ON VORTEX LATTICES

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Submitted to JETP editor July 2, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1875–1883 (December, 1965)

The paper considers some properties of vortex lattices containing one vortex per lattice cell. Their kinematics is studied and their energy and momentum are calculated. It is shown that in superfluid helium inside a rotating cylinder a triangular lattice is energetically the most favorable of all simple lattices. The problem is equivalent to that of determining the distribution of screw dislocations in a single crystal under torsion.

1. IN this paper we consider some of the properties of simple vortex lattices which may possibly be realized in rotating superfluid helium. It is well known that the flow of the superfluid component of the helium is everywhere irrotational except for vortex lines; the circulation around these is quantized.[1] In stationary rotation a system of parallel vortices must rotate as a rigid structure. This puts a severe restriction on the possible configurations of vortices, but nevertheless a complete description seems hardly possible. We shall confine ourselves to the case of vortex lattices, in particular to simple lattices which contain only one vortex per unit cell.

2. We define a complex coordinate z in the plane at right angles to the vortices. We may assume without loss of generality that there is a vortex at $z = 0$. The coordinates of the other vortices will then be $z_{mn} = 2m\omega_1 + 2n\omega_2$, where m and n are arbitrary integers and ω_1 and ω_2 are the half-periods of the lattice, and $\tau = \omega_1/\omega_2$ must not be a real number. Lattices which differ from each other by a rotation in the plane or by a scale factor will be regarded as equivalent, and it is convenient to characterize them by the value of τ. The choice of periods for any given lattice is not unique, and different τ may correspond to the same lattice.

We choose in the complex τ plane a region D_0 defined by the conditions

$$\operatorname{Im}\tau > 0 \quad -1/2 < \operatorname{Re}\tau < 1/2, \quad |\tau| > 1.$$

For any simple lattice the half-periods ω_1 and ω_2 may be chosen in such a way[2] that $\tau = \omega_2/\omega_1$ will lie within the region D_0 or on its edge. In particular for a rectangular lattice $\operatorname{Re}\tau = 0$, for a square lattice $\tau = i$, for the triangular lattice $\tau = e^{\pi i/3}$ and $\tau = e^{2\pi i/3}$. Two points τ_1 and τ_2 on the boundary of the region D_0 for which

$$\operatorname{Im}\tau_1 = \operatorname{Im}\tau_2, \qquad \operatorname{Re}\tau_1 = -\operatorname{Re}\tau_2,$$

belong to the same lattice. We denote by D the set of points which consists of the region D_0 and that part of its boundary for which $\operatorname{Re} \geq 0$. (The set D has the structure of a complex manifold which is homomorphic with a real two-dimensional sphere S_2.) There is a one-to-one correspondence between lattices (apart from orientation and scale) and the points of the set D. We shall always assume $\tau \in D$.

3. It turns out that all simple lattices can undergo rigid rotation (cf. [3]). To prove this statement we consider the velocity of a fluid. We represent the velocity by a complex quantity v(z) whose magnitude and direction in the complex plane give the magnitude and direction of the fluid velocity at z. Because of the irrotational nature of the flow, the complex conjugate to v(z) must be an analytic function[4], which has simple poles at the points z_{mn} where there are vortices, with identical residues, equal to the strength of the vortices; the function has no other singularities in the finite plane (it is meromorphic).

Functions of this kind whose poles form a regular lattice have been studied extensively. Any one of them can, apart from a factor, be written in the form $\zeta(z) + f(z)$. Here f(z) is any entire function and $\zeta(z)$ is the zeta function of Weierstrass:

$$\zeta(z;\omega_1,\omega_2) = \frac{1}{z} - \frac{g_2(\omega_1,\omega_2)}{60}z^3 - \frac{g_3(\omega_1,\omega_2)}{140}z^5\ldots, \quad (1)$$

where $g_2(\omega_1, \omega_2)$ and $g_3(\omega_1, \omega_2)$ are functions of ω_1 and ω_2 (the so-called modular forms). For the square lattice ($\tau = i$) $g_2 \neq 0$, $g_3 = 0$, whereas for the triangular lattice ($\tau = e^{\pi i/3}$) $g_2 = 0$, $g_3 \neq 0$. The velocity field in the liquid is given by $v(z) = \overline{i(\zeta(z)}$

$+\overline{f(z)})$ where the bar denotes the complex conjugate.

The zeta function of Weierstrass has the following properties which are sufficient to prove the statement at the beginning of this section:[5]

$$\zeta(z+2m\omega_1) = \zeta(z) + 2m\zeta(\omega_1), \quad (2)$$

$$\zeta(z+2n\omega_2) = \zeta(z) + 2n\zeta(\omega_2), \quad (3)$$

$$\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \pi i/2. \quad (4)$$

These properties evidently also hold for the function

$$\zeta_0(z) = \zeta(z) + \alpha z \quad (5)$$

with arbitrary α (αz is an entire function).

The velocity of a vortex is defined as the flow velocity at the location of the vortex in the absence of the latter. If the vortex at $z = 0$ is at rest, then the complex velocity of the vortex at z_{mn} under a flow described by $\zeta_0(z)$ is

$$v_{mn} = 2im\overline{\zeta_0(\omega_1)} + 2in\overline{\zeta_0(\omega_2)}. \quad (6)$$

Choose α in such a way that

$$\zeta_0(\omega_1) = \zeta(\omega_1) + \alpha\omega_1 = \Omega\bar{\omega}_1, \quad (7)$$

$$\Omega = \pi/4\,\mathrm{Im}\,(\bar{\omega}_1\omega_2). \quad (8)$$

(Different lattices will have different α.) It then follows from (4) that

$$\zeta_0(\omega_2) = \zeta(\omega_2) + \alpha\omega_2 = \Omega\bar{\omega}_2. \quad (9)$$

Inserting (7) and (9) in (6) we have

$$v_{mn} = i\Omega(2m\omega_1 + 2n\omega_2), \quad (10)$$

i.e., the vortex system rotates rigidly with angular velocity Ω.

The vortex density $n = 1/4\,\mathrm{Im}(\bar{\omega}_1\,\omega_2)$ and the angular velocity Ω are connected by Feynman's relation[1]

$$\Gamma n = 2\Omega; \quad (11)$$

Γ is the vortex strength.

The quantity α turns out to be zero for the square and triangular lattices.

4. Not all simple lattices are equivalent. Their energy E and angular momentum M depend on τ. The energetically most favorable lattice will have the least value of $E - \Omega M$.[6] We shall calculate the energy and angular momentum of the general simple lattice and then select the one with the least value of $E - \Omega M$.

Let

$$v_r(z) = v(z) - i\Omega z = i(\overline{\zeta_0(z)} - \Omega z). \quad (12)$$

Then the energy E_K of a certain region K of the lattice is

$$E_K = \frac{1}{2}\int_K |v_r + i\Omega z|^2\,dS = E_{Kr} + \Omega M_K - \Omega M_{K\Omega} + E_{K\Omega}, \quad (13)$$

where we have used the symbols

$$E_{Kr} = \frac{1}{2}\int_K |v_r|^2\,dS, \qquad M_K = \mathrm{Re}\int_K z\zeta_0(z)\,dS,$$

$$M_{K\Omega} = \Omega\int_K |z|^2\,dS, \qquad E_{K\Omega} = \frac{\Omega^2}{2}\int_K |z|^2\,dS.$$

The last two terms in (13) do not depend on the nature of the lattice. Hence

$$(E - M\Omega)_K = E_{Kr} + \text{const},$$

and we have to find the minimum of E_{Kr}. In a rotating reference frame the velocity $v_r(z)$ is a doubly periodic function. The energy E_{Kr} is therefore for large regions proportional to the area of the region K. For its determination it is therefore sufficient to find the value of E_{Rr}, where R is any unit cell. It is obvious that the value of E_{Rr} does not depend on the choice of the cell.

5. The energy and angular momentum per cell can be calculated in closed form. It is convenient to choose as an elementary cell the parallelogram with corners at $\pm\omega_1$, $\pm\omega_2$. To determine the energy (Appendix I) we must consider the expression for the complex velocity potential

$$\Phi(z) = \int \zeta_0(z)\,dz$$

on the sides of the parallelogram, and expand its transcendental part in a Fourier series. We can then calculate the energy from the usual formula

$$E_R = \frac{1}{2}\oint_R \mathrm{Re}\,\Phi\,d(\mathrm{Im}\,\Phi) - \frac{1}{2}\oint_\varepsilon \mathrm{Re}\,\Phi\,d(\mathrm{Im}\,\Phi).$$

The contour in the first integral is the boundary of the region R and in the second it is the circle $|z| = \varepsilon$ (ε is the cut-off radius). We obtain finally

$$E_R = \frac{\pi^2}{24}\frac{1+|\tau|^2}{y} + \frac{\pi^2}{12}y\left(1+\frac{1}{|\tau|^2}\right) - \pi\ln\frac{\pi\varepsilon}{2|\omega_1\omega_2|^{1/2}}$$
$$-\frac{\pi}{3}\ln 2\left|\theta_1'(0,\tau)\theta_1'\left(0, -\frac{1}{\tau}\right)\right|. \quad (14)$$

Here $\tau = x + iy$ and $\theta_1'(u, \tau)$ is the derivative of the theta function with respect to the first argument.

The angular momentum M_R is obtained from the function

$$G(z) = \int z\zeta_0(z)\,dz,$$

whose transcendental part is also expanded in a Fourier series (Appendix II). The result is

$$\Omega M_R = \frac{\pi^2}{12}\left[\frac{1+|\tau|^2}{y} + y\left(1+\frac{1}{|\tau|^2}\right)\right]. \qquad (15)$$

The calculation of $E_{R\Omega}$ and $M_{R\Omega}$ is elementary:

$$E_{R\Omega} = \frac{\Omega M_{R\Omega}}{2} = \frac{\pi^2}{24}\frac{1+|\tau|^2}{y}. \qquad (16)$$

From these expressions one finds easily

$$E_{Rr} = -\pi \ln \frac{\pi\varepsilon}{2|\omega_1\omega_2|^{1/2}} - \frac{\pi}{3}\ln 2\left|\theta_1'(0,\tau)\theta_1'\left(0, -\frac{1}{\tau}\right)\right|. \qquad (17)$$

For the numerical evaluation it is convenient to make use of the Euler identity

$$\left[\frac{\theta_1'(0|q)}{2q^{1/4}}\right]^{1/3} = \prod_{k=1}^{\infty}(1-q^{2k}) = \sum_{-\infty}^{\infty}(-1)^k q^{3k^2+k}.$$

Assuming the area of the elementary cell $S_R = 1$ and $\tan\phi = y/x$,

$$E_{Rr} = -\pi\ln 2\pi\varepsilon - \frac{\pi}{2}\ln\sin\varphi + \frac{\pi^2}{12}y\left(1+\frac{1}{|\tau|^2}\right)$$
$$-\pi\ln\left|\sum_{-\infty}^{+\infty}(-1)^k q^{3k^2+k}\right|\left|\sum_{-\infty}^{+\infty}(-1)^k q_1^{3k^2+k}\right|.$$
$$q = e^{i\pi\tau}, \quad q_1 = e^{-i\pi/\tau}. \qquad (18)$$

It is now easy to compute the energy E_{Rr} for the square lattice ($-4.117 - \pi\ln\varepsilon$) and for the triangular lattice ($-4.150 - \pi\ln\varepsilon$).

We now discuss the question of the minimum value of E_{Rr}. Obviously at the minimum its first derivative must vanish. The derivative is

$$-\frac{\pi}{3}\frac{d\ln 2\theta_1'(0,\tau)\theta_1'(0,-1/\tau)}{d\ln\tau} = i[\omega_2\zeta(\omega_1)+\omega_1\zeta(\omega_2)]$$
$$= -2i[\alpha\omega_1\omega_2 - \Omega\,\mathrm{Re}(\overline{\omega_1}\omega_2)]. \qquad (19)$$

We make use of the differential equation of the theta function[7, 8]

$$\pi i\frac{\partial^2\theta_1(u,\tau)}{\partial u^2} + 4\frac{\partial\theta_1(u,\tau)}{\partial\tau} = 0 \qquad (20)$$

and the relation

$$\zeta(\omega_1;\ \omega_1,\omega_2) = -\frac{\pi^2}{12\omega_1}\frac{\theta_1'''(0|q)}{\theta_1'(0|q)}$$
$$= \frac{\pi^2}{12\omega_1}\frac{1-3^3q^2+5^3q^6\ldots}{1-3q^2+5q^6\ldots}. \qquad (21)$$

Thus the radial derivative is

$$\partial E_{Rr}/\partial\ln|\tau| = -2\,\mathrm{Im}\,(\alpha\omega_1\omega_2),$$

and the angular derivative

$$\partial E_{Rr}/\partial\varphi = -2\,\mathrm{Re}\,(\alpha\omega_1\omega_2).$$

We see that for a minimum it is necessary that $\alpha = 0$. It is easy to derive from (7) and (21) that α can vanish only for $\tau = i$ and $\tau = e^{i\pi/3}$, which corresponds to the square and triangular lattices. At $\tau = i$ there is a saddle point and at $\tau = e^{i\pi/3}$ a minimum.

This proves that the triangular lattice is energetically more favorable than others. This does not, however, prove that the triangular lattice will be found in reality. This would follow only if one could prove that it represents the absolute minimum of the quantity $E - \Omega M$ compared to all other configurations of vortices. At present even the stability of the triangular lattice (1) has not been proved. On the other hand the proof of the instability (for small disturbances) of the square lattice is relatively easy.

7. The problem discussed above is equivalent to a problem in the continuum theory of dislocations, i.e., the problem of the position of parallel screw dislocations in a twisted single crystal. Indeed the stress field of a screw dislocation is determined by giving the shear stress in the plane at right angles to the dislocations, and this is determined by a vector within that plane.[9] Introducing a complex coordinate z in the plane and a complex stress p (the analog of the complex velocity) for one dislocation at z_k, we may write

$$p_k(z) = ibr/(\bar{z} - \bar{z}_k).$$

Here b is the magnitude of the Burgers vector which characterizes the dislocation, r is the displacement. We shall assume b = 1 and r = 1. For several dislocations the stresses $p_k(z)$ are superimposed. The torsion of the crystal requires the addition of a stress field $p_0(z) = i\gamma z$. The force acting on a dislocation is proportional to the stress at its location in the absence of the dislocation itself.

The analogy between the vortices in the rotating system of reference and the dislocations in the single crystal under torsion turns out to be complete: the velocity $v_r(z)$ corresponds to the total stress p(z), the integral $\frac{1}{2}\int|v_r|^2\,dS$ corresponds to the deformation energy $\frac{1}{2}\int|p|^2\,dS$ and the angular velocity to the quantity γ.

The similar problem concerning the behavior of vortices in a superconductor of type 2 in a magnetic field is, however, not equivalent to the one we have discussed. The most important difference is the finite penetration depth λ for the field of the vortex. Even if λ is much greater than the typical distance between vortices the force acting on a single vortex in the lattice due to all other vortices vanishes while in the case of the vortices in the liquid for which $\lambda = \infty$, the corresponding expression diverges. Nevertheless for the superconductor the triangular lattice is also energetically more favorable than the square one.[10, 11]

The author wishes to thank Professor I. M. Khalatnikov for his helpful interest in this work. He would also like to thank L. P. Pitaevskiĭ, A. F. Andreev, and particularly S. V. Iordanskiĭ for numerous discussions.

APPENDIX I

CALCULATION OF THE ENERGY OF A UNIT CELL

We choose the unit cell as the parallelogram with the corners $\pm\omega_1$, $\pm\omega_2$. The expansion of $\zeta_0(z)$ at the origin along the ω_1 axis may be written:[7]

$$\zeta_0(z) = \frac{\Omega\bar{\omega}_1}{\omega_1}z + \frac{\pi}{2\omega_1}\cot\frac{\pi z}{2\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{\pi n z}{\omega_1}. \tag{I.1}$$

One easily finds a similar expansion for $\zeta_0(z+\omega_1)$ in the ω_2 direction:

$$\zeta_0(z+\omega_1) = \frac{\Omega\bar{\omega}_2}{\omega_2}z + \Omega\bar{\omega}_1 + \frac{2\pi}{\omega_2}\sum\frac{q_1^n}{1-q_1^{2n}}\sin\frac{\pi n z}{\omega_2}. \tag{I.2}$$

Consider now the potential

$$\Phi(z) = \int\zeta_0(z)\,dz.$$

It is given by the expression[5]

$$\Phi(z) = \frac{\Omega\bar{\omega}_1}{\omega_1}\frac{z^2}{2} + \ln\theta_1\left(\frac{\pi z}{2\omega_1}\right) + C_0.$$

Here θ_1 is the theta function. The constant C_0 is chosen in such a way that at the origin $\Phi(z) = \ln z + 0(z)$. Then

$$\Phi(z) = \frac{\Omega\bar{\omega}_1}{\omega_1}\frac{z^2}{2} + \ln\theta_1\left(\frac{\pi z}{2\omega_1}\right) - \ln\frac{\pi}{2\omega_1} - \ln\theta_1'(0).$$

Using the fact that

$$\theta_1(\pi/2) = \theta_2(0), \quad \theta_1'(0) = \theta_2(0)\theta_3(0)\theta_4(0),$$

We have

$$\Phi(\omega_1) = \frac{\Omega\omega_1\bar{\omega}_1}{2} - \ln\frac{\pi}{2\omega_1} - \ln\theta_3(0)\theta_4(0).$$

By integrating (I.2) we find

$$\Phi(\omega_1+z) = \frac{\Omega\bar{\omega}_2}{\omega_2}\frac{z^2}{2} + \Omega\bar{\omega}_1 z - 2\sum_{n=1}\frac{q_1^n}{1-q_1^{2n}}\frac{1}{n}\cos\frac{\pi n z}{\omega_2} + C.$$

By putting $z = 0$ we see that

$$C = \frac{\Omega\omega_1\bar{\omega}_1}{2} - \ln\frac{\pi}{2\omega_1} - \ln\theta_3(0)\theta_4(0) + 2\sum_{n=1}\frac{q_1^n}{1-q_1^{2n}}\frac{1}{n}.$$

For the energy we require the integral

$$E_0 = \frac{1}{2}\oint_R \operatorname{Re}\Phi\, d(\operatorname{Im}\Phi).$$

We first find the integral

$$E_1 = \int\operatorname{Re}\Phi\, d(\operatorname{Im}\Phi)$$

along the interval $(\omega_1 - \omega_2,\ \omega_1 + \omega_2)$. We have

$$\Phi(\omega_1 + \omega_2\xi) = \frac{\Omega\bar{\omega}_2\omega_2}{2}\xi^2 + \Omega\bar{\omega}_1\omega_2\xi - 2\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{\cos\pi n\xi}{n} + C.$$

A simple integration gives the result

$$E_1 = {}^1/_3\Omega^2\omega_2\bar{\omega}_2\operatorname{Im}(\bar{\omega}_1\omega_2) - 4\Omega\operatorname{Re}(\bar{\omega}_1\omega_2) \times \operatorname{Im}\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{(-1)^n}{n} + 2\Omega\operatorname{Re}C\cdot\operatorname{Im}(\bar{\omega}_1\omega_2).$$

The same quantity gives the integral along the side $(-\omega_1+\omega_2,\ -\omega_1-\omega_2)$. The integrals over the sides $(\omega_1+\omega_2, -\omega_1+\omega_2)$ and $(\omega_1-\omega_2, -\omega_1-\omega_2)$ differ from the expression for E_1 only by the replacement $\omega_1 \to \omega_2$, $\omega_2 \to -\omega_1$. In the end we obtain

$$E_0 = \frac{\pi^2}{12}\frac{\omega_1\bar{\omega}_1 + \omega_2\bar{\omega}_2}{\operatorname{Im}(\bar{\omega}_1\omega_2)} - \pi\frac{\operatorname{Re}(\bar{\omega}_1\omega_2)}{\operatorname{Im}(\bar{\omega}_1\omega_2)}\operatorname{Im}\left(\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{(-1)^n}{n} - \sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\frac{(-1)^n}{n}\right) - \frac{\pi}{2}\ln\frac{\pi^2}{4|\omega_1\omega_2|}$$
$$- \frac{\pi}{2}\ln|\theta_3(0|q)\theta_4(0|q)\theta_3(0|q_1)\theta_4(0|q_1)| + \pi\operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{1}{n} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\frac{1}{n}\right). \tag{I.3}$$

It is easy to show by an expansion in power that

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\frac{1}{n} = -\ln\prod_{n=1}^{\infty}(1-q^{2n-1}). \tag{I.4}$$

Moreover from the various properties of the theta function,[5]

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^3(1-q_1^{2n-1})^3 = 2(qq_1)^{1/8}\left[\frac{\theta_2(0|q)\theta_2(0|q_1)}{\theta_3(0|q)\theta_3(0|q_1)}\right]^{1/2}, \tag{I.5}$$

$$\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1+q_1^{2n-1}} = \left(\frac{q}{q_1}\right)^{1/24}. \tag{I.6}$$

Inserting (I.4) to (I.6) in (I.3) and using the relation

$$\theta_2(0|q)\theta_2(0|q_1) = \theta_4(0|q)\theta_4(0|q_1),$$

we obtain $(x + iy = \tau = \omega_2/\omega_1)$

$$E_0 = \frac{\pi^2}{24}\frac{1+|\tau|^2}{y} + \frac{\pi^2}{12}y\left(1+\frac{1}{|\tau|^2}\right) - \pi\ln\frac{\pi}{2|\omega_1\omega_2|^{1/2}}$$
$$-\frac{\pi}{3}\ln 2|\theta_1'(0|q)\theta_1'(0|q_1)|. \qquad (I.7)$$

APPENDIX II

CALCULATION OF THE ANGULAR MOMENTUM OF A UNIT CELL

The angular momentum is

$$M_R = \mathrm{Re}\int z\zeta_0(z)\,dS.$$

If we let $z = \omega_1\mu + \omega_2\nu$ then $dS = \mathrm{Im}(\overline{\omega_1}\,\omega_2)\,d\mu d\nu$. We shall carry out the calculation by means of the function

$$G(z) = \int_0^z z\zeta_0(z)\,dz.$$

in terms of which the angular momentum becomes

$$M_R = \mathrm{Im}(\bar{\omega}_1\omega_2)\,\mathrm{Re}\int_{-1}^{+1}(G_+ - G_-)\frac{d\nu}{\omega_1}$$
$$= 2\,\mathrm{Im}(\bar{\omega}_1\omega_2)\,\mathrm{Re}\int_{-1}^{+1}G_+(\nu)\frac{d\nu}{\omega_1}. \qquad (II.1)$$

Here $G_\pm(\nu)$ is the value of $G(\omega_1\mu + \omega_2\nu)$ for $\mu = \pm 1$.

We now determine G(z). From (I.1) we find

$$G(z) = \int_0^z z\zeta_0(z)\,dz = \frac{\Omega\bar{\omega}_1}{\omega_1}\frac{z^3}{3} + z\ln\sin\frac{\pi z}{2\omega_1} - \int_0^z \ln\sin\frac{\pi z}{2\omega_1}\,dz$$
$$+ 2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\left(-\frac{z}{n}\cos\frac{\pi n z}{\omega_1} + \frac{\omega_1}{\pi n^2}\sin\frac{\pi n z}{\omega_1}\right),$$

$$G(\omega_1) = \frac{\Omega\bar{\omega}_1\omega_1^2}{3} + \omega_1\ln 2 - 2\omega_1\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\frac{(-1)^n}{n}. \qquad (II.2)$$

To find G_+, we use (I.2):

$$G_+(\nu) = \frac{\Omega\bar{\omega}_2\omega_2^3}{3}\nu^3 + \frac{\Omega\bar{\omega}_1\omega_2^2}{2}\nu^2 + \frac{\Omega\bar{\omega}_2\omega_1\omega_2}{2}\nu^2$$
$$+ \Omega\bar{\omega}_1\omega_1\omega_2\nu - 2\omega_1\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{\cos\pi n\nu}{n}$$
$$+ 2\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\left(-\frac{\omega_2\nu\cos\pi n\nu}{n} + \frac{\omega_2}{\pi n^2}\sin\pi n\nu\right)$$
$$+ 2\omega_1\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{1}{n} + G(\omega_1).$$

Finally we find

$$\int_{-1}^{+1}G_+\frac{d\nu}{\omega_1} = \frac{\Omega\bar{\omega}_1\omega_2^2}{3\omega_1} + \frac{\Omega\bar{\omega}_2\omega_2}{3} + 4\sum_{n=1}^{\infty}\frac{q_1^n}{1-q_1^{2n}}\frac{1}{n} + \frac{2G(\omega_1)}{\omega_1}. \qquad (II.3)$$

The real part of (II.3) must not change if we replace ω_1, ω_2 by $\omega_2, -\omega_1$. If we add the two expressions which differ by this replacement and use (I.4) and the relation

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\frac{(-1)^n}{n} = -\ln\prod_{n=1}^{\infty}(1+q^{2n})$$

we find after some straightforward transformation that

$$\Omega M_R = \frac{\pi^2}{12}\left[\frac{1+|\tau|^2}{y} + y\left(1+\frac{1}{|\tau|^2}\right)\right]. \qquad (II.4)$$

Note added in proof (November 1, 1965). A proof of the stability of the triangular lattice has been found and will be published shortly.

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Translated by R. E. Peierls
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