ISING MODEL WITH INTERACTION BETWEEN NONNEAREST NEIGHBORS

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A two-dimensional Ising lattice is considered in which, besides the usual interaction, there is an interaction along diagonals between nodes with equal row-plus-column parities. The free energy and the spontaneous magnetization are found as functions of the temperature. The form of the correlation function at large distances is derived at and close to the phase-transition point.

1. INTRODUCTION

THE Ising model consists of a lattice of dipoles, each of which takes only two positions and interacts only with its nearest neighbors. This model is attracting great interest in connection with the theory of phase transitions of the second kind. It is argued that phase transitions in binary alloys and with changes of crystal symmetry, and also the behavior of substances near the critical point, are described by this model.[1,2] Therefore it is interesting to ascertain how sensitive the results are to the form of the model, and in particular whether there are changes of the nature of the singularity in macroscopic quantities and of the shape of the correlation function when interactions with nonnearest neighbors are included.

In the present paper we consider a two-dimensional lattice, and include in addition to the interaction of nearest neighbors an interaction of certain nonnearest neighbors.

2. CALCULATION OF THE FREE ENERGY

Let us consider a two-dimensional lattice of the Ising type, consisting of two kinds of "atoms" which are arranged in a checkerboard pattern and interact with each other in the way shown in Fig. 1. The interaction energy between different atoms, i.e., along vertical and horizontal directions, is $-J_1$, and that along the diagonals is $-J_2$. The difference between this model and the ordinary Ising lattice is that besides the interaction between nearest neighbors there is also an interaction along the diagonals for the atoms of one kind. For $J_1 = 0$ the system goes over into an ordinary Ising lattice.

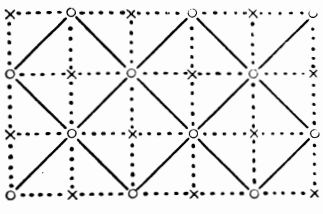

FIG. 1

The partition function is given by the expression

$$Z = \sum_{(\sigma)} \exp\left[\frac{J_1}{T}\sum_{k,l=1}^{L}\sigma_{kl}(\sigma_{kl+1}+\sigma_{k+1l}) + \frac{J_2}{T}\sum_{k,l=1}^{L}\eta_{kl}\sigma_{kl}(\sigma_{k+1,l+1}+\sigma_{k-1,l+1})\right],$$

$$\sigma_{kl} = \pm 1, \quad \eta_{kl} = {}^1/_2[1+(-1)^{k+l}], \tag{1}$$

where L is the number of atoms in a row or column. The expression (1) can be put in the form

$$Z = (1-x^2)^{-N}(1-y^2)^{-N/2}S,$$

$$S = \sum_{(\sigma)}\prod_{k,l}(1+x\sigma_{kl}\sigma_{kl+1})(1+x\sigma_{kl}\sigma_{k+1l})(1+y\eta_{kl}\sigma_{kl}\sigma_{k+1,l+1}) \times (1+y\eta_{kl}\sigma_{kl}\sigma_{k-1,l+1}). \tag{2}$$

Here $x = \tanh(J_1/T)$, $y = \tanh(J_2/T)$, and $N = L^2$ is the total number of atoms. The quantity S is a polynomial in x and y, in which the coefficient g_{nm} of $x^n y^m$ is equal to the number of ways closed polygons can be constructed in which the total number of vertical and horizontal links is n and the total number of diagonal links is m (cf., e.g.,[3]).

It is shown in a paper by Vdovichenko [4] that for the ordinary Ising lattice the quantity g_{nm} can be put in the form of a sum over closed loops, each loop being taken with the factor $(-1)^r$, where r is the number of intersections. Our present case differs from the usual one by the fact that there can be intersection not of only two,

but also of three or even four, lines at a given node. We shall show that in our case each such intersection, which goes into g_{nm} with the weight unity, can be represented as the sum of all possible unions of pairs, if we count each union with the weight $(-1)^r$, where r is the number of intersections.

For this purpose we associate with each line a Fermi operator $\psi = a + a^+$. We have the obvious identity

$$\langle \psi\psi\psi\psi\psi\psi \rangle = \langle \psi^{2n} \rangle = 1,$$

where $\langle \ldots \rangle$ means averaging over the state with occupation number zero. On the other hand, according to Wick's rule [5] (which can be proved here, as in ordinary cases, by induction) such an average is equal to the sum of products of all possible averages of pairs, each equal to unity, multiplied by the parities of the permutations of the operators ψ. But the order of a permutation is equal to the number of intersections, if we let a union of a pair of lines correspond to each such contraction.

It is also shown in Vdovichenko's paper that we do not need to deal with the number of intersections if we take each node in the loop with the weight $\exp(i\varphi/2)$, where φ is the angle of turn at the node, which in our case takes the values $\varphi = 0, \pm\pi/4, \pm\pi/2, \pm 3\pi/4$. Making the further calculations in analogy with [3,4], we get the equation

$$S = 2^N \prod_i (1 - \lambda_i)^{1/4} = 2^N \prod_{p,\,q}^{12} \prod_{s=1} [1 - \lambda_s(p, q)]^{1/4}, \qquad (3)$$

where $p = 2\pi n_1/L$, $q = 2\pi n_2/L$, and $\lambda_S(p, q)$ are the eigenvalues of the matrix $A(p, q)$, which is of the form

	ye_{p+q}		$\alpha^{-2}ye_{p-q}$				$\alpha^2 ye_{q-p}$		$\alpha^{-1}xe_p$	$\alpha^{-3}xe_{-q}$	$\alpha^3 xe_{-p}$	$\alpha x\varepsilon_q$
	αye_{p+q}		$\alpha^{-1}ye_{p-q}$		$\alpha^{-3}ye_{p-q}$		$\alpha^3 ye_{q-p}$		xe_p	$\alpha^{-2}xe_{-q}$		$\alpha^2 x\varepsilon_q$
	$\alpha^2 ye_{p+q}$		ye_{p-q}		$\alpha^{-2}ye_{p-q}$				αxe_p	$\alpha^{-1}xe_{-q}$	$\alpha^3 xe_{-p}$	$\alpha^3 xe_q$
	$\alpha^3 ye_{p+q}$		αye_{p-q}		$\alpha^{-1}ye_{p-q}$		$\alpha^{-3}ye_{q-p}$		$\alpha^2 xe_p$	xe_{-q}	$\alpha^2 xe_{-p}$	
			$\alpha^3 ye_{p-q}$		ye_{p-q}		$\alpha^{-2}ye_{q-p}$		$\alpha^3 xe_p$	αxe_{-q}	$\alpha^{-1}xe_{-p}$	$\alpha^{-3}xe_q$
	$\alpha^{-2}ye_{p+q}$		$\alpha^3 ye_{p-q}$		αye_{p-q}		$\alpha^{-1}ye_{q-p}$			$\alpha^2 xe_{-q}$	xe_{-p}	$\alpha^{-2}xe_q$
	$\alpha^{-1}ye_{p+q}$				$\alpha^3 ye_{p-q}$		ye_{q-p}		$\alpha^{-3}xe_p$	$\alpha^3 xe_{-q}$	αxe_{-p}	$\alpha^{-1}x\varepsilon_q$
	$\alpha^{-1}ye_{p+q}$		$\alpha^{-3}ye_{p-q}$		$\alpha^3 ye_{p-q}$		αye_{q-p}		$\alpha^{-2}xe_p$		$\alpha^2 xe_{-p}$	xe_q
		xe_p		$\alpha^{-2}xe_{-q}$				$\alpha^2 xe_q$				
		$\alpha^2 xe_p$		xe_{-q}		$\alpha^{-2}xe_{-p}$						
				$\alpha^3 xe_{-q}$		xe_{-p}		$\alpha^{-2}xe_q$				
		$\alpha^{-2}xe_p$				$\alpha^2 xe_{-p}$		xe_q				

(4)

where $\alpha = e^{i\pi/8}$, $e_p = e^{ip}$.

The product of eigenvalues which appears in (3) is equal to the determinant of the matrix $1 - A$. Calculating the determinant, we get the expression

$$\begin{aligned} \mathrm{Det}\,(1 - A(p, q)) \equiv D(p, q) = {} & (y^2 + 1)^2(x^8 + 14x^4 + 1) \\ & + 16y(y^2 + y + 1)x^2(x^2 + 1)^2 \\ & + 2(\cos\omega + \cos\rho)(y^2 - 1)(x^2 - 1)^2[2x^2(y^2 + 1) \\ & + y(x^2 + 1)^2] - 4x^2(y^2 - 1)^2(x^2 - 1)^2 \cos\omega \cos\rho; \\ \omega = p + q, \qquad \rho = p - q. & \end{aligned} \qquad (5)$$

By means of Eqs. (5), (3) and (2) we get for the free energy F:

$$F(T) = -T \ln Z = \frac{NT}{2} \ln \frac{(1 - x^2)^2(1 - y^2)}{4}$$

$$- \frac{NT}{4} \int_0^{2\pi} \int_0^{2\pi} \frac{dpdq}{(2\pi)^2} \ln D(p, q). \qquad (6)$$

For $x = 0$ or $y = 0$ Eq. (6) goes over into the usual expression for the free energy of the Ising lattice. At a certain temperature T_c the system undergoes a phase transition of the second kind into a state with ordered "spins"; $F(T)$ has a singularity at this temperature.

The transition temperature is determined from the condition that for the corresponding x, y the determinant $D(p, q)$ must be zero for certain values of p, q. As a function of $\omega = p + q$ and $\rho = p - q$ the expression $D(\omega, \rho)$ of Eq. (5) is a minimum for $\cos\omega = \cos\rho = 1$ or for $\cos\omega = \cos\rho = -1$. These minimum values D_1 and D_2 are squares of polynomials in x and y, and T_c is determined from the condition that D_1 or D_2 be equal to zero:

$$D_1 = [(y + 1)^2(x^2 + 1)^2 - 2(1 - x^2)^2]^2 = 0 \qquad (7a)$$

or

$$D_2 = [(y + 1)^2(x^2 + 1)^2 - 2y^2(1 - x^2)^2]^2 = 0. \qquad (7b)$$

The dependence of T_c on the coupling constants J_1, J_2 which is obtained from these equations is shown in Fig. 2. For $J_2 > -|J_1|$ the phase transition leads to a state in which the spins of the atoms located along the diagonals are parallel. For $J_2 < -|J_1|$ the transition is to a state with antiparallel spins of these atoms. It is interesting to note that in a certain (numerically rather narrow) region of values of J_1 and $J_2 < -|J_1| < J_2 < -0.94|J_1|$ —the system undergoes three successive phase transitions as the temperature is lowered: first from the unordered to the "antiferromagnetic" state, then back from the antiferromagnetic to the unordered state, and

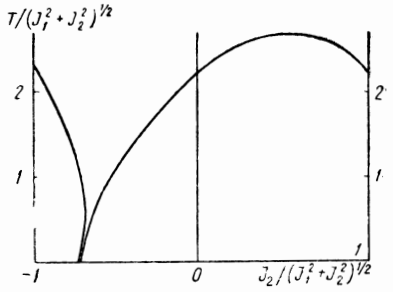

FIG. 2

only thereafter into the ferromagnetic state. In this region a transition to the unordered state occurs not only with increase, but also with decrease, of the temperature. Phase transitions of this sort are very rarely encountered experimentally, but exist, for example, in Rochelle salt.[3,6]

To find the form of the singularity of F(T) at the phase transition point we expand the D(p, q) in (6) in a series near its minimum value:

$$D(p, q) = c_1(T/T_c - 1)^2 + c_2(p^2 + q^2). \quad (8)$$

Integrating over p and q we find that the singularity in the free energy near $T = T_C$ is of the same form as for the ordinary Ising lattice:

$$F = F_0(T) + a(T - T_c)^2 \ln|T - T_c| \quad (9)$$

[where $F_0(T)$ is regular near T_C, and $a > 0$], and the heat capacity has a logarithmic infinity.

3. THE CORRELATION FUNCTION

We shall look for the correlation function of two nodes located on a diagonal line:

$$G(r) = \langle\sigma_{11}\sigma_{1+r,1+r}\rangle = \frac{\mathrm{Sp}(\sigma_{11}\sigma_{1+r,1+r}e^{-H/T})}{\mathrm{Sp}\,e^{-H/T}}. \quad (10)$$

Again following the method of Vdovichenko,[7] we arrive at the formula

$$G(r) = y^r \mathrm{Det}^{1/2}[1 - (y^{-1} - y)(1 - A)^{-1}B], \quad (11)$$

where the matrix A in the momentum representation is of the form (4), and the nonvanishing matrix elements of B in the coordinate representation are given by

$$\langle k+1, k+1, \nu|B|k, k, 1\rangle = (1, \alpha, \alpha^2, \alpha^2, 0, \alpha^{-3}, \alpha^{-2}, \alpha^{-1}),$$
$$\langle k, k, \nu|B|k+1, k+1, 5\rangle = (0, \alpha^{-3}, \alpha^{-2}, \alpha^{-1}, 1, \alpha, \alpha^2, \alpha^3);$$
$$1 \leqslant k \leqslant r-1, \quad 1 \leqslant \nu \leqslant 8. \quad (12)$$

Introducing the notation $1 - (y^{-1} - y) \times (1 - A)^{-1}B = P$, we get for the matrix elements of P in the coordinate representation

$$\langle l, l, \nu|P|k, k, 1\rangle = \delta_{kl}\delta_{\nu 1} - (y^{-1} - y)\int\int \frac{dpdq}{(2\pi)^2 D} e^{i\omega(k-l+1)} \times (d_{1\nu} + \alpha d_{2\nu} + \alpha^2 d_{3\nu} + \alpha^3 d_{4\nu} + \alpha^{-3}d_{6\nu} + \alpha^{-2}d_{7\nu} + \alpha^{-1}d_{8\nu}),$$
$$\langle l, l, \nu|P|k, k, 5\rangle = \delta_{kl}\delta_{\nu 5} - (y^{-1} - y)\int\int \frac{dpdq}{(2\pi)^2 D} e^{i\omega(k-l-1)} \times (d_{5\nu} + \alpha d_{6\nu} + \alpha^2 d_{7\nu} + \alpha^3 d_{8\nu} + \alpha^{-3}d_{2\nu} + \alpha^{-2}d_{3\nu} + \alpha^{-1}d_{4\nu}). \quad (13)$$

Here $d_{\mu\nu}$ is the algebraic complement of the $\mu\nu$ element of the matrix $1 - A$.

It can be verified that for the quantities P_{15}, P_{51} we get integrands which change sign on the interchange $p \rightarrow q$, so that the integrals are zero. Then, in analogy with the work of Vdovichenko,[7] we get

$$G(r) = y^r \mathrm{Det}^{1/2} P_{11} \cdot \mathrm{Det}^{1/2} P_{55}. \quad (14)$$

The expression for P_{55} differs from P_{11} by the replacements $p, q \rightarrow -p, -q$, $k - l \rightarrow l - k$ in the integrand; therefore the determinants of these matrices are equal, and

$$G(r) = y^r \mathrm{Det}\, P_{11} = \mathrm{Det}\,\|c_{k-l}\|, \quad (15)$$

where $c_{k-l} = yP_{11}(k - l)$. After extremely cumbersome calculations we find for the elements c_{k-l} the expression

$$c_{k-l} = \int\int \frac{dpdq}{(2\pi)^2} e^{i\omega(k-l)}\left[y - (1-y^2)\frac{N(\omega,\rho)}{D(\omega,\rho)}\right];$$

$$N(\omega,\rho) = -y(y^2+1)(x^8 + 14x^4 + 1) - 4x^2(3y^2 + 2y + 1)(x^2+1)^2 + e^{-i\omega}(1-y^2)(x^4-1)^2 + 4x^2y\cos\omega(1-y^2)(x^2-1)^2 - 2y\cos\rho(x^2-1)^2[2x^2(y^2+1) + y(x^2+1)^2] + 4x^2y\cos\omega\cos\rho(y^2-1)(x^2-1)^2. \quad (16)$$

Using the fact that the integrand is periodic, when we go over to variables ρ, ω we can restrict the integration to the region $0 < \omega, \rho < 2\pi$. Carrying out the integration over ρ we get

$$c_{k-l} = \int_0^{2\pi} \frac{d\omega}{2\pi} e^{i(k-l)\omega} f(\omega),$$

$$f(\omega) = \left[\frac{(e^{i\omega}\alpha_1 - 1)(e^{i\omega}\alpha_2 - 1)}{(e^{i\omega} - \alpha_1)(e^{i\omega} - \alpha_2)}\right]^{1/2}, \quad (17)$$

$$\alpha_1 = \frac{1+y}{1-y}\frac{(x^2+1)^2 y + 4x^2}{(1-x^2)^2},$$

$$\alpha_2 = y\frac{1-y}{1+y}\frac{(1-x^2)^2}{(x^2+1)^2 + 4x^2y}. \quad (18)$$

Our expressions (15) and (17) agree in form with the analogous expressions for the correlation function in the ordinary Ising model.[7,8] For

$y = 0$ they go over into the expressions found by Onsager and Kaufman [9] for the correlation along a diagonal, and for $x = 0$ into the formulas for the correlation along a row.[7,8] The phase transition points correspond to $\alpha_1 \to 1$ or $\alpha_2 \to -1$, in agreement with the first or second of the equations (4).

4. THE ASYMPTOTIC BEHAVIOR OF THE CORRELATION FUNCTION

Below the transition point infinitely distant correlations appear, and the function $G(r)$ has a finite limit for $r \to \infty$. This limit can be found in the same way as for the ordinary Ising lattice, by using Segö's theorem [7-10] on the limiting values of Toeplitz determinants:

$$\lim_{r\to\infty}\left(\frac{\operatorname{Det} c}{\exp[(r+1)K_0]}\right) = \exp\left(\sum_1^\infty nK_nK_{-n}\right),$$

$$K_n = \int_0^{2\pi} e^{in\omega}\ln f(\omega)\,d\omega/2\pi. \tag{19}$$

For $\alpha_1 > 1$, $\alpha_2 > -1$, i.e., in the ferromagnetic region, represented by the lower right-hand part of Fig. 2, we get from (19)

$$M^2 = \lim_{r\to\infty} G(r) = \left[\frac{(1-\alpha_2^2)(\alpha_1^2-1)}{(\alpha_1-\alpha_2)^2}\right]^{1/4}, \tag{20}$$

where M is the mean spontaneous magnetic moment of the atoms with the diagonal interaction. In the antiferromagnetic—lower left-hand—region of Fig. 2, where $\alpha_2 < -1$, $\alpha_1 < 1$, $G(r)$ has the asymptotic behavior

$$G(r) \underset{r\to\infty}{\to} (-1)^r\left[\frac{(\alpha_2^2-1)(1-\alpha_1^2)}{(\alpha_1-\alpha_2)^2}\right]^{1/4}. \tag{21}$$

In the unordered phase, i.e., for $\alpha_1^2, \alpha_2^2 < 1$, $G(r)$ goes to zero exponentially for $r \to \infty$.[11,12] At the transition point the law of decrease of $G(r)$ for large r becomes a power law. Let us find the asymptotic behavior of G for $T = T_c$.

It is simplest to find the correlation for $y = 0$, i.e., along the diagonal of the ordinary Ising lattice. In this case $\alpha_2 = 0$, α_1 is equal to unity at the transition point, and we find from (17)

$$f = f_0(\omega) = (-e^{-i\omega})^{1/2}, \qquad c^0_{k-l} = \frac{1}{\pi(l-k+1/2)}. \tag{22}$$

The determinant with the elements (22) was calculated by Onsager and Kaufman [9]; for large r its behavior is

$$G(r) = \text{const}\cdot r^{-1/4}. \tag{23}$$

It can be verified that this law holds also in the general case $y \neq 0$. To do so we use a method suggested by Ryazanov.[12] Denoting the matrix $c - c_0$ by b, we put the determinant of the c_{kl} of (17) in the form

$$\ln\frac{\operatorname{Det} c}{\operatorname{Det} c_0} = \operatorname{Sp}\ln(1+bc_0^{-1}) = \operatorname{Sp} bc_0^{-1} - \frac{1}{2}\operatorname{Sp} bc_0^{-1}bc_0^{-1} + \ldots. \tag{24}$$

We shall show that the right member of (24) is finite for $r \to \infty$. We find the part of each term of the series (24) which is of highest degree in r. It can be shown from (17) and (21) that the matrix elements $b_{kl} = b_{k-l}$ fall off as $(k-l)^{-2}$ with increasing $k - l$: The matrix c_0^{-1} is calculated in the Appendix; the matrix elements $(c_0^{-1})_{kl}$ fall off as $(k-l)^{-1}$ with increasing $k - l$ [see Eqs. (A.6), (A.7)]. Therefore the main contribution of order r in each term comes from a region where all the indices k, l of the matrix elements are close to each other. In this region, according to (A.7), the elements $(c_0^{-1})_{kl}$ also depend only on the difference $k - l$. Therefore in calculating each of the traces in (24) we can extend the summation over the differences of indices to infinity, and the one remaining summation gives a factor r. Going over to the Fourier representation, we get

$$\ln\frac{\operatorname{Det} c}{\operatorname{Det} c_0} = r\sum_{n=1}^\infty \frac{(-)^{n+1}}{n}\int_0^{2\pi}\frac{d\omega}{2\pi}[b(\omega)c_0^{-1}(\omega)]^n = r\int_0^{2\pi}\frac{d\omega}{2\pi}\ln\frac{f(\omega)}{f_0(\omega)}, \tag{25}$$

where we have used the relation (A.8). The integral in (25) is equal to zero, and therefore there are no terms linear in r in (24).

We shall now show that the logarithmic terms $\sim \ln r$ also give zero. Such terms can arise only from the region of large values of the indices, $k \gg 1$, $l \gg 1$, so that as before we can use the formula (A.6). Then, however, it is necessary, first, to take into account for ϑc_0^{-1} not only the zeroth approximation (A.7) but also the next term of the expansion in the parameters $(k - l + 1/2)k^{-1}$, $(k - l + 1/2)(r - k)^{-1}$:

$$(c_0^{-1})_{kl}^{(1)} = -\frac{1}{2\pi}\left(\frac{1}{k}+\frac{1}{r-k}\right).$$

This term does not depend on the difference $k - l$, but only on the one index k. Summation over k will give terms $\sim \ln r$, but then each term of the series (24) contains as a factor the sum

$$\sum_l b_{l-m} = b(\omega)\,|_{\omega=0} = f(0) - f_0(0) = 0, \tag{26}$$

so that no logarithms arise from the corrections to c_0^{-1}. Second, since all of the indices vary over the finite range from 1 to r, the summation over the differences of indices does not extend to infinity, but is limited, and logarithmic terms could come in from end effects. Logarithms can appear from the range of indices $1 \ll |k - l| \ll r$.

Let us consider, for example, the second term of (24):

$$\sum_{k_1 l_1 k_2 l_2} (c_0^{-1})_{k_1 l_1}\, b_{l_1 - k_2} (c_0^{-1})_{k_2 l_2}\, b_{l_2 - k_1}. \tag{27}$$

Since b_{l-k} falls off rapidly with increasing $l - k$, the main contribution can come only from the region $k_2 \sim l_1$, $l_2 \sim k_1$, so that (27) takes the form

$$\sum_{k_1 l_1} (c_0^{-1})_{k_1 l_1} (c_0^{-1})_{l_1 k_1} \cdot \sum_m b_m \cdot \sum_n b_n. \tag{28}$$

In the first factor of (28), besides the term $\sim r$ which we have dealt with earlier, there can also be logarithmic terms owing to end effects, but the remaining factor is zero by (26). The logarithmic parts in the other terms of the series (24) vanish in just the same way.

Accordingly there are neither linear nor logarithmic terms in the right member of (24), so that independently of α_2—that is, of the ratio of the coupling constants J_1, J_2—there is a universal law of decrease of the correlation function with distance:

$$G(r) = C(\alpha_2) r^{-1/4}. \tag{29}$$

Numerical methods must be used to determine $C(\alpha_2)$. The particular values of C for $\alpha_2 = 0$ and $\alpha_2 = (2^{1/2} - 1)^2$, which correspond to the correlations along a diagonal and along a row in the ordinary Ising lattice, are 0.703 and $0.703 \times 2^{1/8}$ [13] (this ratio of the constants corresponds to radially symmetrical long-range correlations).

Let us now consider the long-range correlations near the transition point, i.e., in the region $r|1 - \alpha_1| \ll 1$. We get as the expression for $G(r)$ in first order in the distance from the transition point

$$\ln \frac{G(\alpha_1, r)}{G(1, r)} = \operatorname{Sp} b^1 c^{-1} = \sum_{k_1 l_1} b^1_{k_1 - l_1} (c_0^{-1})_{l_1 k_1} - \sum_{k_1 l_1 k_2 l_2} b^1_{k_2 - l_1} (c_0^{-1})_{l_1 k_1}\, b_{k_1 - l_2} (c_0^{-1})_{l_2 k_2} + \ldots, \tag{30}$$

where b is the same as in (24) and the matrix b^1 is given by the formula

$$b^1_{k-l} = \frac{1}{2\pi} \int_0^{2\pi} d\omega \left[\left(\frac{\alpha_1 e^{i\omega} - 1}{e^{i\omega} - \alpha_1} \frac{\alpha_2 e^{i\omega} - 1}{e^{i\omega} - \alpha_2} \right)^{1/2} - \left(\frac{\alpha_2 e^{i\omega} - 1}{e^{i\omega} - \alpha_2} \right)^{1/2} \right] e^{i(k-l)\omega} \approx \frac{\tau}{2\pi} \ln \left| \tau \left(k - l - \frac{1}{2} \right) \right|, \quad \tau \equiv 1 - \alpha_1. \tag{31}$$

In the calculation of (31) we have used the fact that $|\tau(k - l - 1/2)| \ll 1$, and have contented ourselves with logarithmic accuracy with respect to this parameter.

We shall look for only the strongest terms in (30), those linear in r. Then the important values of the indices k and l are those of order r, and we can use (A.6). We shall show that the contribution from the region of small values of the differences $k_i - l_i$ is zero. In fact, in this region we can use for $(c_0^{-1})_{kl}$ the difference expression (A.7), which is odd under the replacement k, $l \to l + 1$, k, and to logarithmic accuracy b^1_{kl} is independent of the indices k and l. Therefore the summation over l_1 will give zero.

For large values of the differences $k_i - l_i \sim r$ the quantities $(c_0^{-1})_{k_i l_i}$ vary slowly with k_i, l_i, and all of the terms in (30) except the first are like the expressions (27), (28) in containing the the zero factor (26). Only the first term in (30) gives a nonzero contribution. Furthermore in the factor b^1_{k-l} from (31) we can to logarithmic accuracy replace $k - l$ by r. Using the formula (A.6) for c_0^{-1}, we get for the right member of (30) the expression

$$\frac{\tau}{2\pi^2} \ln|\tau r| \sum_{k,l=1}^{r} \frac{1}{k - l + 1/2} \left(\frac{k(r-l)}{l(r-k)} \right)^{1/2} = \frac{\tau \ln|\tau r|}{4\pi^2} \sum_{kl} \left\{ \frac{1}{k - l + 1/2} \left[\left(\frac{k(r-l)}{l(r-k)} \right)^{1/2} - \left(\frac{l(r-k)}{k(r-l)} \right)^{1/2} \right] - \frac{1}{(k-l)^2 - 1/4} \left(\frac{l(r-k)}{k(r-l)} \right)^{1/2} \right\}. \tag{32}$$

The last term falls off rapidly with increasing $k - l$, and therefore in it we can replace the root by unity, after which the resulting series gives zero, in agreement with the previously noted cancellation of the contributions from small $k - l$. Replacing summation by integration in the other terms and neglecting all terms $\sim 1/r$, we get

$$\operatorname{Sp} b^1 c^{-1} = \frac{\tau \ln|\tau r|}{4i\pi^2} \sum_{k,l} \int_0^\infty e^{i(k-l)x}\, dx \left[\left(\frac{k(r-l)}{l(r-k)} \right)^{1/2} - \left(\frac{l(r-k)}{k(r-l)} \right)^{1/2} \right] = \frac{\tau \ln|\tau r|}{4i\pi^2} r^2 \int_0^\infty dx e^{-irx} \left[\left(\int_0^1 du e^{irxu} \times \left(\frac{u}{1-u} \right)^{1/2} \right)^2 - \left(\int_0^1 e^{irxu} \left(\frac{1-u}{u} \right)^{1/2} du \right)^2 \right]. \tag{33}$$

After the substitution $u = \cos^2\varphi$ the integrand in (33) is converted into a total derivative with respect to x. The result is

$$\mathrm{Sp}\, b^1 c^{-1} = \frac{1}{4}\tau r \ln|\tau r|. \tag{34}$$

Accordingly, in the region $1 \ll r \ll |\tau|^{-1}$ the correlation function is of the form

$$G(r) = Cr^{-1/4}(1 + {}^1/_4 r\tau \ln|\tau r|), \qquad \tau = 1 - \alpha_1 = B(T/T_c - 1). \tag{35}$$

In the case of the ordinary Ising lattice the constant B is equal to $2 \ln (1 + 2^{-1/2}) = 1.07$ for the row and to $2^{3/2} \ln (1 + 2^{-1/2}) = 1.07 \cdot 2^{1/2}$ for the diagonal, which again corresponds to isotropic correlation at large distances.

We can find from (35) the Fourier representation of G in the region $|\tau| \ll k \ll 1$:

$$G(k) = k^{-7/4}\left(a_1 - a_2 \frac{\tau}{k} \ln \frac{k}{|\tau|}\right) \tag{36}$$

where a_1, a_2 are constants. We note that G(k) has a singularity of the form $\tau \ln|\tau|$ for all k different from zero.

5. CONCLUSION

Accordingly, in the model considered here the singularities in the macroscopic quantities at the transition point are the same as in the Ising model. As in the Ising model, the behavior of the correlation function near the transition point is described by the formulas (34), (35). Of course we cannot exclude the possibility that when other interactions in the lattice are taken into account, for which the problem cannot be solved exactly, the singularities will be of a different form, but this possibility seems to us physically improbable (see also [2]). The only difference between the present model and the Ising model is that for a certain ratio of the constants of the system there are three successive phase transitions as the temperature is changed.

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APPENDIX

We shall find the matrix c_0^{-1} inverse to the matrix c_0 given by (22):

$$c_{kl}^0 = \frac{1}{\pi(l - k + {}^1/_2)}. \tag{A.1}$$

We use the same method as applied to find the determinant c_0 [12,14]. The matrix c_{kl}^0 is of the form

$$c_{kl}^0 = (x_l - y_k)^{-1}; \qquad x_l = \pi(l + {}^1/_4), \quad y_k = \pi(k - {}^1/_4). \tag{A.2}$$

The determinant c^0 contains in the denominator the product of all the factors $x_l - y_k$, and in the numerator the product of all possible differences $x_l - x_{l'}$ and $y_k - y_{k'}$, since the determinant must be zero when two rows or two columns are equal. The constant coefficient can be found, for example, by considering the case $x_k \to y_k$. The result is

$$\mathrm{Det}\, c_0 = (-1)^{r(r-1)/2} \prod_{l<l'} (x_l - x_{l'}) \prod_{k<k'} (y_k - y_{k'}) \times \prod_{k,l} (x_l - y_k)^{-1}. \tag{A.3}$$

The algebraic complement of the k, l element of c_0 is of the same form as (A.3), but does not contain the factors with x_k and y_l. The k, l element of the inverse matrix is equal to the algebraic complement of the l, k element of c_0 divided by Det c_0:

$$(c_0^{-1})_{kl} = (-1)^{r-1}(x_k - y_l)^{-1} \prod_{\substack{m=1 \\ n=1}}^{r} (x_k - y_m)(x_n - y_l) \times \prod_{\substack{m \neq k \\ n \neq l}} (x_m - x_k)^{-1}(y_n - y_l)^{-1}. \tag{A.4}$$

Substituting here the values of x_k, y_l from (A.2), we get

$$(c_0^{-1})_{kl} = \frac{1}{\pi(k - l + {}^1/_2)} \times \frac{\Gamma(k + {}^1/_2)\Gamma(r - k + {}^1/_2)\Gamma(l - {}^1/_2)\Gamma(r - l + {}^3/_2)}{\Gamma(k)\Gamma(r - k + 1)\Gamma(l)\Gamma(r - l + 1)}. \tag{A.5}$$

In the treatment of the behavior of G(r) for small r the important elements $(c_0^{-1})_{kl}$ are those with k, l not close to the edges of the matrix, i.e., such that k, l, r − k, r − $l \gg 1$. We can then use the Stirling formula in (A.5):

$$(c_0^{-1})_{kl} = \frac{1}{\pi(k - l + {}^1/_2)} \left(\frac{(r - l)k}{l(r - k)}\right)^{1/2}. \tag{A.6}$$

If k, l are not too far apart, so that $|k - l| \ll r$, the expression (A.6) becomes a difference formula:

$$(c_0^{-1})_{kl} = 1/\pi(k - l + {}^1/_2). \tag{A.7}$$

We note that the expression (A.7) is the Fourier component of $f_0^{-1}(\omega)$ as in Eq. (22):

$$\frac{1}{\pi} \sum_n \frac{e^{in\omega}}{n + {}^1/_2} = f_0^{-1}(\omega) = (-e^{-i\omega})^{-1/2}. \tag{A.8}$$

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