3a_3\Delta_3^2}{D} + 2g_2\Delta_3^2 + \left[-\frac{Da_3g_1}{2} + \Delta_3^2(a_1a_2 + 2a_3^2 - 2m_3^2a_3)\right]\frac{L}{a_3}\right\},$$

$$d_7 = a_3^2(d_1 - d_3) + d_6 + \frac{a_1a_2a_3}{2D}\left[-2\Delta_3^2 g_2 - a_1a_2\Delta_3^2\frac{L}{a_3}\right],$$

$$d_8 = -a_3^2(d_1 + d_3) - 3a_1a_2d_2 + 3d_4 + d_5 - 2(d_6 + d_7) - \frac{a_1a_2}{D}\left[2\Delta_3^2 a_2 + 2g_2\Delta_3^2(a_1 + 2m_3^2) + g_4\frac{a_2}{a_3}L\right]. \quad (4.7)$$

Here

$$g_1 = \frac{(a_1a_2 + 2a_3^2)\Delta_3^4}{D^2}, \quad g_2 = \left[1 + \frac{4m_3^2 D}{a_1a_2\Delta_3^2}\right]^{-1},$$

$$g_3 = 4[a_1a_2 + \Delta_3^2(a_3 + m_3^2) - 2(a_3 + m_3^2)^2] + \Delta_3^2 D^{-1}[4a_3^2(m_3^2 + a_3) + a_1a_2\Delta_3^2],$$

$$g_4 = a_1\Delta_3^2(a_1 + a_2) + 2m_3^2(2a_1a_3 + 2a_3^2 - a_1a_2 + a_1^2),$$

$$L = \frac{1}{\beta_3\sqrt{D}}\ln\frac{a_3 - \beta_3\sqrt{D}}{a_3 + \beta_3\sqrt{D}}. \quad (4.8)$$

If $\Delta_1^2 = \Delta_2^2 = 0$ the tensor $B^{\sigma\rho\sigma'\rho'}$ is proportional to the cross section for the conversion of a pair of polarized photons into a pair of fermions after integration over the final state of the pair. After averaging over the polarization of the photons we find the well-known result for the total cross section of the process $\gamma + \gamma \to e^- + e^+$[11]:

$$\sigma_{\gamma\gamma} = -\frac{2\pi e^4\beta_3}{\Delta_3^2}(d_1^0 + 2d_2^0 + d_3^0). \quad (4.9)$$

If only one of the photons is real we can easily go over to Eq. (2.4).

As already mentioned in Sec. 2, the lower limit for the parameters $|\Delta_1|^2$ and $|\Delta_2|^2$ is very small (cf. (2.25)) and just the region of small $|\Delta_1|^2$ and $|\Delta_2|^2$ gives the major contribution to the total cross section for diagrams of type 3. For this reason such diagrams may be called peripheral. Since, however, the overlap between the regions of small $|\Delta_1|^2$ and $|\Delta_2|^2$ is small and the other diagrams also contain one small momentum transfer ($|\Delta_1|^2$ or $|\Delta_2|^2$) the magnitude of the contribution from diagram 3 is larger than the others only logarithmically (and not in power-law fashion as in the case of pair creation by photons). Diagram 3 gives the main (logarithm-cubed) contribution to the total cross section in the limit first obtained by Landau and Lifshitz[11]:

$$\sigma = \frac{28}{27\pi}r_0^2\alpha^2\ln^3\frac{s^2}{m^2}, \quad s^2 = (p_1 + p_2)^2. \quad (4.10)$$

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