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QUANTUM EFFECTS IN MULTIPLE SCATTERING

N. P. KALASHNIKOV and M. I. RYAZANOV

Moscow Physics Engineering Institute

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A quantum distribution function is derived which describes the multiple scattering of a beam of charged particles as a function of the deflection angle and the transverse displacement of the particle. In the classical limit the formula is identical with Fermi's classical distribution function for multiple scattering. The possibility of measuring quantum effects in multiple scattering depending on the properties of the measuring apparatus is investigated.

1. INTRODUCTION

THE classical distribution function for charged particles in matter with respect to the direction of momentum, the distance traversed, and the transverse displacement owing to multiple Coulomb scattering was obtained by Fermi as late as 1940.[1] From the quantum mechanical point of view this distribution function is valid for distances which exceed many times the dimensions of the wave packet describing the particle. It is therefore of interest to investigate the form of the distribution function for distances comparable with the dimensions of the wave packet, where the classical picture is inadequate. This problem can be solved with the help of the quantum mechanical kinetic equation of Migdal,[2] which has been developed further in papers devoted to the study of the effect of multiple scattering on bremsstrahlung and pair production at high energies.[3] However, using this method in our case would be a rather complicated procedure because of the necessity to take a wave packet for the wave function of the particle before its entry into the medium (if the impinging particle is described by a plane wave, the distribution function is independent of the transverse displacement). Therefore, we shall solve our problem by the simpler method of the quantum mechanical theory of multiple scattering formulated earlier by the authors.[4]

The method consists in solving the Schrödinger equation for the elastic scattering of a particle on a system of many scatterers in the approximation where the wave function depends multiplicatively on the coordinates of the scatterers. The distribution function is obtained by averaging the density matrix in the mixed representation over the atomic coordinates. The resulting distribution function coincides in the classical limit with the classical distribution function of Fermi. The use of a quantum mechanical distribution function allows one to point out a number of quantum effects in multiple scattering and to investigate the dependence of these effects on the properties of the measuring apparatus and on the original form of the wave packet.

The results obtained can be of interest in connection with the known method of measuring the energy of a particle by the angle of multiple scattering, and also in connection with the experimental indications of deviations from the usual theory of multiple scattering at high energies.[5]

2. WAVE FUNCTION OF A PARTICLE IN MATTER

Let the wave function of a particle in the medium satisfy the equation ($\hbar = c = 1$)

$$(\Delta + p_0^2)\psi(\mathbf{r}) = 2E_0 U(\mathbf{r})\psi(\mathbf{r}), \qquad (2.1)$$

where for relativistic particles with spin 0 and $^1/_2$ $E_0^2 = p_0^2 + m^2 \gg U^2$ and for nonrelativistic particles $E_0 \approx m$, in which case it coincides with the Schrödinger equation. The external potential in which the particle moves is composed of contributions from all atoms of the medium:

$$U(\mathbf{r}) = \sum_a U_0(\mathbf{r} - \mathbf{R}_a). \qquad (2.2)$$

We shall seek the solution of (2.1) in the form

$$\psi_{\mathbf{p}_0}(\mathbf{r}) = e^{i\mathbf{p}_0\mathbf{r}} e^{S(\mathbf{r})}, \qquad (2.3)$$

which after substitution in (2.1) leads to an equation for S:

$$2i\mathbf{p}_0\nabla S + \Delta S + (\nabla S)^2 = 2E_0 \sum_a U_0(\mathbf{r} - \mathbf{R}_a). \qquad (2.4)$$

The usual assumption of multiple scattering theory is that the particle interacts with each particle independently of the other particles. In this case the dependence of the wave function of the particle on the coordinates of the atoms must be multiplicative. This means that the phase of the wave function $S(\mathbf{r})$ must be a sum of phases,

$$S(\mathbf{r}) \cong \sum_a S_1(\mathbf{r}-\mathbf{R}_a).$$

This approximation corresponds to neglecting terms of the type

$$\sum_a \sum_{b\neq a} S_2(\mathbf{r}-\mathbf{R}_a,\ \mathbf{r}-\mathbf{R}_b).$$

We use this approximation in solving (2.4), retaining only a single sum over the atoms and discarding terms with multiple sums; (2.4) then reduces to

$$\sum_a \{2i\mathbf{p}_0\nabla S_1(\mathbf{r}-\mathbf{R}_a)+\Delta S_1(\mathbf{r}-\mathbf{R}_a)+(\nabla S_1(\mathbf{r}-\mathbf{R}_a))^2 - 2E_0U_0(\mathbf{r}-\mathbf{R}_a)\} = 0. \qquad (2.5)$$

This equation must be satisfied for arbitrary values of $\mathbf{R}_\alpha$, which implies that each term of the sum must be zero:

$$2i\mathbf{p}_0\nabla S_1(\mathbf{r}-\mathbf{R}_a)+\Delta S_1(\mathbf{r}-\mathbf{R}_a)+(\nabla S_1(\mathbf{r}-\mathbf{R}_a))^2 = 2E_0U_0(\mathbf{r}-\mathbf{R}_a). \qquad (2.6)$$

Equation (2.6) coincides with the equation for the wave function of the one-center problem; therefore

$$\psi_{\mathbf{p}_0}{}^0(\mathbf{r}) = e^{i\mathbf{p}_0\mathbf{r}} e^{S_1(\mathbf{r}-\mathbf{R}_a)} \qquad (2.7)$$

is the wave function of the one-center problem with the same boundary conditions. Let us introduce the notation

$$e^{S_1(\mathbf{r})} - 1 = (2\pi^2)^{-1}\int d^3\mathbf{q} f(\mathbf{q})\,(q^2+2\mathbf{p}_0\mathbf{q}-i\delta)^{-1}\,e^{i\mathbf{q}\mathbf{r}}; \qquad (2.8)$$

then the scattered wave in the one-center problem for large distances has the form

$$\lim_{r\to\infty} e^{i\mathbf{p}_0\mathbf{r}}\int \frac{d^3\mathbf{q} f(\mathbf{q})\,e^{i\mathbf{q}\mathbf{r}}}{(2\pi^2)(q^2+2\mathbf{p}_0\mathbf{q}-i\delta)} = -f\left(\mathbf{p}_0-\frac{\mathbf{r}}{r}\,p_0\right)\frac{e^{ip_0r}}{r} \qquad (2.9)$$

(the asymptotic form of the integral has been investigated in [6]). It follows from (2.9) that $f(\mathbf{q})$ is the amplitude for the scattering of a particle on an isolated atom and hence, the quantity $S_1(\mathbf{r}-\mathbf{R})$ can be found if the amplitude for one-center scattering is known. This circumstance guarantees the usefulness of the theory described below if the one-center scattering amplitude is known.

Knowing the wave function of the particle in matter, we can obtain the probability distribution for any quantity characterizing the particle. In an amorphous medium only the calculation of quantities averaged over the distribution of the atoms is of interest. The average over the distribution of the atoms reduces to the average of expressions of the form

$$\left\langle \exp\Bigl(\sum_a Q_a\Bigr)\right\rangle = \langle e^{Q_a}\rangle^N = \left(1+\frac{1}{N}\left\langle\sum_a (e^{Q_a}-1)\right\rangle\right)^N.$$

The sign $\langle \dots \rangle$ indicates averaging over the atomic coordinates; we assume that the coordinates of the atoms are independent. When the number of atoms, N, goes to infinity, we have the formula

$$\left\langle \exp\left(\sum_a Q_a\right)\right\rangle = \exp\left\langle \sum_a (e^{Q_a}-1)\right\rangle \qquad (2.10)$$

(a more detailed consideration of the averaging process can be found in [4]).

It should be noted that the wave function of the particle in the medium

$$\psi(\mathbf{r},t) = \exp(i\mathbf{p}_0\mathbf{r}-iE_0t)\exp\left(\sum_a S_1(\mathbf{p}_0,\mathbf{r}-\mathbf{R}_a)\right) \qquad (2.11)$$

corresponds to a completely definite boundary condition: before the entry into the medium the state of the particle is described by the plane wave $\exp(i\mathbf{p}_0\cdot\mathbf{r} - iE_0t)$. The solution to the Schrödinger equation corresponding to another boundary condition can be obtained by a superposition of the solution (2.11), for example,

$$\psi(\mathbf{r},t) = \int d^3\mathbf{p}_0 a(\mathbf{p}_0)\exp(i\mathbf{p}_0\mathbf{r}-iE_0t) \times \exp\left[\sum_a S_1(\mathbf{p}_0,\mathbf{r}-\mathbf{R}_a)\right], \qquad (2.12)$$

where the wave function of the particle before the entry into the medium has the form

$$\int d^3\mathbf{p}_0 a(\mathbf{p}_0)\exp(i\mathbf{p}_0\mathbf{r}-iE_0t).$$

3. QUANTUM DISTRIBUTION FUNCTION

In order to compare the results of the quantum theory of multiple scattering with the classical distribution function, it is convenient to formulate the results of the quantum theory in a form which is as close as possible to the method of the classical kinetic equation. As is known, the quantum analog to the classical distribution function is the density matrix in the mixed representation:[7]

$$f(\bar{\mathbf{p}},\bar{\mathbf{r}},t) = (2\pi)^{-3}\int d^3\mathbf{r}' e^{-i\mathbf{p}\mathbf{r}'}\left\langle \psi^*\left(\mathbf{r}-\frac{\mathbf{r}'}{2},t\right) \times \psi\left(\mathbf{r}+\frac{\mathbf{r}'}{2},t\right)\right\rangle. \qquad (3.1)$$

The quantum distribution function is not a directly observable quantity and is, in general, not positive definite. Directly observable quantities are the average values of physical quantities,

$$\bar{Q} = \int d^3\mathbf{p}d^3\mathbf{r}\; Q(\mathbf{p}, \mathbf{r}) f(\mathbf{p}, \mathbf{r}, t) \tag{3.2}$$

or the probability densities of the momenta or the coordinates,

$$w(\mathbf{p}) = \int d^3\mathbf{r} f(\mathbf{p}, \mathbf{r}, t); \quad w(\mathbf{r}) = \int d^3\mathbf{p} f(\mathbf{p}, \mathbf{r}, t);$$
$$w(\mathbf{p}_\perp, r_\parallel) = \int d^2\mathbf{r}_\perp dp_\parallel f(\mathbf{p}, \mathbf{r}, t). \tag{3.3}$$

The quantum distribution function (3.1) must satisfy the inequality

$$\int d^3\mathbf{r} d^3\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) f(\mathbf{p}, \mathbf{r}, t) \leqslant 1. \tag{3.4}$$

The form of f(**p**, **r**, t) can in general be determined in two ways. In the first method an equation for f(**p**, **r**, t) is derived from the Schrödinger equation for the wave functions; this is the quantum mechanical kinetic equation, which is to be solved. The second method, used below, consists in first solving the Schrödinger equation in the corresponding approximation and then calculating the quantum distribution functions with the help of (3.1). This method of calculating the distribution function was used in [4]. We calculate in this fashion the quantum distribution function for the case where the state of the particle before entry into the medium is described by a wave packet $(\mathbf{p}_\perp \cdot \mathbf{p}_0 = 0)$

$$\psi(\mathbf{r}, t) = \int dE \eta_1(E - E_0) \int d^2\mathbf{p}_\perp \eta_2(\mathbf{p}_\perp) e^{i\mathbf{pr} - iEt}. \tag{3.5}$$

It follows from the discussion above that the wave function of the particle in the medium corresponding to the initial condition (3.5) has the form

$$\psi(\mathbf{r}, t) = \int dE \eta_1(E - E_0) \int d^2\mathbf{p}_\perp \eta_2(\mathbf{p}_\perp)$$
$$\times \exp\Big[i\mathbf{pr} - iEt + \sum_a S_1(\mathbf{p}, \mathbf{r} - \mathbf{R}_a) \Big]. \tag{3.6}$$

Substituting this expression in (3.1) we obtain after averaging over the atomic coordinates, using (2.10),

$$f(\mathbf{q}, \mathbf{x}, t) = (2\pi)^{-3} \int d^3\mathbf{y} dE dE' d^2\mathbf{p}_\perp d^2\mathbf{p}_\perp{}' \eta_1(E - E_0)$$
$$\times \eta_1^*(E' - E_0)\, \eta_2(\mathbf{p}_\perp) \eta_2^*(\mathbf{p}_\perp{}') \exp\Big\{ -i\mathbf{yq} - i(E - E')t$$
$$+ i(\mathbf{p} - \mathbf{p}')\mathbf{x} + i\frac{\mathbf{p} + \mathbf{p}'}{2}\mathbf{y} + F(\mathbf{x}, \mathbf{y}; \mathbf{p}, \mathbf{p}') \Big\},$$

$$F(\mathbf{x}, \mathbf{y}; \mathbf{p}, \mathbf{p}') = \int \frac{d^3\mathbf{q} f(\mathbf{q}, E)\, e^{i\mathbf{q}(\mathbf{x}+\mathbf{y}/2)} \Big\langle \sum_a e^{-i\mathbf{q}\mathbf{R}_a} \Big\rangle}{(2\pi^2)(q^2 + 2\mathbf{pq} - i\delta)}$$

$$+ \int \frac{d^3\mathbf{q} f^*(\mathbf{q}, E')\, e^{-i\mathbf{q}(\mathbf{x}-\mathbf{y}/2)} \Big\langle \sum_a e^{i\mathbf{q}\mathbf{R}_a} \Big\rangle}{(2\pi^2)(q^2 + 2\mathbf{p}'\mathbf{q} + i\delta)}$$

$$+ \int \frac{d^3\mathbf{q} d^3\mathbf{q}' f(\mathbf{q}, E) f^*(\mathbf{q}'E')\, e^{i\mathbf{q}(\mathbf{x}+\mathbf{y}/2) - i\mathbf{q}(\mathbf{x}-\mathbf{y}/2)} \Big\langle \sum_a e^{-i(\mathbf{q}-\mathbf{q}')\mathbf{R}_a} \Big\rangle}{(2\pi^2)^2 (q^2 + 2\mathbf{pq} - i\delta)(q^2 + 2\mathbf{p}'\mathbf{q} + i\delta)} \tag{3.7}$$

Using (3.7), we can obtain the distribution of the charged particles in the medium with respect to all three coordinates and the transverse momenta:

$$f(\mathbf{x}, \mathbf{q}_\perp) \equiv f(\mathbf{z}, \mathbf{r}_\perp, \mathbf{q}_\perp) = \int dt dq_\parallel f(\mathbf{x}, \mathbf{q}, t). \tag{3.8}$$

We assume that the relative spread of the energies in the wave packet is small and that the initial spread of the transverse momentum has Gaussian form

$$\eta_2(\mathbf{p}_\perp) = r_0^2 \pi^{-1} \exp(-\mathbf{p}_\perp{}^2 r_0^2), \tag{3.9}$$

where r_0 is the mean transverse spatial dimension of the packet. After the integration over dt and $dq_{||}$ in (3.8), we obtain easily

$$f(z, \mathbf{r}_\perp, \mathbf{q}_\perp) = \mathrm{const} \int d^2\mathbf{p}_\perp d^2\mathbf{k}_\perp d^2\mathbf{y}_\perp$$
$$\times \exp\Big[-i\mathbf{q}_\perp \mathbf{y}_\perp - 2k^2 r_0^2 - \frac{p^2}{2} r_0^2 \Big]$$
$$\times \exp\Big[i\mathbf{p}_\perp \mathbf{x}_\perp + i\mathbf{k}_\perp \mathbf{y}_\perp - i\mathbf{k}_\perp \mathbf{p}_\perp \frac{z}{p_0} \Big] \exp[L_1 + L_2 + L_3],$$
$$L_1 + L_2 = -n_0 \sigma_t \tilde{L},$$

$$L_3 = n_0 \int |f(\mathbf{s}_\perp)|^2 \exp(i\mathbf{s}_\perp \mathbf{y}_\perp) d^2\mathbf{s}_\perp$$
$$\times \int_0^{\tilde{L}} \exp\Big[-i\frac{\mathbf{s}_\perp \mathbf{p}_\perp (z - \xi)}{p} \Big] d\xi,$$
$$\tilde{L} = \min(L, z). \tag{3.10}$$

Let us now determine the angular distribution in space of the charged particles, assuming that the deviations from the initial deflection are small. In this case the full angle of deflection is the cumulative effect of small deflections at each scatterer and the momentum transfer to the individual scatterer is much smaller than the total momentum transfer. This permits a considerable simplification of the expression for L_3. Assuming $\mathbf{s}_\perp \cdot \mathbf{y}_\perp \ll 1$, we find

$$L_3 = n_0 \sigma_s \tilde{L} - n_0 \sigma_s \frac{\langle \theta_0^2 \rangle}{4} \mathbf{y}_\perp{}^2 p_0^2 \tilde{L} - n_0 \sigma_s \frac{\langle \theta_0^2 \rangle}{4} \mathbf{p}_\perp{}^2 \int_0^{\tilde{L}} (z - \xi)^2 d\xi$$

$$+ n_0\sigma_s \frac{\langle\theta_0^2\rangle}{4} p_0 \mathbf{y}_\perp \mathbf{p}_\perp \int_0^{\tilde{L}} (z-\xi)\, d\xi = n_0\sigma_s \tilde{L}$$

$$- \beta \mathbf{y}_\perp^2 - \alpha \mathbf{p}_\perp^2 + \gamma \mathbf{y}_\perp \mathbf{p}_\perp,$$

$$\langle\theta_0^2\rangle = p_0^{-2} \int q_\perp^2 |f(\mathbf{q}_\perp)|^2 d^2\mathbf{q}_\perp. \qquad (3.11)$$

Substituting (3.11) in (3.10) and performing the necessary integrations,[8] we obtain

$$f(z, \mathbf{r}_\perp, \mathbf{q}_\perp) = \frac{\text{const}}{4r_\theta^2(4AB - C^2)}$$

$$\times \exp\left\{ - r_\perp^2 \frac{B}{4AB - C^2} - q_\perp^2 \frac{A}{4AB - C^2} + q_\perp r_\perp \frac{C}{4AB - C^2} \right\};$$

$$A = \frac{r_0^2}{2} + \frac{z^2}{8p_0^2 r_0^2} + n_0\sigma_s \frac{\langle\theta_0^2\rangle}{4} \int_0^{\tilde{L}} (z-\xi)^2\, d\xi,$$

$$B = \frac{1}{8r_0^2} + n_0\sigma_s \frac{\langle\theta_0^2\rangle}{4} p_0^2 \tilde{L}, \qquad C = \frac{z}{4p_0 r_0^2}$$

$$+ n_0\sigma_s \frac{\langle\theta_0^2\rangle}{2} p_0 \int_0^{\tilde{L}} (z-\xi)\, d\xi. \qquad (3.12)$$

The function f(z, $\mathbf{r}_\perp$, pϑ)($\mathbf{q}_\perp \equiv$ pϑ) is the quantum analog of the classical Fermi distribution function.[1] Moreover, in the limit ħ → 0 the distribution function (3.12) coincides with the classical distribution function.[1] We emphasize that, as already noted, the quantum distribution function is not a directly observable quantity. Therefore it is not meaningful to compare (3.12) with the classical distribution function.

It is interesting to compare, for example, the spatial distribution of the particles at the depth z independently of the angle. In the classical theory it is given by

$$\frac{1}{2}\sqrt{\frac{3}{\pi}} \left(\frac{L_{\text{rad}}}{z}\right)^{3/2} \left(\frac{2E}{E_s}\right) \exp\left[- \frac{3E^2 L_{\text{rad}} y^2}{E_s^2 z^3} \right],$$

i.e., by a Gaussian distribution in which the mean square deviation increases like the cube of the traversed depth z. The integration of (3.12) over $\mathbf{q}_\perp$ also yields a Gaussian distribution, but the mean square deviation depends on z in a much more complicated fashion:

$$\langle r_\perp^2 \rangle \sim \left[n_0\sigma_s \langle\theta_0^2\rangle \int_0^{\tilde{L}} (z-\xi)^2\, d\xi + \frac{r_0^2}{2} + \frac{z^2}{8k_0^2 r_0^2} \right]^{1/2}.$$

Finally, we can compare the classical result with the experimentally observed distribution function f(z, $\mathbf{r}_\perp^{(0)}$, $\mathbf{q}_\perp$), which can be obtained from f(z, $\mathbf{r}_\perp$, $\mathbf{q}_\perp$) by averaging over the region of sensitivity of the measuring apparatus, i.e., with the help of the relation

$$f(z, \mathbf{r}_\perp^{(0)}, \mathbf{q}_\perp) = \int d^2\mathbf{x}_\perp P(\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}) f(\mathbf{q}_\perp, z, \mathbf{x}_\perp), \qquad (3.13)$$

where P($\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}$) is a function defining the sensitivity of the detector [sometimes P($\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}$) is called the transmission of the aperture]. With good accuracy P($\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}$) can be approximated either by the step function

$$P(\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}) = \begin{cases} \text{const}, & |\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}| \leqslant a \\ 0, & |\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}| > a \end{cases}, \qquad (3.14)$$

or by the Gauss function

$$P(\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)}) = \text{const} \exp\left[-(\mathbf{x}_\perp - \mathbf{r}_\perp^{(0)})^2 a^{-2} \right], \qquad (3.14a)$$

where a is a characteristic dimension of the aperture.

With the help of the quantum distribution function one can determine any integral feature of the multiple scattering of charged particles. We illustrate this by the example of the mean square of the scattering angle on the axis of the beam. Using the expression for the distribution function for charged particles on the axis of the beam, f(z, 0, $\mathbf{q}_\perp$), we easily find

$$\langle\theta^2\rangle = p_0^{-2} \left\{ 4B - \frac{a^2 C^2}{4A^2} e^{-a^2/4A} [1 - e^{-a^2/4A}]^{-1} \right\}. \qquad (3.15)$$

It should be noted that for a → ∞

$$\langle\theta^2\rangle = 4B / p_0^2, \qquad (3.16)$$

which is in complete agreement with the results of the classical multiple scattering theory[9] for ħ → 0. On the other hand, for a → 0,

$$\langle\theta\rangle^2 = p_0^{-2} \left\{ \frac{4AB - C^2}{A} + \frac{C^2}{8A^2} a^2 \right\}. \qquad (3.17)$$

The functions $\langle\theta^2\rangle = \Phi_a(z)$ for different apertures lie between the straight line (4.16) and the curve (3.17).

Thus the function f(z, $\mathbf{r}_\perp$, $\mathbf{q}_\perp$) provides a complete quantum mechanical solution of the multiple scattering problem.

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[1] B. Rossi and K. Greisen, Interaction of Cosmic Rays with Matter (Russ. Transl.), IIL, [probably Revs. Modern Phys. **13**, 240 (1941)].

[2] A. B. Migdal, DAN SSSR **96**, 49 (1954). Yu. N. Gnedin and A. Z. Dolginov, JETP **45**, 1136 (1963), Soviet Phys. JETP **18**, 784 (1964).

[3] A. B. Migdal, JETP **32**, 633 (1957), Soviet Phys. JETP **5**, 527 (1957). F. F. Ternovskiĭ, JETP **37**,

1010 (1959) and **39**, 491 (1960), Soviet Phys. JETP **10**, 718 (1960) and **12**, 344 (1961).

[4] N. P. Kalashnikov and M. I. Ryazanov, JETP **47**, 1055 (1964), Soviet Phys. JETP **20**, 707 (1965).

[5] P. K. Aditya, Nuovo Cimento **31**, 473 (1964).

[6] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Systems, in Handbuch der Physik **35**, Springer, Berlin (1957), Russ. Transl. Fizmatgiz (1960).

[7] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[8] I. M. Ryzhik and I. S. Gradshteĭn, Tablitsy integralov, ryadov, summ i proizvedeniĭ (Tables of Integrals, Series, Sums, and Products), Fizmatgiz (1962).

[9] W. T. Scott, Revs. Modern Phys. **35**, 231 (1963).

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