EFFECT OF INHOMOGENEITIES ON THE OPERATION REGIME OF SOLID-STATE LASERS

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A set of two equations is derived in the geometric optics approximation for the electromagnetic field and for the inverse population that describes nonstationary processes in lasers in the presence of inhomogeneities of the complex dielectric constant perpendicular to the laser axis (it is assumed that the active medium is homogeneous along the laser axis). The conditions are found under which the field produced can be represented as a superposition of the natural oscillation modes of a cavity resonator. The equations obtained are used to calculate the operating conditions of a laser with a thin active rod along the axis. Operating modes consisting of undamped relaxation oscillations are found.

USUALLY the field generated inside the resonator of a solid state laser is represented in the form of a superposition of the natural oscillations of the empty resonator. This approach presupposes that the complex dielectric constant of the active medium filling the resonator is homogeneous in space. However, the inhomogeneity of the crystal used in real lasers, the nonstationary nature of the generation, and the inhomogeneity of the pump cause the real and imaginary parts of the complex dielectric constant to be, generally speaking, functions of the coordinates and of the time. The spatial inhomogeneities influence strongly the operating modes of solid-state lasers. Thus, Leontovich and Veduta[1] have shown experimentally that the inhomogeneities of the refractive index of the crystal, in a direction transverse to the generator axis, determine to a great degree the distribution of the field amplitude over the end face of the crystal, and also the beam divergence. The inhomogeneity of the imaginary part of the dielectric constant in a direction transverse to the resonator axis causes the generated radiation to have a whole set of wave-vector directions concentrated in a certain solid angle.

Galanin et al.[2] indicate that the effective lifetime of such a radiation in an active volume cannot be completely characterized by the Q concept, and is determined also by the preceding instants of generation. If the radiation is pulsating in form, this signifies physically that radiation is lost from the active medium not only as a result of the absorption and transmission of the mirrors, but also as a result of "oozing" of the radiation in a transverse direction.

It is of interest to investigate theoretically the influence of these effects on the laser generation conditions. Even a qualitative solution of the problem in general form is a complicated matter. We confine ourselves to an analysis of the effect produced on the generation modes by inhomogeneities of the dielectric constant only in a direction transverse to the resonator axis. This analysis is based on the following premises. It is assumed that the active medium is homogeneous along the generator axis and is described by a complex dielectric constant. The imaginary part of the dielectric constant is determined by the population of the levels between which generation takes place.

1. DERIVATION OF EQUATIONS

We consider an infinite plane-parallel layer of a dielectric of thickness l with ideally reflecting boundaries. We choose the coordinate system in such a way that the xy plane coincides with the lower boundary of the layer, and the z axis is perpendicular to it. Without loss of generality, we assume for simplicity that the dielectric constant varies in the layer only in the x direction and is of the form

$$\varepsilon(x, t) = \varepsilon'(0) + \delta\varepsilon'(x, t) + i\varepsilon''(x, t). \quad (1)$$

Since usually $|\delta\varepsilon'| \sim 10^{-4}$ and $|\varepsilon''| \sim 10^{-6}$, we henceforth assume throughout that

$$|\delta\varepsilon'|,\quad |\varepsilon''| \ll 1. \tag{2}$$

We assume also that the dimensions Δx of the inhomogeneities in the x direction are large compared with the wavelength of the generated radiation $\lambda \sim 10^{-4}$ cm:

$$\Delta x \gg \lambda. \tag{3}$$

The reflection loss in the mirrors of real lasers are included in $\varepsilon''(x, t)$, that is, we assume them to be uniformly distributed along the z axis.

From Maxwell's equations, with the material equations in the form

$$\mathbf{D} = \varepsilon\mathbf{E},\qquad \mathbf{B} = \mu\mathbf{H}, \tag{4}$$

and assuming henceforth $\mu = 1$, we obtain

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} = \frac{1}{c^2}\frac{\partial^2(\varepsilon\mathbf{E})}{\partial t^2}. \tag{5}$$

We shall consider waves whose propagation direction lies in the xz plane. In this case none of the quantities depend on the coordinate y. We must distinguish here between two independent cases of polarization. If the vector **E** is perpendicular to the xz plane, Eq. (5) takes the form

$$\Delta\mathbf{E} = \frac{1}{c^2}\frac{\partial^2(\varepsilon\mathbf{E})}{\partial t^2}. \tag{6}$$

An equation similar to (6) can be obtained for the case when the vector **E** lies in the propagation plane by using the conditions (2) and (3).

Further, we neglect in the right side of (6) small terms containing the derivative of ε, bearing in mind that in real solid-state lasers

$$\frac{\partial E}{\partial t} \sim \omega_0 E,\qquad \omega_0 \sim 10^{15},\qquad |\varepsilon''| \sim 10^{-6},$$

$$\left|\frac{\partial\varepsilon''}{\partial t}\right| \sim 10^2. \tag{7}$$

Then (6) takes the form

$$\Delta\mathbf{E} = \frac{1}{c^2}\varepsilon(x,t)\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{8}$$

Since the cavities of lasers have high Q, and their radiation is characterized by a narrow spectrum and high directivity in the z direction, we seek the solution of (6) in the form

$$E(x,z,t) = \sum_n U_n(x,t)\exp\{-i\alpha_n(x,t)\}$$

$$\times \sin\frac{\pi nz}{l}\exp\{-i\omega_n{}^0 t\}. \tag{9}$$

Here $U_n(x,t)$ and $\alpha_n(x,t)$ are respectively the slowly-varying amplitude and phase of the field, so that

$$\left|\frac{\partial\alpha_n(x,t)}{\partial t}\right|,\qquad \left|\frac{1}{U_n}\frac{\partial U_n}{\partial t}\right| \ll \omega_n{}^0. \tag{10}$$

The factor $\sin(\pi nz/l)$ describes a standing wave along the generator axis such that (9) satisfies the boundary conditions $E = 0$ on the mirrors; ω_n^0—the critical frequency corresponding to the x-independent solution of Eq. (6) with $\varepsilon(x,t) = \varepsilon'(0)$. For $\omega_n{}^0$ we have

$$\omega_n{}^0 = nc/2l\sqrt{\varepsilon'(0)}. \tag{11}$$

The summation in (9) must be carried out over all n for which $\omega_n{}^0$ lies in the amplification band of the active medium, but since the active medium is homogeneous along the z axis, we confine ourselves in the sum (9) to a single term, for which $\omega_n{}^0$ lies at the maximum of the amplification band; the index n will henceforth be omitted.

We seek the final solution in the form

$$E(x,z,t) = U(x,t)e^{-i\alpha(x,t)}\sin\frac{\pi nz}{l}e^{-i\omega_0 t}. \tag{12}$$

After substituting (12) in (6), we separate the real and imaginary parts and, in accordance with (10), (7), and (2), discarding small terms, we obtain two equations:

$$U\frac{\partial\alpha}{\partial t} = \frac{c^2}{2\varepsilon'(0)\omega_0}\left[\left(\frac{\partial\alpha}{\partial x}\right)^2 U - \frac{\partial^2 U}{\partial x^2}\right] + \frac{\omega_0\delta\varepsilon'(x,t)}{2\varepsilon'(0)}U,$$

$$\frac{\partial U}{\partial t} = \frac{c^2}{2\varepsilon'(0)\omega_0}\left[2\frac{\partial\alpha}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial^2\alpha}{\partial x^2}U\right] - \frac{\varepsilon''(x,t)\omega_0}{2\varepsilon'(0)}U. \tag{13}$$

Assuming that

$$U(x,t)e^{-i\alpha(x,t)} = F(x,t), \tag{14}$$

we rewrite (13) in the form of a single equation with complex coefficients:

$$\frac{\partial F}{\partial t} = \frac{ic^2}{2\varepsilon'(0)\omega_0}\frac{\partial^2 F}{\partial x^2} - \frac{i\omega_0}{2\varepsilon'(0)}(\delta\varepsilon'(x,t) - i\varepsilon''(x,t))F. \tag{15}$$

Equation (15) must be solved simultaneously with the equation for the inverse population

$$\frac{\partial N(x,t)}{\partial t} = n(x)\left(\rho(x,t) - \frac{1}{\tau}\right) - N(x,t)\left(\rho(x,t) + \frac{1}{\tau}\right)$$

$$- 2w_i N(x,t)\overline{U^2(x,t)}. \tag{16}$$

Here $N(x,t) = n_2(x,t) - n_1(x,t)$, where $n_2(x,t)$ and $n_1(x,t)$ are the concentrations of the atoms in the upper and lower generation levels, n(x) is the concentration of the active atoms, τ the lifetime of the atom at the upper generation level with respect to spontaneous emission, w_i the stimulated-emission probability, $\rho(x,t)$ the probability of transition of the atom to the excited state under the influence of the pump field, and

$$\overline{U^2(x,t)} = {}^1/_2 U^2(x,t) \tag{17}$$

is the square of the field amplitude inside the layer averaged over the z coordinate. If the absorption line has a dispersion contour, the imaginary part of the dielectric constant $\varepsilon''(x,t)$, is connected with the inverse population and with the losses in the resonator by the relation

$$\varepsilon''(x,t) = R(x) - N(x,t)\frac{4\pi e^2 f\sqrt{\varepsilon'(0)}}{m\gamma\omega_l} \times\left[1+\left(2\frac{\omega_l-\omega_0}{\gamma}\right)^2\right]^{-1}. \tag{18}$$

Here R(x) characterizes the losses on the mirrors, e is the charge, m the mass of the electron, γ the width of the absorption line at half height, f the transition oscillator strength, and ω_l is the frequency of the absorption-band maximum, assumed to be close to the generation frequency ω_0.

Equations (15) and (16) can be directly extended to include the two-dimensional case. Under the approximations employed, they describe completely, together with (18), the kinetics of the electromagnetic fields and the inverse population in the layer.

2. ENERGY BALANCE EQUATION

Multiplying (15) by $\varepsilon'(0)F^*(x,t)/8\pi$, multiplying the complex conjugate of (15) by $\varepsilon'(0)F(x,t)/8\pi$, and adding the resultant equations, we obtain

$$\frac{\partial w(x,t)}{\partial t} = -\frac{\omega_0}{\varepsilon'(0)}\varepsilon''(x,t)w(x,t) + \frac{c^2}{\varepsilon'(0)\omega_0}\frac{\partial}{\partial x}\left(w(x,t)\frac{\partial\alpha(x,t)}{\partial x}\right), \tag{19}$$

where w(x,t) is the energy density of the electromagnetic field averaged over the z coordinate. The physical meaning of the terms in the right side of (19) is as follows. The first term determines amplification (absorption), and the second is connected with the dependence of the electromagnetic-field flux density on the x coordinate. We note that in the absence of the second term Eq. (19), together with (16), is equivalent to the system of equations obtained in [3]. The second term denotes that the loss of radiation from the active volume is due not only to the absorption and transmission of the mirrors, but also to the "oozing" of the radiation in the transverse direction. This can lead to a mode of undamped relaxation oscillations. We shall verify this later with a concrete example.

3. BOUNDED LAYER. STATIONARY CASE

Let us consider a case when the active medium is a strip bounded by the planes $|x| = R$. We assume that when $|x| < R$ the dielectric constant is defined by Eq. (1), and that $\varepsilon = 1$ when $|x| > R$. We shall solve (15) with boundary conditions corresponding to reflection from the wall:

$$F(-R) = F(R) = 0. \tag{20}$$

When $\varepsilon''(x,t) = 0$ and $\delta\varepsilon'$ does not depend on the time, Eq. (15) takes the form

$$\frac{\partial F}{\partial t} = \frac{ic^2}{2\varepsilon'(0)\omega_0}\frac{\partial^2 F}{\partial x^2} - \frac{i\omega_0}{2\varepsilon'(0)}\delta\varepsilon'(x)F. \tag{21}$$

The stationary solutions of (21) with boundary conditions (20) are the natural modes of the strip and constitute a discrete frequency spectrum. The eigenfunctions $U_k^{(0)}(x)$ describe the field distribution on the ends, while the frequencies of the natural oscillations $\omega_k^{(0)}$ are determined by the relations

$$\omega_k^{(0)} = \omega_0 + \Omega_k^{(0)}, \tag{22}$$

where $\Omega_k^{(0)}$ are the eigenvalues of (21).

We now find the approximate solutions of (15) with time-independent $\delta\varepsilon'(x)$ and $\varepsilon''(x) \neq 0$. Regarding $\varepsilon''(x)$ as a perturbation, we have for the eigenfunctions and the eigenvalues (see [4]):

$$\Omega_k = \Omega_k^{(0)} - \frac{i\omega_0}{2\varepsilon'(0)}\int_{-R}^{R} U_k^{(0)2}(x)\varepsilon''(x)\,dx,$$

$$U_k = U_k^{(0)} - \frac{i\omega_0}{2\varepsilon'(0)}\sum_{m\neq k}\int_{-R}^{R} U_k^{(0)}\varepsilon'' U_m^{(0)}\,dx\,\frac{U_m^{(0)}}{\Omega_k^{(0)}-\Omega_m^{(0)}} \tag{23}$$

The imaginary addition to the frequency characterizes the time variation of the natural modes. The presence of an imaginary correction to the eigenfunctions signifies that the end is no longer an equal-phase surface. The condition for the applicability of (23)

$$\left|\omega_0\int U_k^{(0)}\varepsilon'' U_m^{(0)}\,dx/2\varepsilon'(0)\,(\Omega_k^{(0)}-\Omega_m^{(0)})\right| \ll 1 \tag{24}$$

defines the limits within which the field generated inside the cavity can be represented by a superposition of natural modes of the empty cavity.

In second-order perturbation theory we obtain the corrections $\Delta\Omega_k$ to the natural frequencies:

$$\Delta\Omega_k = -\sum_{m\neq k}\left|\int U_m^{(0)}\varepsilon'' U_k^{(0)}\,dx\right|^2\frac{1}{\Omega_k^{(0)}-\Omega_m^{(0)}}. \tag{25}$$

We note that the presence of the inhomogeneity of $\varepsilon''(x)$ leads to a pulling together of the frequencies of the closest mode groups. This follows directly from (25).

4. BOUNDED LAYER. NONSTATIONARY CASE

In the case when ε'' is a function of the coordinates and of the time, we seek the solution of (15) in the form

$$F(x,t)=\sum_k a_k(t)U_k\exp[-i(\omega_0+\Omega_k^{(0)})t]. \tag{26}$$

Here $U_k^{(0)}\Omega_k^{(0)}$ are the eigenfunctions and the eigenvalues of Eq. (21) with boundary conditions (20), and $a_k(t)$ are the time-dependent complex amplitudes. For the complex amplitudes $a_k(t)$ we obtain from (15) in the usual manner the system of equations

$$\frac{da_k(t)}{dt}=\frac{\omega_0}{2\varepsilon'(0)}\sum_m \varepsilon_{km}''(t)\exp[i(\Omega_k^{(0)}-\Omega_m^{(0)})t]\,a_m(t), \tag{27}$$

$$\varepsilon_{km}''(t)=-\int_{-R}^{R}U_k^{(0)}\varepsilon''(x,t)\,U_m^{(0)}\,dx. \tag{28}$$

We shall use (27) together with (16) to calculate the generation modes of an idealized laser model with localized inhomogeneity and population inversion, when the activated region is bounded by the planes

$$|x|=r,\quad r\ll R. \tag{29}$$

The dielectric constant is defined as follows:

$$\varepsilon=\begin{cases}\varepsilon_0'+i\varepsilon_0'', & r<|x|<R\\ \varepsilon_0'+i\varepsilon_0''-i\varepsilon_1'', & |x|<r\end{cases}. \tag{30}$$

Here ε''_0 is responsible for the mirror-reflection loss and is connected with the lifetime T_0 of the photon in the resonator by the relation

$$\varepsilon_0''=1/T_0\omega_0, \tag{31}$$

while $\varepsilon''_1(t)$ at the center of the amplification line is expressed in terms of the inverse population by the relation

$$\varepsilon_1''(t)=\frac{4\pi e^2 f}{m\gamma\omega_0}N(t). \tag{32}$$

For the natural modes $F_k(x,t)$ that are symmetrical with respect to the center of the laser we have

$$F_k(x,t)=\frac{1}{\sqrt{R}}\cos\frac{(2k-1)}{2R}x\exp[-i(\omega_0+\Omega_k^{(0)})t],$$

$$\Omega_k^{(0)}=\frac{1}{2\omega_0\varepsilon'(0)}\left[\frac{\pi c(2k-1)}{2R}\right]^2. \tag{33}$$

No asymmetrical modes will be excited. For the chosen form of $\varepsilon''(x,t)$, it is easy to calculate the coefficients ε''_{mk}:

$$\varepsilon_{mk}''(t)=-\varepsilon_0''\delta_{mk}+\frac{2r}{R}\varepsilon_1''(t). \tag{34}$$

This expression is valid so long as the wavelength of the excited modes in the x direction is large compared with the inhomogeneity dimension 2r. We shall consider a case when the effective dimension of the field in the x direction is large compared with 2r. In this case the terms with large values of k in the expansion (26) have small amplitudes. This can be verified analytically. We shall verify this directly with numerical calculations.

For convenience we make in (27) a change in variable:

$$a_k(t)\exp[i(\Omega_1^{(0)}-\Omega_k^{(0)})t]=A_k'(t)+iA_k''(t). \tag{35}$$

We shall use the calculation (34) and, separating the real and imaginary parts, obtain the system of equations:

$$\frac{dA_k'(t)}{dt}=-\frac{\omega_0\varepsilon_0''}{2\varepsilon_0'}A_k'+\frac{\omega_0 r}{\varepsilon_0'R}\varepsilon_1''(t)\sum_m A_m(t)+\Omega_{1k}A_k''(t),$$

$$\frac{dA_k''(t)}{dt}=-\frac{\omega_0\varepsilon_0''}{2\varepsilon_0'}A_k''+\frac{\omega_0 r}{\varepsilon_0'R}\varepsilon_1''(t)\sum_m A_m''(t)-\Omega_{1k}A_k'(t). \tag{36}$$

We neglect in (16) the small term $N(t)(\rho+1/\tau)$ and, assuming the pump to be homogeneous in space and constant in time, we obtain an equation for the inverse population in the active region of the resonator:

$$\frac{dN(t)}{dt}=n\left(\rho-\frac{1}{\tau}\right)-\frac{e^2 fN(t)}{4m\gamma\omega_0\hbar}E^2(0,t). \tag{37}$$

The system (36) together with (37) was solved numerically for the first five interacting modes. The calculations were made using the following typical parameters for ruby: $f=10^{-5}$, $\omega_0=3\times10^{15}\ \mathrm{sec}^{-1}$, $\gamma=3\times10^{12}\ \mathrm{sec}^{-1}$, $n=2\times10^{19}\ \mathrm{cm}^3$, $\tau=3\times10^{-3}$ sec for values $R=0.35$ cm and $r=0.01$ cm. Figures 1a, b, and c show three solutions obtained at a pump parameter $\rho=2/\tau$ and at ε''_0 values of 7×10^{-9}, 5×10^{-9}, and 3×10^{-9} respectively. The abscissas represent the time and the ordinates the total energy of the electromagnetic field in the resonator, in arbitrary units. We see thus that in a generator with localized inhomogeneity there exist generation modes in the form of undamped regular pulsations (Fig. 1a).

Figure 2 shows the distribution of the square of

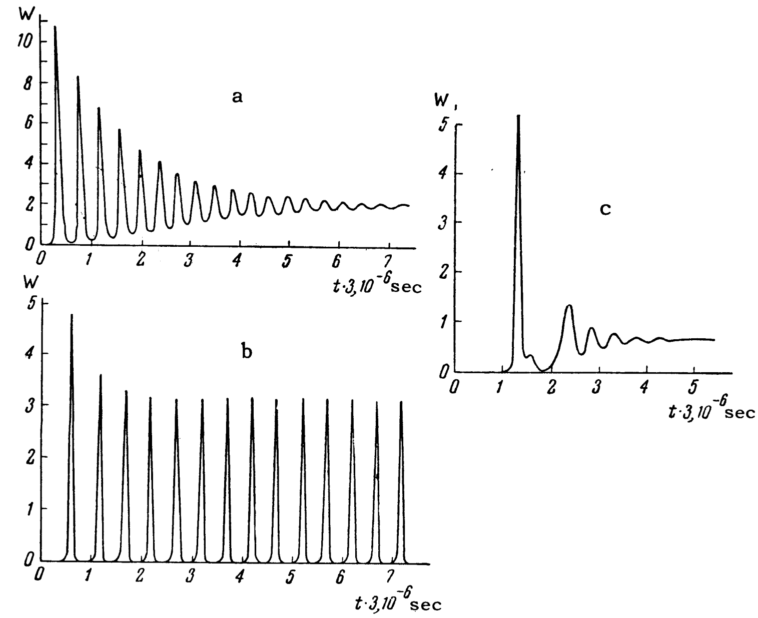

FIG. 1. Total energy W of electromagnetic field in the resonator vs. the time t, for the following values of the parameter ϵ''_0: a – 7×10^{-9}, b – 5×10^{-9}, c – 3×10^{-9}.

the field amplitude F*F in a direction perpendicular to the generator axis, such that the integral $\int F^*F dx$ is normalized to unity. The distribution of the square of the field amplitude on Fig. 2a corresponds to the instant of time preceding the start of the next generation pulse, when the inverse population is maximal, while that in Fig. 2b corresponds to the end of the generation pulse, when the inverse population is minimal. The spreading of the field in the transverse direction leads to a nonlinear energy loss of the electromagnetic field from the active volume, so that undamped pulsations become possible. This result confirms the suggestion made in [2] that account must be taken of the angular distribution of the radiation in order to explain the undamped pulsations.

The solutions represented in Figs. 1a and c begin with a transient process and then go over into a stationary mode. Figures 3a and b show the complex amplitudes A_k corresponding to the steady-state solutions shown respectively in Figs. 1a and c. It is seen from Figs. 3a and b that for the chosen values of ε''_0 the first five symmetrical modes suffice for a description of the electromagnetic field in the assumed model of generator with localized homogeneity $\varepsilon''(x,t)$. All the numerical calculations were made with the M-20 electronic computer.

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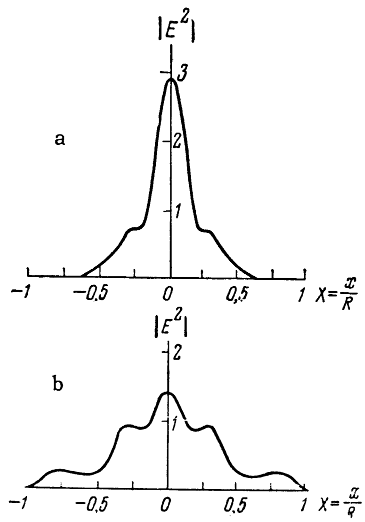

FIG. 2. Distribution of the square of the field amplitude F*F in the direction perpendicular to the generator axis X = x/R: a – at the instant of time preceding the start of the generation pulse; b – at the end of the generation pulse.

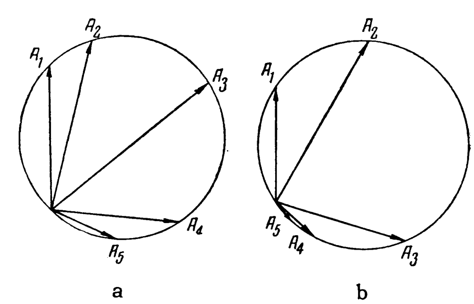

FIG. 3. Relative values of the complex amplitudes $A_k = A'_k + iA''_k$ for stationary generation modes: a – for $\epsilon''_0 = 7 \times 10^{-9}$, b – for $\epsilon''_0 = 3 \times 10^{-9}$.

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